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Applications of Horadam Polynomials for Bi-univalent Functions associated with Gamma-Starlike Function

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ABSTRACT

This study presents a novel subclass of analytic and bi-univalent functions within the open unit disk, utilizing Horadam polynomials related to Gamma-starlike functions. Our primary focus is on deriving upper bounds for the second and third Taylor-Maclaurin coefficients of functions belonging to this subclass. We employ subordination techniques and properties of Horadam polynomials to establish these coefficient estimates.

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1 . Introduction

Define A as the class of all normalized analytic functions ξ within the open unit disk $\mathcal{Q} = \{\mathfrak{z}: \mathfrak{z} \in \mathbb{C}, |\mathfrak{z}| < 1\}$ of the form:

$$\xi(\mathfrak{z}) = \mathfrak{z} + \sum_{n=2}^{\infty} a_n \mathfrak{z}^n, \qquad (\mathfrak{z} \in \mathcal{Q}). \tag{1.1}$$

A function ξ possesses an inverse ξ^{-1} such that $\xi^{-1}\big(\xi(\mathfrak{z})\big) = \mathfrak{z}$, where $\mathfrak{z} \in \mathcal{Q}$

$$\xi\big(\xi^{-1}(\omega)\big) = \omega, \qquad \left(|\omega| < r_0(\xi), r_0(\xi) \geq \frac{1}{4}\right),$$

where

$$g(\omega) = \xi^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots, (\omega \in \mathcal{Q}). \tag{1.2}$$

When ξ and ξ^{-1} are both univalent functions in Q, therefore ξ is classified as bi-univalent in Q, and the set of bi-univalent functions defined in Q is represented by $\sum$. Refer to [12].

An analytic function ξ is subservient to another function g if there exists an analytic function $\omega: Q \longrightarrow Q$ such that $\omega(0) = 0$ Let $\xi(z) = g(\omega(z))$ for $z \in Q$, denoted as $\xi \prec g$.

Should the function g be univalent in Q, the following equivalence holds

$$\xi(z) \prec g(z) \Leftrightarrow \xi(0) = g(0) \text{ with } \xi(Q) \subset g(Q).$$

Ma & Minda [14] proposed a classification of starlike as well as convex functions by the technique of subordination. They studied the classes $S^*(\phi)$ and $G^*(\phi)$, that are defined by

$$S^*(\phi) = \left\{\xi \in A: \frac{z\xi'(z)}{\xi(z)} \prec \phi(z), z \in Q\right\},$$

and

$$G^*(\phi) = \left\{\xi \in A: 1 + \frac{z\xi''(z)}{\xi'(z)} \prec \phi(z), z \in Q\right\}.$$

We indicate by $S^*_{\Sigma}(\phi)$ and $G^*_{\Sigma}(\phi)$ the classes of bi-starlike and bi-convex functions, respectively, where f is classified as bi-starlike and bi-convex of Ma-Minda type [14].

The pioneering research of Shakir et al. [15] has significantly revitalized the study of bi-univalent functions in recent years. For a succinct historical summary and numerous compelling instances of functions within the class Σ, one may consult this foundational research. Many authors have suggested and analyzed various subclasses of Σ, in which they established non-sharp bounds for the initial Taylor-Maclaurin coefficients. For further information, refer to sources [1,2,3,4,7,8,9,10,16].

Hörcum and Kocer [12] examined the Horadam polynomials $\mathcal{k}_n(r)$, which are characterized by the subsequent recurrence relation (see also to [11]):

$$\mathcal{k}_n(r) = er\mathcal{k}_{n-1}(r) + d\mathcal{k}_{n-2}(r) \quad (r \in \mathbb{R}, n \in \mathbb{N}),$$

with

$$\mathcal{k}_1(r) = a \quad \text{and} \quad \mathcal{k}_2(r) = br, \tag{1.3}$$

The production function for the Horadam polynomials $\mathcal{k}_n(r)$ is delineated in reference [5].

$$\prod(r, z) = \sum_{n=1}^{\infty} \mathcal{k}_n(r) z^{n-1} = \frac{a + (b - ae)rz}{1 - erz - dz^2}. \tag{1.4}$$

Definition (1.1) [6]. A function $\xi \in A$ is classified as a Gamma-starlike function, represented as $\xi \in ST_\sigma$ $(0 \leq \sigma \leq 1)$, if and only if:

$$Re\left[\left(\frac{zf'(z)}{f(z)}\right)^{1-\sigma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\sigma}\right] \geq 0 \ (z \in Q). \tag{1.5}$$

This class was delineated and analyzed by Lewandowski, Miller, and Zlotkiewicz [13]. It is noted that for $\sigma = 0$ with $\sigma = 1$, we obtain $ST_0 = ST$ and $ST_1 = CV$, respectively.

2- Main result

Definition (2.1). A function $\xi \in \sum$, as defined by (1.1), is classified within the class $\mathcal{HG}_{\sum}(\sigma, r)$ if it fulfills the following issues such as

$$\left(\frac{z\xi'(z)}{\xi(z)}\right)^{1-\sigma}\left(1+\frac{z\xi''(z)}{\xi'(z)}\right)^{\sigma} \prec \prod(r, z) + 1 - a, \tag{2.1}$$

and

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)^{1-\sigma}\left(1+\frac{\omega g''(\omega)}{g'(\omega)}\right)^{\sigma} \prec \prod(r, \omega) + 1 - a, \tag{2.2}$$

where $(0 \leq \sigma \leq 1, r \in \mathbb{R})$, $z, \omega \in Q$ and $g = \xi^{-1}$ is given by (1.2).

By substituting $\sigma = 0$ in Definition (2.1), we derive the subsequent Remark indicating that $\mathcal{HG}_{\sum}(\sigma, r) = \mathcal{HG}_{\sum}(r)$.

Remark (2.1). A function $\xi \in \sum$, as indicated by (1.1), is categorized into the class $\mathcal{HG}_{\sum}(r)$ if it meets the subsequent criteria:

$$\frac{z\xi'(z)}{\xi(z)} \prec \prod(r, z) + 1 - a, \tag{2.3}$$

and

$$\frac{\omega g'(\omega)}{g(\omega)} \prec \prod(r, \omega) + 1 - a, \tag{2.4}$$

where $z, \omega \in Q$ and $g = \xi^{-1}$ is given by (1.2).

By substituting $\sigma = 1$ in Definition (2.1), we derive the subsequent Remark indicating that $\mathcal{HG}_{\sum}(\sigma, r) = \mathcal{HG}_{\sum}(1, r)$.

Remark (2.2). A function $\xi \in \sum$, as indicated by (1.1), is categorised within the class $\mathcal{HG}_{\sum}(1, r)$ if it satisfies the subsequent conditions:

$$1+\frac{z\xi''(z)}{\xi'(z)} \prec \prod(r, z) + 1 - a, \tag{2.5}$$

and

$$1+\frac{\omega g''(\omega)}{g'(\omega)} \prec \prod(r,\omega)+1-a, \tag{2.6}$$

where $z, \omega \in Q$ and $g=\xi^{-1}$ is given by (1.2).

Theorem (2.1). Given $0 \leq \sigma \leq 1$ with $r \in \mathbb{R}$, and consider $\xi \in A$ that belongs to the class $\mathcal{HG}_{\Sigma}(\sigma, r)$. Subsequently

$$|a_2| \leq \frac{|br|\sqrt{|br|}}{\sqrt{|(\sigma^2+\sigma+2)b^2r^2-2(1+\sigma)^2(ebr^2+da)|}}, \tag{2.7}$$

and

$$|a_3| \leq \frac{|br|}{2(1+2\sigma)}+\frac{b^2r^2}{(1+\sigma)^2}. \tag{2.8}$$

Proof. Suppose that f belongs to $\mathcal{HG}_{\Sigma}(\sigma, r)$ let g be the inverse of f. Subsequently, there exist two analytic functions, s and $t: Q \to Q$ defined by

$$s(z)=s_1 z+s_2 z^2+s_3 z^3+\cdots, \qquad (z \in Q) \tag{2.9}$$

and

$$t(\omega)=t_1\omega+t_2\omega^2+t_3\omega^3+\cdots, \qquad (\omega \in Q) \tag{2.10}$$

such that $s(0)=0$ with $t(0)=0$, $|s(z)|<1$, $|t(\omega)|<1$, and

$$\left(\frac{z\xi'(z)}{\xi(z)}\right)^{1-\sigma}\left(1+\frac{z\xi''(z)}{\xi'(z)}\right)^{\sigma}=\prod(r,z)+1-a,$$

and

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)^{1-\sigma}\left(1+\frac{\omega g''(\omega)}{g'(\omega)}\right)^{\sigma}=\prod(r,\omega)+1-a$$

Or, equivalently

$$\left(\frac{z\xi'(z)}{\xi(z)}\right)^{1-\sigma}\left(1+\frac{z\xi''(z)}{\xi'(z)}\right)^{\sigma}=1+k_1(r)-a+k_2(r)s(z)+k_3(r)s^2(z)+\cdots \tag{2.11}$$

and

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)^{1-\sigma}\left(1+\frac{\omega g''(\omega)}{g'(\omega)}\right)^{\sigma}=1+k_1(r)-a+k_2(r)t(\omega)+k_3(r)t^2(\omega)+\cdots. \tag{2.12}$$

Combining (2.9), (2.10), (2.11) and (2.12), yield

$$\left(\frac{z\xi'(z)}{\xi(z)}\right)^{1-\sigma}\left(1+\frac{z\xi''(z)}{\xi'(z)}\right)^{\sigma}=1+k_2(r)s_1z+[k_2(r)s_2+k_3(r)s_1^2]z^2+\cdots \quad (2.13)$$

and

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)^{1-\sigma}\left(1+\frac{\omega g''(\omega)}{g'(\omega)}\right)^{\sigma}=1+k_2(r)t_1\omega+[k_2(r)t_2+k_3(r)t_1^2]\omega^2+\cdots. \quad (2.14)$$

It is quite well-known that if $|s(z)|<1, |t(\omega)|<1, z, \omega \in Q$ we get

$$|s_i|<1, |t_i|<1 (\forall \text{ i} \in \mathbb{N}). \quad (2.15)$$

By comparing the corresponding coefficients in (2.13) as well as (2.14), oversimplification yields

$$(1+\sigma)a_2=k_2(r)s_1, \quad (2.16)$$

$$2(1+2\sigma)a_3+\left(\frac{\sigma^2-7\sigma-2}{2}\right)a_2^2=k_2(r)s_2+k_3(r)s_1^2, \quad (2.17)$$

$$-(1+\sigma)a_2=k_2(r)t_1, \quad (2.18)$$

and

$$\left(\frac{\sigma^2+9\sigma+6}{2}\right)a_2^2-2(1+2\sigma)a_3=k_2(r)t_2+k_3(r)t_1^2. \quad (2.19)$$

From (2.16) and (2.18), we have

$$s_1=-t_1, \quad (2.20)$$

and

$$2(1+2\sigma)^2a_2^2=k_2^2(r)(s_1^2+t_1^2). \quad (2.21)$$

If we add (2.17) to (2.19), we deduce that

$$(\sigma^2+\sigma+2)a_2^2=k_2(r)(s_2+t_2)+k_3(r)(s_1^2+t_1^2). \quad (2.22)$$

By putting the numerical value of $s_1^2+t_1^2$ given (2.21) with the correct side of (2.22), we ascertain that

$$a_2^2=\frac{k_2^3(r)(s_2+t_2)}{(\sigma^2+\sigma+2)k_2^2(r)-2(1+2\sigma)^2k_3(r)}. \quad (2.23)$$

Further computations using (1.3), (2.15) and (2.23), we get

$$|a_2| \le \frac{|b\mathcal{r}|\sqrt{|b\mathcal{r}|}}{\sqrt{|(\sigma^2+\sigma+2)b^2\mathcal{r}^2 - 2(1+\sigma)^2(eb\mathcal{r}^2+da)|}}.$$

Now, if we subtract (2.19) from (2.17), we get

$$4(1+2\sigma)(a_3 - a_2^2) = \mathcal{k}_2(\mathcal{r})(\mathcal{s}_2 - \mathcal{t}_2) + \mathcal{k}_3(\mathcal{r})(\mathcal{s}_1^2 - \mathcal{t}_1^2). \tag{2.24}$$

In view of (2.20) and (2.21), we obtain from (2.24)

$$a_3 = \frac{\mathcal{k}_2(\mathcal{r})(\mathcal{s}_2 - \mathcal{t}_2)}{2(1+2\sigma)} + \frac{\mathcal{k}_2^2(\mathcal{r})(\mathcal{s}_1^2 + \mathcal{t}_1^2)}{2(1+2\sigma)^2}. \tag{2.25}$$

Applying (1.3), we can easily see that

$$|a_3| \le \frac{|b\mathcal{r}|}{2(1+2\sigma)} + \frac{b^2\mathcal{r}^2}{(1+\sigma)^2}.$$

By setting $\sigma = 1$ within Theorem (2.1), we obtain the subsequent corollary:

Corollary (2.1). Given $\mathcal{r} \in \mathbb{R}$, assume f $\in$ A belong to the class $\mathcal{HG}_{\Sigma}(1,\mathcal{r})$. Subsequently

$$|a_2| \le \frac{|b\mathcal{r}|\sqrt{|b\mathcal{r}|}}{2\sqrt{|b^2\mathcal{r}^2 - (eb\mathcal{r}^2+da)|}}, \tag{2.26}$$

and

$$|a_3| \le \frac{|b\mathcal{r}|}{6} + \frac{b^2\mathcal{r}^2}{4}. \tag{2.27}$$

By putting $\sigma = 0$ in Theorem(2.1), we get the next Corollary:

Corollary (2.2). For $\mathcal{r} \in \mathbb{R}$, and consider $f \in A$ that belongs to the class $\mathcal{HG}_{\Sigma}(\mathcal{r})$. Then

$$|a_2| \le \frac{|b\mathcal{r}|\sqrt{|b\mathcal{r}|}}{\sqrt{|2b^2\mathcal{r}^2 - 2(eb\mathcal{r}^2+da)|}}, \tag{2.28}$$

and

$$|a_3| \le \frac{|b\mathcal{r}|}{2} + b^2\mathcal{r}^2. \tag{2.29}$$

In the next theorem, we present the “Fekete-Szegö inequality” for $\xi \in \mathcal{HG}_{\Sigma}(\sigma,\mathcal{r})$.

Theorem (2.2). Given $0 \le \sigma \le 1$ as well as $\mu \in \mathbb{R}$, let $\xi \in A$ belong to the class $\mathcal{HG}_{\Sigma}(\sigma,\mathcal{r})$. Subsequently

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{|b\mathscr{r}|}{2(1+2\sigma)}\;; \\ \left(|\mu-1| \le \left|\dfrac{(\sigma^2+\sigma+2)b^2\mathscr{r}^2-2(1+\sigma)^2(eb\mathscr{r}^2+da)}{4(1+2\sigma)b^2\mathscr{r}^2}\right|\right), \\ \dfrac{2|\mu-1||b^3\mathscr{r}^3|}{|(\sigma^2+\sigma+2)b^2\mathscr{r}^2-2(1+\sigma)^2(eb\mathscr{r}^2+da)|}\;; \\ \left(|\mu-1| \ge \left|\dfrac{(\sigma^2+\sigma+2)b^2\mathscr{r}^2-2(1+\sigma)^2(eb\mathscr{r}^2+da)}{4(1+2\sigma)b^2\mathscr{r}^2}\right|\right). \end{cases}$$

Proof. From (2.23) and (2.25), we get

$$a_3-\mu a_2^2 = \frac{\mathscr{k}_2(\mathscr{r})}{4(1+2\sigma)}(\mathscr{s}_2-\mathscr{t}_2)+(1-\mu)\frac{\mathscr{k}_2^3(\mathscr{r})(\mathscr{s}_2+\mathscr{t}_2)}{(\sigma^2+\sigma+2)\mathscr{k}_2^2(\mathscr{r})-2(1+\sigma)^2\mathscr{k}_3(\mathscr{r})}$$

$$= \mathscr{k}_2(\mathscr{r})\left[\left(\mathrm{G}(\mu;\mathscr{r})+\frac{1}{4(1+2\sigma)}\right)\mathscr{s}_2+\left(\mathrm{G}(\mu;\mathscr{r})-\frac{1}{4(1+2\sigma)}\right)\mathscr{t}_2\right],$$

where

$$G(\mu;\mathscr{r})\frac{(1-\mu)\mathscr{k}_2^2(\mathscr{r})}{(\sigma^2+\sigma+2)\mathscr{k}_2^2(\mathscr{r})-2(1+\sigma)^2\mathscr{k}_3(\mathscr{r})}.$$

Thus, according to (1.3), we have

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{|b\mathscr{r}|}{2(1+2\sigma)}\;; & \left(0 \le |G(\mu;\mathscr{r})| \le \dfrac{1}{4(1+2\sigma)}\right), \\ 2|b\mathscr{r}||G(\mu;x)|\;; & \left(|G(\mu;\mathscr{r})| \ge \dfrac{1}{4(1+2\sigma)}\right), \end{cases}$$

hence, after some calculations, gives

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{|b\mathscr{r}|}{2(1+2\sigma)}\;; \\ \left(|\mu-1| \le \left|\dfrac{(\sigma^2+\sigma+2)b^2\mathscr{r}^2-2(1+\sigma)^2(eb\mathscr{r}^2+da)}{4(1+2\sigma)b^2\mathscr{r}^2}\right|\right), \\ \dfrac{2|\mu-1||b^3\mathscr{r}^3|}{|(\sigma^2+\sigma+2)b^2\mathscr{r}^2-2(1+\sigma)^2(eb\mathscr{r}^2+da)|}\;; \\ \left(|\mu-1| \ge \left|\dfrac{(\sigma^2+\sigma+2)b^2\mathscr{r}^2-2(1+\sigma)^2(eb\mathscr{r}^2+da)}{4(1+2\sigma)b^2\mathscr{r}^2}\right|\right). \end{cases}$$

By substituting $\sigma = 1$ in Theorem (2.2), we derive the subsequent result:

Corollary (2.3). If $\xi \in A$ defined by (1.1) be in the class $\mathcal{HG}_{\Sigma}(1, r)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|br|}{6}\,; & \left(|\mu - 1| \leq \left|\dfrac{b^2 r^2 - 2(ebr^2 + da)}{3b^2 r^2}\right|\right), \\ \\ \dfrac{|\mu - 1||b^3 r^3|}{|2b^2 r^2 - 4(ebr^2 + da)|}\,; & \left(|\mu - 1| \geq \left|\dfrac{b^2 r^2 - 2(ebr^2 + da)}{3b^2 r^2}\right|\right). \end{cases}$$

By substituting $\sigma = 0$ in Theorem (2.2), we derive the subsequent result:

Corollary (2.4). If $\xi \in A$ defined by (1.1) be in the class $\mathcal{HG}_{\Sigma}(r)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|br|}{2}\,; & \left(|\mu - 1| \leq \left|\dfrac{b^2 r^2 - (ebr^2 + da)}{2b^2 r^2}\right|\right), \\ \\ \dfrac{|\mu - 1||b^3 r^3|}{|b^2 r^2 - (ebr^2 + da)|}\,; & \left(|\mu - 1| \geq \left|\dfrac{b^2 r^2 - (ebr^2 + da)}{2b^2 r^2}\right|\right). \end{cases}$$

Conclusions

This study presents a novel subclass of analytic and bi-univalent functions utilizing Horadam polynomials related to Gamma-starlike functions. This work's primary accomplishments are determining upper limits for the second and third Taylor-Maclaurin coefficients of functions within this category. These results contribute significantly to the understanding of coefficient bounds in bi-univalent function theory, an area of study that remains rich with challenges and open problems. The findings also extend earlier work in this field by providing sharper bounds under specific mathematical conditions.

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