# SUBCLASS OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY KOMATU OPERATOR

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**Abstract:** In this paper ,we introduce some properties of the class $T(p,A,B,\alpha,c,\delta)$ for multivalent functions with negative coefficients defined by Komatu operator .We obtain coefficient estimates ,growth and distortion theorem, radius of convexity for the class $T(p,A,B,\alpha,c,\delta)$, closure theorems and convolution property.

**Key words and phrases:** Multivalent Functions, Komatu Operator, Coefficient Estimates, Distortion Theorem, Radius of Convexity, Closure Theorems, Convolution Property.
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## 1. Introduction

Let W(p),(p? 1) denote the class of functions of the form:

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n,\ p\in \mathbb{N}=\{1,2,3,....\}, \tag{1}$$

which are analytic and multivalent in the open unit disk U= {z ∈ ¢: |z|<1}

if a function f is given by (1) and g is defined by $g(z)=z^p+\sum_{n=p+1}^{\infty}b_nz^n$ (2)

is in W(p),then convolution or Hadamard product of f(z) and g(z) is defined by

$$(f*g)(z)=z^p+\sum_{n=p+1}^{\infty}a_nb_nz^n,z\in U \tag{3}$$

Let T(p) denote the subclass of W(p) or which is consisting of function of the form

$$f(z)=z^p-\sum_{n=p+1}^{\infty}a_nz^n,(a_n\geq 0) \tag{4}$$

**Definition 1:**The integral operator of $f\in T(p)$ $for$ $c>-p,\delta\geq 0,$ is defined by G and defined as following :

$$\begin{aligned}G(z)&=\frac{(c+p)^{\delta}}{\Gamma(\delta)}\int_0^1 t^c(\log\frac{1}{t})^{\delta-1}\frac{f(tz)}{t}dt,c>-p,\delta\geq 0\\ &=z^p-\sum_{n=p+1}^{\infty}(\frac{c+p}{c+n})^{\delta}a_nz^n\end{aligned} \tag{5}$$

The operator defined by (5) known as the Komatu operator [1].

**Defination 2 :**The function f(z) is said to be subordinate to g(z) in U written $f(z)\prec g(z)$ ,if there exist a function W(z) analytic in U such that W(0)=0,and |W(z)|<1,such that f(z)=g(W(z)).

**Definition 3**:For A,B arbitrary fixed real number,-1? B<A? 1, a function $f(z)\in T(p)$ defined by (4) is said to be in the class $T(p,A,B,\alpha,c,\delta)$ *if it satisfies*

$$\frac{zG'(z)}{G(z)} \prec \frac{1+[(A-B)(p-\alpha)+B]z}{1+Bz}, (z \in U), \tag{6}$$

where $0 \le \alpha \le p$, and G(z) is defined in definition(1).The condition(6) is equivalent to

$$\left| \frac{\frac{zG'(z)}{pG(z)} - 1}{B + (A-B)(p-\alpha) - B\frac{zG'(z)}{pG(z)}} \right| < 1, z \in U. \tag{7}$$

for other subclasses of multivalent functions , we can see the recent works of authors[2],[3].

## 2. Coefficient Estimates

**Theorem (1):** A function f (z) defined by (4) belongs to the class $T(p, A, B, \alpha, c, \delta)$ if and only if

$$\sum_{n=p+1}^{\infty}[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta} a_n \le p(A-B)(p-\alpha), \tag{8}$$

the result is sharp.

**Proof**: Assume that the inequality (8) holds true and let $|z|=1$, then from (7) and (5), we have

$$\left|\frac{zG'(z)}{pG(z)} - 1\right| - \left|B + (A-B)(p-\alpha) - B\frac{zG'(z)}{pG(z)}\right| \tag{9}$$

$$= \left| -\sum_{n=p+1}^{\infty}(n-p)(\frac{c+p}{c+n})^{\delta} a_n z^n \right|$$

$$- \left| p(A-B)(p-\alpha)z^P + \sum_{n=p+1}^{\infty}(B(n-p)-p(A-B)(p-\alpha))(\frac{c+p}{c+n})^{\delta} a_n z^n \right|$$

$$\le \sum_{n=p+1}^{\infty}(n-p)(\frac{c+p}{c+n})^{\delta} a_n + \sum_{n=p+1}^{\infty}(p(A-B)(p-\alpha)-B(n-p))(\frac{c+p}{c+n})^{\delta} a_n - p(A-B)(p-\alpha)$$

$$= \sum_{n=p+1}^{\infty}((1-B)(n-p)+p(A-B)(p-\alpha))(\frac{c+p}{c+n})^{\delta} a_n - p(A-B)(P-\alpha) \le 0.$$

Hence by the principle of maximum modulus , $f(z) \in T(p, A, B, \alpha, c, \delta)$ . Conversely, assume that f(z) defined by (4) is in the class $T(p, A, B, \alpha, c, \delta)$ . Then from (5), we have

$$\left| \frac{\frac{zG'(z)}{pG(z)} - 1}{B + (A-B)(p-\alpha) - B\frac{zG'(z)}{pG(z)}} \right|$$

$$= \left| \frac{-\sum_{n=p+1}^{\infty}(n-p)(\frac{c+p}{c+n})^{\delta} a_n z^n}{p(A-B)(p-\alpha)z^p + \sum_{n=p+1}^{\infty}(B(n-p)-p(A-B)(p-\alpha))(\frac{c+p}{c+n})^{\delta} a_n z^n} \right|.$$

Since |Re (z) |? |z| for all z, we have

$$\operatorname{Re}\left\{\frac{\sum_{n=p+1}^{\infty}(n-p)(\frac{c+p}{c+n})^{\delta}a_n z^n}{p(A-B)(p-\alpha)z^p+\sum_{n=p+1}^{\infty}(B(n-p)-(p(A-B)(p-\alpha))(\frac{c+p}{c+n})^{\delta}a_n z^n}\right\}<1.$$

Choose the values of z on the real axis so that $\frac{zG'(z)}{G(z)}$ is real. Upon clearing the denominator of (9) and letting z? 1 through real values ,we get

$$\sum_{n=p+1}^{\infty}(n-p)(\frac{c+p}{c+n})^{\delta}a_n \le p(A-B)(p-\alpha)z^p+\sum_{n=p+1}^{\infty}(B(n-p)-p(A-B)(p-\alpha))(\frac{c+p}{c+n})^{\delta}a_n,$$

which implies the inequality (8).
Sharpness of the result follows setting

$$f(z)=z^p-\frac{p(A-B)(p-\alpha)}{((1-B)(n-p)+p(A-B)(p-\alpha))(\frac{c+p}{c+n})^{\delta}}z^n, n\ge p+1 \qquad (10)$$

**Corollary (1):** Let the function f(z) defined by (4) be in the class $T(p,A,B,\alpha,c,\delta)$ ,then

$$a_n \le \frac{p(A-B)(p-\alpha)}{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}}, n\ge p+1 \qquad (*)$$

The equality in (*) is attained for the function f (z) given by (10).

## 3. Distortion and Growth Theorems

**Theorem (2):** Let the function f (z) defined by (4) be in the class $T(p,A,B,\alpha,c,\delta)$. Then, for |z|=r (0<r<1).

$$r^p-\frac{p(A-B)(p-\alpha)r^{p+1}}{((1-B)+(p(A-B)(p-\alpha))(\frac{c+p}{c+p+1})^{\delta}}\le|f(z)|\le r^p+\frac{p(A-B)(p-\alpha)r^{p+1}}{((1-B)+p(A-B)(p-\alpha))(\frac{c+p}{c+p+1})^{\delta}} \qquad (11)$$

for $z\in U$ .The result (11) is sharp.

**Proof:** Since $f(z)\in T(p,A,B,\alpha,c,\delta)$, in view of Theorem (1), we have

$$[(1-B)+p(A-B)(p-\alpha)](\frac{c+p}{c+p+1})^{\delta}\sum_{n=p+1}^{\infty}a_n$$
$$\le \sum_{n=p+1}^{\infty}[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}a_n \le p(A-B)(p-\alpha), \qquad (12)$$

which immediately yields

$$\sum_{n=p+1}^{\infty}a_n \le \frac{p(A-B)(p-\alpha)}{[(1-B)+p(A-B)(p-\alpha)](\frac{c+p}{c+p+1})^{\delta}} \qquad (13)$$

Consequently, for |z|= r (0<r<1),we obtain

$$|f(z)| \geq r^p - r^{p+1} \sum_{n=p+1}^{\infty} a_n$$

$$\geq r^p - \frac{p(A-B)(p-\alpha)}{\left[(1-B)+p(A-B)(p-\alpha)\right](\frac{c+p}{c+p+1})^{\delta}} r^{p+1} \tag{14}$$

*and* $$|f(z)| \leq r^p + r^{p+1} \sum_{n=p+1}^{\infty} a_n$$

$$\leq r^p + \frac{p(A-B)(p-\alpha)}{\left[(1-B)+p(A-B)(p-\alpha)\right](\frac{c+p}{c+p+1})^{\delta}} r^{p+1}, \tag{15}$$

*for* $z \in U$.

This completes the proof of Theorem (2).Finally, by taking the function

$$f(z) = z^p - \frac{p(A-B)(p-\alpha)}{((1-B)+p(A-B)(p-\alpha))(\frac{c+p}{c+p+1})^{\delta}} z^{p+1} \tag{16}$$

we can show that the result of Theorem (2) is sharp.

**Corollary (2):** Under the hypothesis of Theorem (2), f (z) is included in a disk with its center at the origin and Radius $r_1$ given by

$$r_1 = 1 + \frac{p(A-B)(p-\alpha)}{\left[(1-B)+p(A-B)(p-\alpha)\right](\frac{c+p}{c+p+1})^{\delta}} \tag{17}$$

**Theorem (3):** Let the function f (z) defined by (4) be in the class $T(p, A, B, \alpha, c, \delta)$ .Then, for |z|=r (0<r<1).

$$pr^{p-1} - \frac{p(p+1)(A-B)(p-\alpha)r^p}{((1-B)+p(A-B)(p-\alpha))(\frac{c+p}{c+p+1})^{\delta}} \leq |f'(z)| \leq pr^{p-1} + \frac{p(p+1)(A-B)(p-\alpha)r^p}{((1-B)+p(A-B)(p-\alpha))(\frac{c+p}{c+p+1})^{\delta}} \tag{18}$$

for $z \in U$ .The result (18) is sharp.

**Proof:** In view of Theorem (1), we have

$$\frac{\left[(1-B)+p(A-B)(p-\alpha)\right]\left(\frac{c+p}{c+p+1}\right)^{\delta}}{(p+1)} \sum_{n=p+1}^{\infty} na_n \leq \sum_{n=p+1}^{\infty} \left[(1-B)(n-p)+p(A-B)(p-\alpha)\right] \cdot \left(\frac{c+p}{c+n}\right)^{\delta} a_n \leq p(A-B)(p-\alpha), \tag{19}$$

which readily yields

$$\sum_{n=p+1}^{\infty} na_n \leq \frac{p(p+1)(A-B)(p-\alpha)}{\left[(1-B)+p(A-B)(p-\alpha)\right]\left(\frac{c+p}{c+p+1}\right)^{\delta}} \tag{20}$$

Consequently, for |z|=r (0<r<1), we obtain

$$\left|f'(z)\right| \geq pr^{p-1} - r^p \sum_{n=p+1}^{\infty} na_n \geq pr^{p-1} - \frac{p(p+1)(A-B)(p-\alpha)}{\left[(1-B)+p(A-B)(p-\alpha)\right]\left(\frac{c+p}{c+p+1}\right)^{\delta}} r^p \tag{21}$$

$$and \quad \left|f'(z)\right| \leq pr^{p-1} + r^p \sum_{n=p+1}^{\infty} na_n$$

$$\leq pr^{p-1} + \frac{p(p+1)(A-B)(p-\alpha)}{\left[(1-B)+p(A-B)(p-\alpha)\right]\left(\frac{c+p}{c+p+1}\right)^{\delta}} r^p \tag{22}$$

For $z \in U$ .Further the result of Theorem (3) is sharp for the function f(z) given by (16).

**Corollary (3):** Under the hypothesis of Theorem (3) ,f '(z) is included in a disk with its center at the origin and radius $r_2$ given by

$$r_2 = p + \frac{p(p+1)(A-B)(p-\alpha)}{\left[(1-B)+p(A-B)(p-\alpha)\right](\frac{c+p}{c+p+1})^{\delta}} \tag{23}$$

the result is sharp with extremal function f(z) given by (16)

## 4. Raduis of Convexity for the Class $T(p, A, B, \alpha, c, \delta)$

**Theorem (4):** Let the function f(z) defined by (4) be in the class $T(p, A, B, \alpha, c, \delta)$ Then f(z) is p-valently convex in the disk $|z|<WA_p$, where

$$WA_p = \inf_{n \geq p+1} \left\{ \frac{p^2\left[(1-B)(n-p)+p(A-B)(p-\alpha)\right](\frac{c+p}{c+n})^{\delta}}{n^2 p(A-B)(p-\alpha)} \right\}^{\frac{1}{n-p}} \tag{24}$$

The result is sharp.

**Proof:** To prove Theorem (4) ,it is sufficient to show that

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right| \leq p, \; for \quad |z|<WA_p .$$

Indeed, we have

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right| = \left|\frac{-\sum_{n=p+1}^{\infty} n(n-p)a_n z^{n-p}}{p-\sum_{n=p+1}^{\infty} na_n z^{n-p}}\right| \leq \frac{\sum_{n=p+1}^{\infty} n(n-p)a_n |z|^{n-p}}{p-\sum_{n=p+1}^{\infty} na_n |z|^{n-p}} .$$

Thus

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right| \leq p \quad if \quad \frac{\sum_{n=p+1}^{\infty} n(n-p)a_n |z|^{n-p}}{p-\sum_{n=p+1}^{\infty} na_n |z|^{n-p}} \leq p,$$

that is, if $\sum_{n=p+1}^{\infty}\left(\frac{n}{p}\right)^2 a_n |z|^{n-p} \le 1$ .

But, from Theorem (1), we obtain

$$\sum_{n=p+1}^{\infty}\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}}{p(A-B)(p-\alpha)}a_n \le 1.$$

Hence the function f(z) is p-valently convex if

$$\left(\frac{n}{p}\right)^2 |z|^{n-p} \le \frac{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}}{p(A-B)(p-\alpha)}, n \ge p+1$$

that is ,if

$$|z| \le \left\{\frac{p^2[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}}{n^2 p(A-B)(p-\alpha)}\right\}^{\frac{1}{n-p}} \quad , n \ge p+1$$

This evidently completes the proof of Theorem (4). The result is sharp with extremal function f (z) given by (10)

## 5. A set of Closure Theorem:

Here, we shall prove that the class $T(p, A, B, \alpha, c, \delta)$ is closed under arithmetic mean and under Convex linear combinations.

**Theorem (5):** Let $f_j(z) = z^p - \sum_{p+1}^{\infty} a_{n,j} z^n$ (25)

$(a_n \ge 0, p \in \mathbb{N}, j = 1,2,\ldots\ldots,m)$. *If* $f_j(z) \in T(p, A, B, \alpha, c, \delta)$ $(j = 1,2,\ldots.,m)$,

then the function $g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n$

also belongs to the class $T(p, A, B, \alpha, c, \delta)$ , where $b_n = \frac{1}{m}\sum_{j=1}^{m} a_{n,j}$ .

**Proof:** Since $f_j(z) \in T(p, A, B, \alpha, c, \delta)$ $(j = 1,2,\ldots.,m)$, It follows from Theorem (1) that

$$\sum_{n=p+1}^{\infty}[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta} a_{n,j} \le p(A-B)(p-\alpha), j = 1,2,\ldots m$$

Therefore, we have

$$\sum_{n=p+1}^{\infty}[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta} b_n$$

$$= \sum_{n=p+1}^{\infty}[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}\left\{\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right\}$$

$$= \frac{1}{m}\sum_{j=1}^{m}\sum_{n=p+1}^{\infty}[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta} a_{n,j} \le p(A-B)(p-\alpha).$$

Hence by ,Theorem (1) , $g(z) \in T(p, A, B, \alpha, c, \delta)$ ,which completes the proof of Theorem(5).

**Theorem (6):** $T(p,A,B,\alpha,c,\delta)$ is closed under convex linear combination .

**Proof:** Let the function $f_j(z)(j=1,2)$ defined by (25) be in the class $T(p,A,B,\alpha,c,\delta)$ it is sufficient to show that the function h(z) defined by

$$h(z)=\gamma f_1(z)+(1-\gamma)f_2(z),(0\le\gamma\le 1)$$

is in the class $T(p,A,B,\alpha,c,\delta)$ since for $(0\le\gamma\le 1)$ .

$$h(z)=z^p-\sum_{n=p+1}^{\infty}[\gamma a_{n,1}+(1-\gamma)a_{n,2}]z^n$$

By applying Theorem(1),we have

$$\sum_{n=p+1}^{\infty}\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}}{p(A-B)(p-\alpha)}[\gamma a_{n,1}+(1-\gamma)a_{n,2}]$$

$$=\gamma\sum_{n=p+1}^{\infty}\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)]}{p(A-B)(p-\alpha)}a_{n,1}+(1-\gamma)\sum_{n=p+1}^{\infty}\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)]}{p(A-B)(p-\alpha)}a_{n,2}\le 1.$$

Which implies that h(z) is in $T(p,A,B,\alpha,c,\delta)$ and this completes the proof.

**Theorem (7):** Let $f_p(z)=z^p$ and

$$f_n(z)=z^p-\frac{p(A-B)(p-\alpha)}{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}}z^n, n\ge p+1.$$

Then $f(z)\in T(p,A,B,\alpha,c,\delta)$ if and only if it can be expressed in the form:

$$f(z)=\lambda_p f_p(z)+\sum_{n=p+1}^{\infty}\lambda_n f_n(z),\ \text{also}\ (\lambda_n\ge 0;\lambda_p+\sum_{n=p+1}^{\infty}\lambda_n=1)$$

**Proof:** Assume that

$$f(z)=\lambda_p f_p(z)+\sum_{n=p+1}^{\infty}\lambda_n f_n(z)$$

$$=z^p-\sum_{n=p+1}^{\infty}\frac{p(A-B)(p-\alpha)}{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}}\lambda_n z^n$$

Then, since

$$\sum_{n=p+1}^{\infty}[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}\cdot\frac{p(A-B)(p-\alpha)}{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}}\lambda_n$$

$$=p(A-B)(p-\alpha)\sum_{n=p+1}^{\infty}\lambda_n\le p(A-B)(p-\alpha),$$

we conclude that $f(z)\in T(p,A,B,\alpha,c,\delta)$, by virtue of Theorem (1).

Conversely,let $f(z)\in T(p,A,B,\alpha,c,\delta)$ .It follows then from Corollary (1) that

$$a_n\le\frac{p(A-B)(p-\alpha)}{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}}, n\ge p+1.$$

Setting

$$\lambda_n = \frac{\left[(1-B)(n-p)+p(A-B)(p-\alpha)\right]\left(\frac{c+p}{c+n}\right)^{\delta}}{p(A-B)(p-\alpha)} a_n$$

*and* $\lambda_p = 1 - \sum_{n=p+1}^{\infty} \lambda_n$ ,

we have $f(z) = \lambda_p f_p(z) + \sum_{n=p+1}^{\infty} \lambda_n f_n(z)$ ,

which completes the proof of Theorem (7).

## 6. Convolution Property:

Here, we prove the convolution result for functions belongs to the class $T(p,A,B,\alpha,c,\delta)$ .

**Theorem (8):** Let $f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n$ *and* $g(z) = z - \sum_{n=p+1}^{\infty} b_n z^n$

belong to $T(p,A,B,\alpha,c,\delta)$ , then $(f*g)(z) \in T(p,A,B,\alpha,c,\delta)$ , where

$$c_1 < \inf_n \left[ \frac{n\left[\frac{[(1-B)+p(A-B)(p-\alpha)]\left(\frac{c+p}{c+n}\right)^{2\delta}}{p(A-B)(p-\alpha)}\right] - p}{1+\left[\frac{[(1-B)+p(A-B)(p-\alpha)]\left(\frac{c+p}{c+n}\right)^{2\delta}}{p(A-B)(p-\alpha)}\right]} \right]$$

**Proof:** By the hypothesis, we can write

$$\sum_{n=p+1}^{\infty}\left[\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)]\left(\frac{c+p}{c+n}\right)^{\delta}}{p(A-B)(p-\alpha)}\right] a_n < 1$$

$$and \sum_{n=p+1}^{\infty}\left[\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)]\left(\frac{c+p}{c+n}\right)^{\delta}}{p(A-B)(p-\alpha)}\right] b_n < 1$$

and by applying the Cauchy-Schwarz inequality, we have

$$\sum_{n=p+1}^{\infty}\left[\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)]\left(\frac{c+p}{c+n}\right)^{\delta}}{p(A-B)(p-\alpha)}\right] \sqrt{a_n b_n}$$

$$\leq \left(\sum_{n=p+1}^{\infty}\left[\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)]\left(\frac{c+p}{c+n}\right)^{\delta}}{p(A-B)(p-\alpha)}\right] a_n\right)^{\frac{1}{2}} \cdot \left(\sum_{n=p+1}^{\infty}\left[\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)]\left(\frac{c+p}{c+n}\right)^{\delta}}{p(A-B)(p-\alpha)}\right] b_n\right)^{\frac{1}{2}}$$

However, we obtain

$$\sum_{n=p+1}^{\infty}\left[\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}}{p(A-B)(p-\alpha)}\right]\sqrt{a_n b_n}<1\cdot \qquad (26)$$

Now, we want to prove

$$\sum_{n=p+1}^{\infty}\left[\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c_1+p}{c_1+n})^{\delta}}{p(A-B)(p-\alpha)}\right]a_n b_n<1. \qquad (27)$$

Let (27) holds true .Then we have

$$\sum_{n=p+1}^{\infty}\left[\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)]}{p(A-B)(p-\alpha)}\right]\sqrt{a_n b_n}\left(\frac{c+p}{c+n}\right)^{\delta}\sqrt{a_n b_n}\frac{\left(\frac{c_1+p}{c_1+n}\right)^{\delta}}{\left(\frac{c+p}{c+n}\right)^{\delta}}<1. \qquad (28)$$

Therefore (28)(consequently (27))holds true if

$$\sqrt{a_n b_n}<\frac{\left(\frac{c+p}{c+n}\right)^{\delta}}{\left(\frac{c_1+p}{c_1+n}\right)^{\delta}} \qquad (29)$$

but from (26) we conclude that

$$\sqrt{a_n b_n}<\frac{p(A-B)(p-\alpha)}{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}} \qquad (30)$$

In view of (30) the inequality (29) holds true if

$$\frac{p(A-B)(p-\alpha)}{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{\delta}}<\frac{\left(\frac{c+p}{c+n}\right)^{\delta}}{\left(\frac{c_1+p}{c_1+n}\right)^{\delta}}$$

or equivalently $c_1<\dfrac{nY^{\frac{1}{\delta}}-p}{1+Y^{\frac{1}{\delta}}}$ ,where

$$Y=\frac{[(1-B)(n-p)+p(A-B)(p-\alpha)](\frac{c+p}{c+n})^{2\delta}}{p(A-B)(p-\alpha)}$$

and this inequality gives the required result.

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