On $\Theta-$ Convergent Filters in Bitopological Spaces

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Abstract. We study the concepts of $\theta-$convergence and $\theta-$accumulates of filters and filter bases in bitopological spaces and we study some properties of these concepts and then we find a relation between the concepts $\theta-$convergence and $\theta-$Hausdorff bitopological space.

1. Introduction .The setting of convergence in terms of filters was sketched by Cartan (1939) and (1939a), and was developed by Bourbaki (1940) (filters first appeared in Riesz (1908)).The equivalence of both theories was noted by Bartle in (1955) as well as by Bruns and Schmidt in (1955) [3]. The definition of filter and ultrafilter given here are those of Sharma[6].***A bitopological space*** [4] , is a non-empty set X with two non identical topologies σ and ρ denoted by (X,σ,ρ). Let Φ and Ψ are two filter bases on X . Ψ is said to be ***subordinate to*** Φ[2] ,if for every $A\in\Phi$ there exists $B\in\Psi$ such that $B\subset A$.

2.Main results .

Definition 2.1 [5] . *Let (X,σ,ρ) be a bitopological spaces . A filter $\mathscr{F}$ on X is said to be $(\sigma-\rho)-\theta-$**converges** to a point $x\in X$, if every $\sigma-$neighborhood N of x, $\rho-Cl(N)\in\mathscr{F}$. We say that x is a $(\sigma-\rho)-\theta-$**limit point** of $\mathscr{F}$.*

Definition 2.2 [5] . *Let (X,σ,ρ) be a bitopological spaces . A filter base Φ on X is said to be $(\sigma-\rho)-\theta-$**converges** to a point $x\in X$, if the filter whose base is Φ $(\sigma-\rho)-\theta-$converges to a point x.We say that x is a $(\sigma-\rho)-\theta-$**limit** point of Φ.*

Proposition 2.3. *Let (X,σ,ρ) be a bitopological space and $\mathscr{F}$ be a filter on X . Then the following statements are equivalent.*
(i) $\mathscr{F}$ is $(\sigma-\rho)-\theta-$converges to a point $x\in X$.
(ii) $\mathscr{F}$ is finer than the collection $\Omega=\{\rho-Cl(N): N \text{ is } \sigma-nhd \text{ of } x\}$
(iii) for every $\sigma-nhd$ N of x, there is $F\in\mathscr{F}$ such that $F\subset\rho-Cl(N)$.

Proof . $(i)\Leftrightarrow(ii)$.Suppose (i). $\rho-Cl(N)\in\mathscr{F}$ for every $\sigma-nhd$ N of x. Hence $\mathscr{F}$ is finer than the collection Ω.
Suppose (ii). $\Omega\subset\mathscr{F}$, then $\rho-Cl(N)\in\mathscr{F}$ for every $\sigma-nhd$ N of x.
$(i)\Leftrightarrow(iii)$. Suppose (i).Since $\rho-Cl(N)\in\mathscr{F}$ for every $\sigma-nhd$ N of x. Then there exists $F\in\mathscr{F}$ such that $F\subset\rho-Cl(N)$. Then $\rho-Cl(N)\in\mathscr{F}$. ■

Corollary 2.4. *Let* (X,σ,ρ) *be a bitopological space. Then the* $\sigma-nhd$ *filter of a point* $x\in X$ *is* $(\sigma-\rho)-\theta-$*converges to* x.

Proof . This follows immediately from Proposition 2.3.■

Proposition 2.5. *Let* (X,σ,ρ) *be a bitopological space. Let* σ *be an indiscrete topology on* X *. Then every filter on* X *is* $(\sigma-\rho)-\theta-$*converges to a point* x *in* X .

Proof. Let $\mathscr{F}$ be a filter on X . Since X is the only non empty member of σ (since σ is an indiscrete topology), $x\in X$ and $X\in\mathscr{F}$. Then $\rho-Cl(X)=X\in\mathscr{F}$ ■ .

Proposition 2.6. *Let* (X,σ,ρ) *be a bitopological space . If a filter* $\mathscr{F}$ *on* X *is* $(\sigma-\rho)-\theta-$*converges to a point* x *in* X *, then every filter* $\mathscr{F}^*$ *finer than* $\mathscr{F}$ *also* $(\sigma-\rho)-\theta-$*converges to a point* x.

Proof . Suppose that $\mathscr{F}$ is $(\sigma-\rho)-\theta-$converges to a point x in X , then for every $\sigma-nhd$ N of $x,\rho-Cl(N)\in\mathscr{F}$. Since $\mathscr{F}^*$ is finer than $\mathscr{F}$, hence $\rho-Cl(N)\in\mathscr{F}^*$. Thus $\mathscr{F}^*$ is $(\sigma-\rho)-\theta-$converges to a point x.■

Proposition 2.7. *Let* Φ *be the collection of all filters on a bitopological space* (X,σ,ρ) *which is* $(\sigma-\rho)-\theta-$*converges to the same point* x *in* X *.Then the intersection* $\mathscr{F}$ *of all filters in* Φ *also* $(\sigma-\rho)-\theta-$*converges to a point* x.

Proof . First note that $\mathscr{F}$ is actually a filter on X . Since all the filters in Φ are $(\sigma-\rho)-\theta-$converges to the same point x in X . Then the collection $\Omega=\{\rho-Cl(N):N\ \ is\ \ \sigma-nhd\ \ of\ \ x\}$ containing in each filter in Φ, and consequently Ω containing in $\mathscr{F}$ .Hence $\mathscr{F}$ also $(\sigma-\rho)-\theta-$converges to a point x.■

Proposition 2.8.*Let* (X,σ,ρ) *be a bitopological space. A filter* $\mathscr{F}$ *on* X *is* $(\sigma-\rho)-\theta-$*converges to a point* x *in* X *if and only if every ultrafilter containing* $\mathscr{F}$ *is* $(\sigma-\rho)-\theta-$*converges to a point* x.

Proof. If $\mathscr{F}$ is $(\sigma-\rho)-\theta-$converges to a point x in X , then evidently every ultrafilter containing $\mathscr{F}$ also $(\sigma-\rho)-\theta-$converges to a point x by Proposition 2.6.Conversely, suppose that every ultrafilter containing $\mathscr{F}$ also $(\sigma-\rho)-\theta-$converges to a point x.Now, $\mathscr{F}$ is the intersection of all ultrafilters on X and so $\mathscr{F}$ is $(\sigma-\rho)-\theta-$converges to a point x by Proposition 1.7.■

Example 2.9. Let (X,σ,ρ) be a bitopological space, where $X=\{a,b,c\}$, $\sigma=\{\phi,\{a\},\{a,b\},X\}$,and $\rho=\{\phi,\{a\},\{b\},\{a,b\}X\}$.Let $\mathscr{F}=\{\{b,c\},X\}$ be the filter on X . Then b and c are both $(\sigma-\rho)-\theta-$ limit points of $\mathscr{F}$.

Definition.2.10.[1] *A bitopological space* (X,σ,ρ) *is said to be* $(\sigma-\rho)-\theta-$ ***Hausdorff*** *, if for every two distinct points* x *and* y *in* X *,there exist two* σ*-open sets* U *and* V *such that* $\rho-Cl(U)\bigcap\rho-Cl(V)=\phi$.

Proposition 2.11.*If a bitopological space* (X,σ,ρ) *is* $(\sigma-\rho)-\theta-$ *Hausdorff , then every* $(\sigma-\rho)-\theta-$*converges filter on* X *has a unique* $(\sigma-\rho)-\theta-$*limit point* .

Proof . Suppose that (X,σ,ρ) be a $(\sigma-\rho)-\theta-$Hausdorff and let $\mathscr{F}$ be a $(\sigma-\rho)-\theta-$converges filter on X .Assume contrary that the two distinct points x and y are $(\sigma-\rho)-\theta-$limit points of $\mathscr{F}$ .Then for every $\sigma-nhd$ N of x and for every $\sigma-nhd$ M of y we have $\rho-Cl(N)\in\mathscr{F}$ and $\rho-Cl(M)\in\mathscr{F}$ .By hypothesis there exist two σ-open sets U and V such that $\rho-Cl(U)\bigcap\rho-Cl(V)=\phi$ .But that is a contradiction since $\rho-Cl(U)\in\mathscr{F}$ and $\rho-Cl(V)\in\mathscr{F}$ .Thus $\mathscr{F}$ has a unique $(\sigma-\rho)-\theta-$limit point .■

Note: Converse of the above Proposition is false as is seen from example below.

Example 2.12. Let (X,σ,ρ) be a bitopological space where $X=\{a,b,c\}$, $\sigma=\{\phi,\{a\},\{c\},\{a,c\},\{a,b\},X\}$ and $\rho=\{\phi,\{a\},\{b\},\{a,b\},X\}$. Let $\mathscr{F}=\{\{a\},\{a,b\},X\}$ be a filter on X . Then b is a unique $(\sigma-\rho)-\theta-$ limit point of $\mathscr{F}$. But (X,σ,ρ) is not $(\sigma-\rho)-\theta-$Hausdorff.

Proposition 2.13. *Let* (X,σ,ρ) *be a bitopological space .Then the filter base* Φ *on* X *is* $(\sigma-\rho)-\theta-$*converges to a point* $x\in X$ *if and only if the* $\rho-$*closure of every member of the* σ*-open local base at* x *contains a member of* Φ.

Proof .Suppose that Φ be a filter base on X which is $(\sigma-\rho)-\theta-$ converges to a point $x\in X$. Then the filter $\mathscr{F}$ generated by Ω is also $(\sigma-\rho)-\theta-$converges to a point $x\in X$ .Also every member of $\mathscr{F}$ contains a member of Ω. Thus $\rho-Cl(N)\in\mathscr{F}$ for every $N\in\Omega$, where Ω is the $\sigma-$open local base at x, and consequently for every $N\in\Omega$, there exists $B\in\Phi$ such that $B\subset\rho-Cl(N)$.
Conversely , suppose that for every $N\in\Omega$ there exists $B\in\Phi$ such that $B\subset\rho-Cl(N)$. Let $\mathscr{F}$ be a filter generated by Φ. Then

$$\mathscr{F}=\{F\subset X : F\supset B, B\in\Phi\}.$$

Hence $\rho-Cl(N)\in\mathscr{F}$. Now , let U be a $\sigma-nhd$ of x. Then there exists a member $N\in\Omega$ such that $N\subset U$, then $\rho-Cl(N)\subset\rho-Cl(U)$ .Hence $\rho-Cl(U)\in\mathscr{F}$. It follows that $\mathscr{F}$ is $(\sigma-\rho)-\theta-$converges to $x\in X$.Accordingly $\Phi$is $(\sigma-\rho)-\theta-$converges to $x\in X$.■

Corollary 2.14. *Let* (X,σ,ρ)*be a bitopological space .Then the filter base* Φ*on* X *is* $(\sigma-\rho)-\theta-$*converges to a point* $x\in X$ *if and only if for every* $\sigma-nhd$ V *of* x, *there exists some* $B\in\Phi$ *such that* $B\subset\rho-Cl(V)$.

Proof . Suppose that $\Phi\,(\sigma-\rho)-\theta-$converges to a point $x\in X$ and let V be a $\sigma-nhd$ of x. Then there exists $N\in\Omega$. Then

$$\rho-Cl(N)\subset\rho-Cl(V).$$

Hence by Proposition 1.14. there exists $B\in\Phi$such that $B\subset\rho-Cl(N)$.Thus $B\subset\rho-Cl(V)$.Conversely, if for every $\sigma-nhd$ V of x, there exists some $B\in\Phi$ such that $B\subset\rho-Cl(V)$ .Then surly for every $N\in\Omega$ there exists $B\in\Phi$ such that $B\subset\rho-Cl(N)$. Hence by Proposition 1.14. Φ is $(\sigma-\rho)-\theta-$ converges to $x\in X$.■

Definition 2.15 [5] . *Let* (X,σ,ρ)*be a bitopological spaces . A filter (filter base)* $\Re$*on a set* X *is said to be* $(\sigma-\rho)-\theta-$***accumulates*** *at a point* $x\in X$ *if every* $\sigma-$*neighborhood* N *of* x, *and every* $F\in\mathscr{F}$ *the intersection* $\rho-Cl(N)\bigcap F\neq\phi$. *We say that* x*is a* $(\sigma-\rho)-\theta-$***cluster point*** *of* $\mathscr{F}$.

Proposition 2.16. *Let* (X,σ,ρ)*be a bitopological space and let* $\mathscr{F}$ *be a filter on* X . *If a point* $x\in X$ *is* $(\sigma-\rho)-\theta-$*limit point of* $\mathscr{F}$, *then it is* $(\sigma-\rho)-\theta-$*cluster point of* $\mathscr{F}$.

Proof .Suppose that x is a $(\sigma-\rho)-\theta-$limit point of a filter $\mathscr{F}$ on X . Then for every $\sigma-nhd$ N of x, we have $\rho-Cl(N)\in\mathscr{F}$.Thus $\rho-Cl(N)$intersect every member of $\mathscr{F}$. Hence x is $(\sigma-\rho)-\theta-$cluster point of $\mathscr{F}$.■

Note: Converse of the above proposition is false as is seen from example below.

Example 2.17. Let (X,σ,ρ)be a bitopological space where $X=\{a,b.c\}$, $\sigma=\{\phi,\{a\},\{a,b\},X\}$, and $\rho=\{\phi,\{a\},\{b\},\{a,b\},X\}$.And let $\mathscr{F}=\{\{b,c\},X\}$ be a filter on X .Then a is $(\sigma-\rho)-\theta-$cluster point of $\mathscr{F}$ but not $(\sigma-\rho)-\theta-$limit point of $\mathscr{F}$.

Proposition2.18.*Let* Φ*be a filter base on a bitopological space* (X,σ,ρ).*Then a point* $x\in X$ *is a* $(\sigma-\rho)-$*limit point of* Φ*if and only if the* $\rho-$*closure of every member of the* $\sigma-$*local base* Ω *at* x *intersect every member of* Φ.

Proof .A point $x\in X$is a $(\sigma-\rho)-$limit point of Φif and only if for every $\sigma-nhd$ N of x and every $B\in\Phi$, $\rho-Cl(N)\bigcap B\neq\phi$,if and only if for every $M\in\Omega$, $\rho-Cl(M)\bigcap B\neq\phi$.■

Proposition 2.19. *Let* (X,σ,ρ) *be a bitopological space . If a filter base* Φ *on* X *is* $(\sigma-\rho)-\theta-$ *converges to* $x\in X$, *then it is* $(\sigma-\rho)-\theta-$*accumulates at* $x\in X$ *and in a* $(\sigma-\rho)-\theta-$*Hausdorff space , at no point other than* x

Proof . Given $\sigma-nhd\ U$ of x, there is some $B\in\Phi \ni B\subset\rho-Cl(U)$; since each $A\in\Phi$ intersect B,it follows that $\forall A: A\cap\rho-Cl(U)\neq\phi$,so $\Phi$is $(\sigma-\rho)-\theta-$ accumulates at $x\in X$ i.e. x is $(\sigma-\rho)-\theta-$cluster point of Φ.Now Let (X,σ,ρ) be a $(\sigma-\rho)-\theta-$Hausdorff space , and let x and y be two distinct points of X ,then there exist two $\sigma-nhds$ N of x and M of y such that $\rho-Cl(N)\cap\rho-Cl(M)=\phi$. Thus there must be $B\in\Phi, B\subset\rho-Cl(N)$; then $B\cap\rho-Cl(M)=\phi$, and so Φcannot be $(\sigma-\rho)-\theta-$accumulates at y .■

Proposition 2.20. Let (X,σ,ρ) *be a bitopological space and let* Ψ *is subordinate to* Φ *If* Φ*is* $(\sigma-\rho)-\theta-$ *converges to* $x\in X$ *,then*Ψ *is* $(\sigma-\rho)-\theta-$ *converges to* x.

Proof. Suppose that$\Phi$is $(\sigma-\rho)-\theta-$converges to $x\in X$, then for every $\sigma-nhd\ N$ of x there exists $A\in\Phi\ni A\subset\rho-Cl(N)$; since Ψ is subordinate to Φ, there is some $B\in\Psi\ni B\subset A$,so Ψ is $(\sigma-\rho)-\theta-$ converges to x also.■

Proposition 2.21. *Let* (X,σ,ρ) *be a bitopological space and let* Ψ *is subordinate to* Φ *If* Ψ *is* $(\sigma-\rho)-\theta-$ *accumulates at* $x\in X$ *,then*Φ*is* $(\sigma-\rho)-\theta-$ *accumulates at* x.

Proof. Suppose that Ψ is $(\sigma-\rho)-\theta-$ accumulates at $x\in X$ .Given $\sigma-nhd\ N$ of x and $A\in\Phi$,there is some $B\in\Psi\ni B\subset A$,and since for every $B\in\Psi, B\cap\rho-Cl(N)\neq\phi$, we find for every $A\in\Phi, A\cap\rho-Cl(N)\neq\phi$,which proves Φ is $(\sigma-\rho)-\theta-$ accumulates at x .■

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حول تقارب المرشحات في الفضاءات الثنائية التبولوجيا

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الملخّص. قدمنا في هذا البحث مفهوم تقاربِ المرشحاتِ في الفضاءات الثنائية التبولوجيا وبينا بَعْض خواص هذا التقارب و العلاقة بين هذا المفهوم و مفهوم فضاء θ - هاوزدورف في الفضاءات الثنائية التبولوجيا.