On The λ- Choquet Integral with Respect to λ- Fuzzy Measure

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Abstract

We define the λ- fuzzy measure and the λ- Choquet integral for a measurable function with respect to λ- fuzzy measure. Also the relation between this integral and plausibility (belief) measure was given. In addition we explain every λ- fuzzy measure is fuzzy measure.

1-Introduction

The Choquet integral [4] for a non-negative measurable function can be taken with respect to λ- fuzzy measure. The term $\Im$ is referred to a δ-algebra on a set X, where (X, $\Im$) a measurable space.

We say that f :X→[0,∞] is measurable [5] with respect to $\Im$ if

For any r ∈[0,∞)

$$f^{-1}([r,\infty)) = \{x \in X; f(x) \geq r\} \in \Im$$

If (X,$\Im$) a measurable space, a function β:$\Im$→[0,1] is called to be belief measure[2] if it verifying the following properties:

$1-\beta(\phi)=0.$

$2-\beta(x)=1.$

$3-\beta(A\cup B)\geq \beta(A)+\beta(B), \forall A,B\in\Im$

A function p: $\Im$ →[0,1] is called to be plausibility measure[2] if it verifying the following properties:

$1-p(\phi)=0.$

$2-p(X)=1.$

$3-p(A\cup B)\leq p(A)+p(B), \forall A,B\in\Im.$

2-Main results

Definition 2.1 [2].A collection of subset of a set X is called a δ-algebra (algebra) on X if:

$1-X\in\Im.$

$2-If \quad A\in\Im then \quad A^c\in\Im.$

$$3-If \quad A_i\in\Im then \quad \bigcup_{i=1}^{\infty}A_i\in\Im(\bigcup_{i=1}^{n}A_i\in\Im). i=1,2,\ldots.$$

Definition 2.2 [2]. A measurable space is a pair (X, $\Im$) where X is a non-empty set and $\Im$ is a δ-algebra on X .A subset A of X is called measurable if A ∈$\Im$. i.e. any number of $\Im$ is called a measurable set.

Definition 2.3 [5]. Let($X_1,\Im_1$) and ($X_2,\Im_2$) be two measurable spaces .A function $f:X_1\rightarrow X_2$ is called measurable (relative to $\Im_1$ and $\Im_2$) if

$$f^{-1}(B)\in\Im_1, \quad \forall B\in\Im_2 .$$

Definition2.4 [4]. Let ($X,\Im$) be a measurable space. A fuzzy measure μ is an extended real valued set function, $\mu:\Im\rightarrow\Re^+$ with the following properties:
1-$\mu(\phi)=0$.
2-$\mu(A)\leq\mu(B)$ whenever $A\subseteq B$, where A, B $\in\Im$ and $\Re^+=[0,\infty]$.

Definition 2.5. Let($X,\Im$) be a measurable space. A λ-fuzzy measure is an extended real valued set function,$\lambda:\Im\rightarrow\Re^+$ with the following properties:
$1-\lambda(\phi)=0.$
$2-A\subseteq B \quad implies \lambda(A)\leq\lambda(B).$
$3-\lambda(\bigcup_{i=1}^{n}A_i)\leq\sum_{i=1}^{n}\lambda(A_i).$

Proposition 2.6. Every λ-fuzzy measure is fuzzy measure.
Proof. It is follows by definition 2.5 and definition 2.4.

Definition 2.7 [5].let $A_1,A_2,\ldots$be subsets of a set X . If $A_1\subset A_2\subset\ldots$and

$$\bigcup_{n=1}^{\infty}A_n=A$$

we say that the A_n form an increasing sequence of sets with limit A, or that the A_n increase to A, we write $A_n\uparrow A$ **.**

Definition 2.8[3]. A fuzzy measure $\mu:\Im\rightarrow\Re^+$ is called lower continuous if
$A_n\uparrow A$, $(A_n)\subset A$, $A\in\Im\Rightarrow$
, and denote it by $\bigvee_{n=1}^{\infty}\mu(A_n)=\mu(\bigcup_{n=1}^{\infty}A_n)$, where $\bigvee_{n=1}^{\infty}\mu(A_n)=\mu(A)$
$\mu(A_n)\uparrow\mu(A)$.

Definition 2.9.A λ-fuzzy measure,$\lambda:\Im\rightarrow\Re^+$ is called lower continuous if
$A_n\uparrow A$, $(A_n)\subset A$, $A\in\Im\Rightarrow\lambda(A_n)\uparrow\lambda(A)$ **.**

Definition 2.10[3].A fuzzy measure $\mu:\Im\rightarrow\Re^+$ is called a belief measure if for any $n\in N$ and any $A_1,A_2,\ldots,A_n\in\Im$

$$\mu(\bigcup_{i=1}^{n}A_i)\leq\sum_{I}(-1)^{|I|+1}\mu(\bigcap_{i\in I}A_i)$$

Where the summations is taken over all non-empty subsets I of $\{1,2,3,\ldots n\}$ and |I| denotes the cardinal number of I .

Definition 2.11. A λ- fuzzy measure $\lambda: \Im \to \Re^+$ is called a belief measure if for any $n \in N$ and any $A_1, A_2 \ldots A_n \in \Im$

$$\lambda(\bigcup_{i=1}^{n} A_i) \leq \sum_{I} (-1)^{|I|+1} \lambda(\bigcap_{i \in I} A_i)$$

Definition 2.12[3]. A fuzzy measure $\mu:\Im \to \Re^+$ is called a plausibility measure if for any $n \in N$ and any $A_1, A_2 \ldots A_n \in \Im$

$$\mu(\bigcap_{i=1}^{n} A) \leq \sum_{I} (-1)^{|I|+1} \mu(\bigcup_{i \in I} A_i)$$

Definition 2.13. A λ- fuzzy measure $\lambda:\Im \to \Re^+$ is called a plausibility measure if for any $n \in N$ and any $A_1, A_2, \ldots, A_n \in \Im$

$$\lambda(\bigcap_{i=1}^{n} A) \leq \sum_{I} (-1)^{|I|+1} \lambda(\bigcup_{i \in I} A_i)$$

Definition 2.14. Let f: $(X,\Im) \to \Re^+$ be a measurable function, $\lambda: \Im \to \Re^+$ be a λ –fuzzy measure and $A \in \Im$, then we define the λ – Choquet integral

$$(C)\int_A f d\lambda$$

By the formula

$$(C)\int_A f d\lambda = \int_0^{\infty} \lambda(\{x \in A, f(x) > r\}) dr$$

$$= \int_0^{\infty} \lambda(A \cap \{x \in X, f(x) > r\}) dr.$$

Definition 2.15 [1]. A non-negative finite –valued function $f(x)$, taking only a finite number of different values, is called a simple function .If $a_1, a_2 \ldots a_m$ are the distinct values taken by f and $A_i = \{x \mid f(x) = a_i\}$, then

.And the integral of f with respect to μ is given by $f(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$

$$\int f d\mu = \sum_{i=1}^{m} a_i \mu(A_i).$$

Theorem 2.16. let f be anon-negative real measurable function with respect to a measurable space $(X,\Im)$, $X \in \Im$, $\lambda:\Im \to \Re^+$ be a λ-fuzzy measure.
Define $g:\Im \to \Re^+$ by the formula

$$g(A) = (C)\int_A f d\lambda.$$

Then

1-g is a λ –fuzzy measure.
2-g is lower continuous, wherever λ is lower continuous.
3-If λ is lower continuous and plausibility measure, then g is plausibility measure, too.
4-If λ is lower continuous and belief measure, then g is belief measure, too.

Proof:

(1)

$$g(\phi)=\int_0^\infty \lambda(\phi)dr=0.$$

If A⊆B, then

$$g(A)=\int_0^\infty \lambda(A\cap\{x\in X;\, f(x)>r\})dr$$

$$\leq \int_0^\infty \lambda(B\cap\{x\in X;\, f(x)>r\})dr=g(B).$$

$$let \qquad Ai\in\Im, i=1,2,\ldots,n$$

$$then \qquad g(\bigcup_{i=1}^{n} A_i)=\int_0^\infty \lambda\left((\bigcup_{i=1}^{n} A_i)\bigcap\{x\in X;\, f(x)>r\}\right)dr$$

$$=\int_0^\infty \lambda(\bigcup_{i=1}^{n}(A_i\bigcap\{x\in X,\, f(x)>r\}))dr$$

$$\leq \int_0^\infty \sum_{i=1}^{n}\lambda(Ai\bigcap\{x\in X;\, f(x)>r\})dr$$

$$=\sum_{i=1}^{n}\int_0^\infty \lambda(A_i\bigcap\{x\in X;\, f(x)>r\})dr$$

$$=\sum_{i=1}^{n} g(A_i).$$

(2)

let λ be lower continuous and let $A_n \uparrow A$

$$\bigvee_{n=1}^{\infty} g(A_n)=\bigvee_{n=1}^{\infty}\int_0^\infty \lambda(A_n\bigcap\{x\in X;\, f(x)>r\})dr$$

$$=\int_0^\infty \bigvee_{n=1}^{\infty}\lambda(A_n\bigcap\{x\in X;\, f(x)>r\})dr$$

$$=\int_0^\infty \lambda(\bigcup_{n=1}^{\infty}(A_n\bigcap\{x\in X;\, f(x)>r\}))dr$$

$$=\int_0^\infty \lambda((\bigcup_{n=1}^{\infty}A_n)\bigcap\{x\in X,\, f(x)>r\})dr$$

$$=\int_0^\infty \lambda(A\bigcap\{x\in X,\, f(x)>r\})dr$$

$$.=(C)\int_A fd\lambda$$

$$=g(A)$$

(3)

Let λ be lower continuous and plausibility measure. Take simple functions f_n

$f_n = \sum_{i=1}^{m} a_i X_{A_I}, 0 = a_0 < a_1 < \ldots < a_m$, such that $f_n \uparrow f$.Fixed n and put

A_i disjoin .Denote

$$g_n(A) = (C)\int_A f_n d\lambda$$

$$= \int_0^\infty \lambda(A \cap \{x \in X;\, f_n(x) > r\})dr$$

$$= \sum_{i=1}^{m} (a_i - a_{i-1})\lambda(A \cap (A_i \cup A_{i+1} \ldots \cup A_m))$$

$$= \sum_{i=1}^{m} (a_i - a_{i-1})\lambda(A \cap B_i);$$

$$where \qquad B_i = A_i \cup A_{i+1} \cup \ldots \cup A_m. \quad then$$

$$g_n(\bigcap_{j=1}^{k} C_j) = \sum_{i=1}^{m} (a_i - a_{i-1})\lambda(\bigcap_{j=1}^{k} C_j \cap B_i)$$

$$\le \sum_{i=1}^{m} (a_i - a_{i-1})\sum_I (-1)^{|I|+1}\lambda(\bigcap_{j=1}^{k}((\bigcup_{j\in I} C_j) \cap B_i))$$

$$= \sum_I (-1)^{|I|+1}\sum_{i=1}^{m} (a_i - a_{i-1})\lambda(\bigcap_{j=1}^{k}((\bigcup_{j\in I} C_j) \cap B_i))$$

$$= \sum_I (-1)^{|I|+1} g_n(\bigcup_{j\in I} C_j),$$

hence

$$g_n(\bigcap_{j=1}^{k} C_j) \le \sum_I (-1)^{|I|+1} g_n(\bigcup_{j\in I} C_j) \qquad (1)$$

now

$$\bigvee_{n=1}^{\infty} g_n(A) = \bigvee_{n=1}^{\infty} (C)\int_A f_n d\lambda$$

$$= \bigvee_{n=1}^{\infty}\int_0^\infty \lambda(A \cap \{x \in X;\, f_n(x) > r\})dr$$

$$= \int_0^\infty \bigvee_{n=1}^{\infty}\lambda(A \cap \{x \in X;\, f_n(x) > r\})dr$$

$$= \int_0^\infty \lambda(A \cap \bigcup_{n=1}^{\infty}\{x \in X;\, f_n(x) > r\})dr$$

$$= \int_0^\infty \lambda(A \cap \{x \in X;\, f(x) > r\})dr$$

$$= g(A). \qquad ..(2)$$

hence $g_n(A) \uparrow g(A)$ for every $A \in \mathfrak{I}$.

By (1) and (2) we obtain

$$g(\bigcap_{j=1}^{K} C_j) \le \sum_{I} (-1)^{|I|+1} g(\bigcup_{j\in I} C_j).$$

(4)

If λ is lower continuous and belief measure. Then

$$g_n(\bigcup_{j=1}^{k} C_j) = \sum_{i=1}^{m} (a_i - a_{i-1}) \lambda(\bigcup_{j=1}^{k} C_j \cap B_j)$$

$$\le \sum_{i=1}^{m} (a_i - a_{i-1}) \sum_{I} (-1)^{|I|+1} \lambda(\bigcup_{j=1}^{k} ((\bigcap_{j\in I} C_j) \cap B_i))$$

$$= \sum_{I} (-1)^{|I|+1} \sum_{i=1}^{m} (a_i - a_{i-1}) \lambda(\bigcup_{j=1}^{k} ((\bigcap_{j\in I} C_j) \cap B_i))$$

$$= \sum_{I} (-1)^{|I|+1} g_n(\bigcap_{j\in I} C_j),$$

hence

$$g_n(\bigcup_{j=1}^{k} C_j) \le \sum_{I} (-1)^{|I|+1} g_n(\bigcap_{j\in I} C_j); \qquad (3)$$

by (1) and (3) we obtain

$$g(\bigcup_{j=1}^{k} C_j) \le \sum_{I} (-1)^{|I|+1} g(\bigcap_{j\in I} C_j).$$

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الخلاصة

تم في هذا البحث تعريف القياس الضبابي λ وتكامل λ- جوكي للدالة القابلة للقياس بعلاقة مع القياس الضبابي λ. واعطيت العلاقة بين هذا التكامل وقياس الامكانية (الاعتقاد). بالاضافة الى ذلك تم توضيح ان كل قياس ضبابي λ هو قياس ضبابي.