The relation between equivalent measures and the bipolar theorem

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Abstract :

In this paper *if* we have Q and P are equivalent measures on the δ-field F and $C \subseteq L^{\circ}_{+}(\Omega, F, P)$. We define the polar of C with respect to Q, denoted by $C^{\circ}(Q)$, and define bipolar $C^{\circ\circ}(Q)$ of C with respect to Q .In this paper we study the relation between equivalent measures (p and Q) and bipolar $C^{\circ\circ}$ of C. Also we prove $C^{\circ\circ}(Q) = C^{\circ\circ}(P)$.

1. Introduction :

Bipolar theorem which states that the bipolar of subset of a locally convex vector space equals its closed convex hull [4]. Let (Ω, F, P) be a probability space and denote by $L^{\circ}(\Omega, F, P)$ the linear space of equivalent classes of *IR* – Valued random variables on (Ω, F, P) with the topology of convergence in measure . although this space fails to be locally convex i.e it hasn't a neighborhood base at 0 consisting of convex sets[2] the bipolar theorem can be obtained for subsets of $L^{\circ}(\Omega, F, P)$. Let P and Q be two probability measures on (Ω, F) then we say P equivalent to Q denoted by $P \approx Q$ if they have the same null – set . in this paper we study the bipolar theorem when we replace P by an equivalent measure Q and define the polar of subset of $L^{\circ}(\Omega, F, P)$ with respect to Q . we prove the bipolar with respect to Q . and prove the bipolar with respect to Q coincides with the bipolar with respect to P .

2. Elementary definitions and concepts :

In this section we introduce some basic definitions in functional analysis which we need it in this paper :

2.1 Definition [1]. A collection F of subsets of a non–empty set Ω is called σ –field or σ – algebra on Ω if

1. $\Omega \in F$.
2. if $A \in F$ then $A^c \in F$.
3. if $\{A_n\}$ is a sequence of sets in F then $\bigcup_{n=1}^{\infty} A_n \in F$.

2.2. Definition [1]. A measurable space is a pair (Ω, F) where Ω is a non-empty set and F is a δ –field on Ω .

2.3 Definition [1]. any member of δ- field F is called a measurable set or (measurable with respect to the δ- field F).

2.4 Definition [6] . A measure on a δ- field F is a non-negative extended real valued function μ on F .

such that whenever A_1, A_2 , A_3 , … Form a finite or countably infinite collection of disjoint sets in F , We have $\mu(\bigcup_n A_n) = \sum \mu(A_n)$.

If $\mu(\Omega) = 1$, μ is called a probability measure .a measure space is a triple (Ω, F, μ) Where Ω is a set , F is a $\delta -$field of subsets of Ω, and μ is a measure on F. if μ is a probability measure on F then the triple (Ω, F, μ) is called probability space.

2.5 Definition [6]. let (Ω_1, F_1) and (Ω_2, F_2) be two measureable spaces a function $f : \Omega_1 \to \Omega_2$ is said to be measureable function (relative to F_1 and F_2) if $f^{-1}(B) \in F_1 \ \forall B \in F_2$.

we say f is Borel measureable function on (Ω_1, F_1) if F_2 is the set of all open set on Ω_2.

2.6 Definition [6]. let P and Q be a probability measure on measureable space (Ω, F) then we say P equivalent to Q, denoted by $P \approx Q$ if they have the same null – set i.e $P (A) = 0$ iff $Q(A)=0 \ \forall A \in F$.

2.7Definition. [6] A random variable X on a probability space(Ω, F, P) is a borel measurable function from Ω to IR. $X:\Omega \to IR$ is random variable iff $\forall a \in IR, \{X \leq a\} \in F$.

2.8Definition. [6] A sequence $\{x_n\}$of random variable is said to be converge Almost every where(surely) to a random variable x, written $x_n \xrightarrow{a.s} x$ or $x_n \xrightarrow{a.e} x$ if $P\{\lim_{n\to\infty} x_n = x\} = 1$.

2.9 Definition. [1] Let F be a $\delta-$field of subsets of a set Ω and μ satisfy

1. $0 \leq \mu(A) \leq +\infty$ for every $A \in F$.
2. if $A_n \in F$, n=1,2,... and $A_i \cap A_j = 0$, $i \neq j$, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.
3. there is a sequence T_n ,n=1,2,...in F such that $\Omega = \bigcup_{n=1}^{\infty} T_n$ and $\mu(T_n) < \infty$,n=1,2,...

then μ is still called a measure on F but the measure space(Ω, F, μ) is called δ-finite.

2.10 Definition. [1] Let Ω be a set and F a $\delta-$field of subsets of Ω and Let μ be a real valued function on F such that if $A_n \in F$, n=1,2,... and $A_i \cap A_j = 0$ whenever $i \neq j$ then

$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ such a function μ is called a finite signed measure on *F*,and we write (Ω, F, μ) for the signed measure.

2.11Definition **.** [6] Let (Ω, F) be a measurable space and let μ be a measure on $\delta-$field *F*,and Q is a signed measure on *F* we say that Q is absolutely continuous with respect to μ (notation $Q << \mu$).iff $\mu(\mathrm{A}) = 0$ implies $Q(\mathrm{A}) = 0$ $(\mathrm{A} \in F)$.

2.12 theorem . [6] Radon – Nikodym theorem

let µ be a $\delta-$finite measure and *Q* be a signed measure on the δ- field *F* of subsets of Ω.assume that *Q* is absolutely continuous with respect to μ.then there is Borel measureable function *f* on Ω such that $Q(A) = \int_A f \, d\mu, A \in F$ if g is another such function,then *f=g a.e[* μ *].*

The function *f* is called the radon – Nikodym derivative or density of *Q* with respect to µ and is denoted by *dQ/d*μ .

2.13 Definition [7]. let (Ω, F, P) be a probability space a vector space $L^{\circ}(\Omega, F, P)$ is the space of (equivalence classes of) real –

valued measureable functions defined on (Ω, F, P), which we equip with the topology of convergence in measure i.e.

$$L^{\circ}(\Omega, F, P) = \{f : \Omega \to IR, f \ measureable \ function\}$$

$$with \ topoLogy \ converge \ in \ measure$$

$$or L^{\circ}(\Omega, F, P) = \{f : \Omega \to IR, f \ random \ variable\}$$

with topology converge in probability

Also we denote the positive orthant of $L^{0}(\Omega, F, P)$ by $L^{0}_{+}(\Omega, F, P)$ i.e $L^{0}_{+}(\Omega, F, P) = \{f \in L^{\circ}(\Omega, F, P), f \geq 0\}$.

<u>2.14 Definition</u> [5]. $L'(\Omega, F, P)$ or , in short $L'(\Omega)$ the set of all real – valued , *F*- measurable function *f* defined *P*-a.e on Ω such that |*f*| is *p*-integrable over Ω i.e .

$$L'(\Omega, F, P) = \{f : \Omega \to IR, f \text{ measurable function } \}$$

$$\text{Such that } \int | f | dp < \infty$$

<u>2.15 Definition</u> [5]. Let (Ω, F, P) be a probability space a *f-measurable* function *f* defined on Ω is said to be essentially bounded if there exists a constant α such that $| f | < \alpha$ *P*-a.e now $L^{\infty}(\Omega, F, P)$, or in short , $L^{\infty}(\Omega)$ is the set of all *f* -measurable , essentially bounded functions defined *P*-a.e on Ω .

3. main Result :

Before stating the main result we recall the following definitions and introduce some theorems :

3.1 Definition [3]. let $C \subseteq L_+^\circ$ we define the polar $C^\circ \, of \, C$ by $C^\circ = \{ \, g \in L_+^\circ : E[f.g] \leq 1 \, \forall \, f \in C \, \}$ and bipolar $C^{\circ\circ}$ of C by $C^{\circ\circ} = \{ \, f \in L_+^0 : E[f.g] \leq 1 \forall \, g \in c^\circ \, \}$

3.2 Definition[3]. we call a subset $C \subseteq L_+^\circ$ solid , if $f \in C$ and $0 \leq g \leq f$ implies that $g \in C$ the set C is said to be closed in probability or simply closed , if it is closed with respect to the topology of convergence in probability .

3.3 Definition[4]. A set $D \subset L^\circ$ is convex if $\lambda \, f_1 + (1-\lambda) \, f_2 \in D \forall \, f_1, f_2 \in D and \, 0 \leq \lambda \leq 1.$ or D is convex if $\lambda D + (1-\lambda) D \subset D \; forall 0 \leq \lambda \leq 1$. for all $f_1, f_2 \in D$.

3.4 Bipolar theorem [7].

For a set $C \subset L_+^\circ(\Omega, F, P)$ the polar $C^\circ(P)$ is a closed , convex , solid subset of $L_+^\circ(\Omega, F, P)$. The bipolar $C^{\circ\circ}(P)$ is the smallest closed , convex , solid set in $L_+^\circ(\Omega, F, P)$ containing C .

3.5 Definition [7]. for $A \in F$, where F is a $\sigma-$field , we denote by $C \backslash A$ the restriction of C to A , i.e $\{\gamma x_A, \gamma \in C\}$ with x_A= 1 on A and 0 other wise . We denote similarly $P \backslash A$ the restriction of P to A.

3.6 Definition [3]. A subset $C \subseteq L^{\circ}(\Omega, F, P)$ is bounded in probability if , for all $\in > 0$, there is $M > 0$ such that $P[\|f\| > M] < \in$ for $f \in C$.

3.7 Definition [7]. we say that C is hereditarily unbounded on a set $B \in F$ if , for every $A \subset B$ with p(A)>0 , the restriction of C to A fails to be bounded in probability .

3.8 Lemma [7]. Let C be a convex subset of $L^{\circ}_{+}(\Omega, F, P)$. There exists a partition of Ω in to disjoint sets Ωu, $\Omega_b \in F$ such that :

1. C is hereditarily unbounded in probability on Ω_u .
2. the restriction $C \backslash \Omega_u$ of C to Ω_u is bounded in probability . The partition { Ω_u, Ω_b } is the unique partition of Ω satisfying (1) and (2) .

<u>3.9 theorem</u> [7].let C be a convex set in $L_+^\circ(\Omega, F, Q')$ such that $Q' = Q \setminus \Omega_b$.

1. If $p(\Omega_b) > 0$ then there exists probability measure P equivalent to Q' such that C is bounded in $L'(\Omega, F, P)$.
2. Let D be a smallest closed , convex solid set containing C then $D = D \setminus \Omega_b \oplus L_+^\circ(\Omega, F, Q) \setminus \Omega_a$.

We now introduce the main result in this paper theorem(3-11) and to prove it we need the following lemaa:

<u>3.10 lemaa</u> . if $Q \approx P$ are equivalent probability measure and $h = dQ/dp$ is the radon – Nikodym derivative of Q with respect to P then $E_p[f.g] = E_Q[f.h^{-1}g]\ f, g \in L_+^\circ$.

Proof :

Since $h = dQ/dp \Rightarrow dQ = hdp$

$$E_p[f.g] = \int f.g\,dp = \int f.h.h^{-1}g\,dp = \int f.h^{-1}.g\,dQ = E_Q[f.h^{-1}.g]$$

$$hence E_p[f.g] = E_Q[f.h^{-1}.g]$$

<u>3.11theorem</u>. let Q be an equivalent measure to P and let $C \subseteq L_+^\circ$ and let $h = dQ/dp$ is radon – Nikodym derivative then the polar of C with respect to Q is $C^\circ(Q) = h^{-1}C^\circ(p) = \{\ h^{-1}g \in L_+^\circ : E[h^{-1}.g.f] \le 1\ \forall f \in C(Q)\ \}$

and it is closed , convex , solid subset of L_+° and the bipolar of C with respect to Q is

$$C^{\circ\circ}(Q) = C^{\circ\circ}(P) = \{ f \in L_+^\circ : E[h^{-1}.g.f] \le 1 \forall h^{-1}g \in C^\circ(Q)\}$$

Is the smallest closed , convex , solid set in L_+° containing C .

proof :

To prove $C^\circ(Q) = h^{-1} C^\circ(P)$,by using lemaa (3.10) since

$$C^\circ(P) = \{ \ g \in L_+^\circ : E_p[f.g] \le 1 \ \forall f \in C \ \}$$
$$= \{g.h.h^{-1} \in L_+^\circ : E_P[f.h.h^{-1}.g] \le 1 \ \forall f \in C\}$$
$$= h\{g.h^{-1} \in L_+^\circ : E_Q[f.h^{-1}.g] \le 1 \ \forall \ f \in C\}$$
$$= hC^\circ(Q)$$

Hence $C^\circ(P) = hC^\circ(Q)$ then $C^\circ(Q) = h^{-1}C^\circ(P)$.

Now to prove $C^\circ(Q)$ closed subset of L_+° let { τ_n } be a sequence in $C^\circ(Q)$ such that $\tau_n = h^{-1}g_n$ and $g \in L_+^\circ$, such that $\tau_n \to \tau \, and \tau = h^{-1} g$, then there exists subsequence $\{\tau_{nm}\} = \{h^{-1}g_{nm}\} in C^\circ(Q)$ such that $\tau_{nm} \to \tau \ a.e$.

Hence $E[f \, \tau_{nm}] \to E[f\tau] \ a.e$ since $\{\tau_n\}$ in $C^\circ(Q)$ then $E[f\tau_{nm}] \le 1 \ \forall f \in C$ hence $E[f \, \tau] \le 1 \ \forall \ f \in C$ then we get $\tau \in C^\circ(Q)$ closed . hence $C^\circ(Q)$ is closed .

Now to prove $C^\circ(Q)$ is convex subset of L_+° let $h_1, h_2 \in C^\circ(Q)$ and $0 \le \lambda \le 1$.

such that $h_1 = h^{-1}g_1$ and $h_2 = h^{-1}g_2$

$$\lambda h_1 + (1-\lambda)h_2 = \lambda h^{-1}g_1 + (1-\lambda)h^{-1}g_2$$
$$= \lambda h^{-1}g_1 + h^{-1}g_2 - \lambda h^{-1}g_2$$
$$= \lambda h^{-1}(g_1 - g_2) + h^{-1}g_2$$

Since

$$g_1, g_2 \in L_+^\circ \Rightarrow g_1 - g_2 \in L_+^\circ \Rightarrow$$
$$\lambda h^{-1}(g_1 - g_2) \in L_+^\circ \; then \; \lambda h^{-1}(g_1 - g_2) \in C^\circ(Q) \; and \; h^{-1}g_2 \in C^\circ(Q).$$

$$then \lambda h^{-1}(g_1 - g_2) + h^{-1}g_2 = \lambda h_1 + (1-\lambda)h_2 \in C^\circ(Q)$$
$$Hence C^\circ(Q)$$

is convex Subset of L_+°

To prove $C^\circ(Q)$ is solid subset of L_+° let $h_1 = h^{-1} f \in C^\circ(Q)$ such that $h_2 \le h_1$, $h_2 = h^{-1} g$ since $h_1 \in C^\circ(Q) \Rightarrow E[h_1.h] \le 1 \; \forall h \in C$ and $h_2 \le h_1$ hence

$$E[h_2.h] \le E[h_1.h] \le 1 \;\; \forall h \in C$$
$$\therefore h_2 \in C^\circ(Q) \Rightarrow C^\circ(Q)$$

solid subset of L_+°

Now to prove $C^{\circ\circ}(Q) = C^{\circ\circ}(P)$ by using lemaa (3.10)

$$C^{\circ\circ}(P) = \{ f \in L_+^\circ : Ep[f.g] \le 1 \;\; \forall g \in C^\circ(P)\}$$
$$= \{ f \in L_+^\circ : E_Q[f.h^{-1}g] \le 1 \;\; \forall h^{-1}g \in C^\circ(Q)\}$$
$$= C^{\circ\circ}(Q)$$

Hence we get the bipolar of C with respect to Q is coincide to the bipolar of C with respect to P .

Now To prove $C^{\circ\circ}(Q)$ is the smallest closed , convex , solid set containing C , Since $C^{\circ\circ}(Q)=(C^{\circ}(Q))^{\circ}$ is the polar of $C^{\circ}(Q),$ then $C^{\circ\circ}(Q)$ is closed , convex , solid set in L_{+}° containing C .

Let B be a smallest closed , convex , solid in $L_{+}^{\circ}(\Omega,F,Q)$ containing $C.$ then $B\subseteq C^{\circ\circ}(Q)$. To prove $C^{\circ\circ}(Q)\subseteq B$, we will prove by the contradiction method , let $f_{\circ}\in C^{\circ\circ}(Q)$ Such that $f_{\circ}\notin B,$ if $P(\Omega_{b})>0$ then by theorem 3.9 there exists probability measure μ equivalent to $\mu'=Q\setminus\Omega_{b}$ such that B is bounded in $L'(\Omega,F,\mu)$.

Now let $B_{b}=\{f\setminus\Omega b: f\in B\}$ then $B_b\subset B$, B_b closed , bounded and convex set in $L_{+}^{\circ}(\Omega,F,\mu)$ Put

$$B_{b}^{*}=\{K\in L_{+}^{1}(\Omega,F,\mu):\exists g\in B_{b}\ \ s.t\ \ K\leq g\ \ \mu-a.s\}$$

Then $B_{+}^{*}\subset B, B_{b}^{*}$ closed , convex in $L_{+}^{1}(\Omega,F,\mu)$ put $f_{b}=f_{\circ}\setminus\Omega_{b}$ To prove $f_{b}\in B$ (equivalently in B_b or B_{b}^{*}) .

Let $f_{b}\in L_{+}^{1}(\Omega,F,\mu)$ But $f_{b}\in B_{b}^{*}$, Since B_{b}^{*} closed , convex in $L_{+}^{1}(\Omega,F,\mu)$ then we get $I\in L_{+}^{\infty}(\Omega,F,\mu)$ and $I = h^{-1}J$ such that $E[f_{b}.I]>1$ but $E[K.I]\leq 1\ \ \forall K\in B_{b}^{*}\ ,C\subseteq B_{b}^{*}$ to prove that let $f\in C,$ since $C\subset B\Rightarrow f\in B$ then $f\setminus\Omega_{b}\in B_{b}$ and since $f\leq f\setminus\Omega_{b}$ then $f\in B_{b}^{*}$ then

$E[I.n]\leq 1 \;\; \forall n\in C(Q), I\geq 0$ considering I as an element of $L_{+}^{\circ}(\Omega, F, Q)$ we therefore have that $I\in C^{\circ}(Q)$ and this contradiction $E[f_b.I]>1$ then $f_b \notin C^{\circ\circ}(Q)$ then $f_{\circ}\notin C^{\circ\circ}(Q)$ and this contradiction .

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