SOME PROPERTIES OF A NEW SUBCLASS OF MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY RUSCHEWEYH DERIVATIVE I

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Abstract : In the present paper, we have studied a new subclass of meromorephic univalent functions with positive coefficients defined by Ruscheweyh derivative in the punctured unit disk $U^*_= \{z : 0 < |z| < 1\}$ and obtain some sharp results including coefficient estimates , growth and distortion bounds and closure theorems.

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1. Introduction :

Let $\sum$ denote the class of functions f(z) of the from :

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1}$$

which are analytic and meromorphic univalent in the punctured unit disk $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$.

Consider a subclass M of functions of the form :

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0. \tag{2}$$

We aim to study the class $H(\alpha, \mu, \beta, \lambda)$ consisting of functions $f \in M$ and satisfying :

$$\left| \frac{\alpha\left(z^2 \left(D^\lambda f(z)\right)' + zD^\lambda f(z)\right)}{\mu z^2 \left(D^\lambda f(z)\right)' + \mu\alpha z D^\lambda f(z)} \right| < \beta, \tag{3}$$

for $0 \leq \alpha < 1, 0 < \beta \leq 1, 0 \leq \mu \leq 1$ and $D^\lambda f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n D_n(\lambda) z^n$ (4)

(Ruscheweyh derivative of f of order λ [6]) , where

$$D_n(\lambda)=\frac{(\lambda+1)(\lambda+2).......(\lambda+n+1)}{(n+1)!},\lambda>-1, z\in U^*. \qquad (*)$$

Some another classes studied by S.M. Khairnar and Meena more [4], W.G. Atshan and S.R. Kulkarni [2], S.R. Kulkarni; and Mrs. S.S. Joshi [5] , N.E. Cho et al. [3], M.K. Aouf [1] and H.M. Srivastava and S. Owa [7] consisting of meromorphic univalent or meromorphic multivalent functions.

2.Coefficient estimates

In the following theorem , we obtain a coefficient inequality for functions in H$(\alpha,\mu,\beta,\lambda)$.

Theorem 1 : A function f(z) defined by (2) belongs to the class H$(\alpha,\mu,\beta,\lambda)$ if and only if

$$\sum_{n=1}^{\infty}[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)a_n\leq\beta\mu(1-\alpha) \qquad (5)$$

The result is sharp.

Proof : Assume that the inequality (5) holds true and let $|z|=1$, then from (3), we have.

$$\left|\alpha\left(z^2\left(D^\lambda f(z)\right)'+zD^\lambda f(z)\right)\right|-\beta\left|\mu z^2\left(D^\lambda f(z)\right)'+\mu\alpha zD^\lambda f(z)\right|$$

$$=\left|\alpha\left(\sum_{n=1}^{\infty}(n+1)D_n(\lambda)a_nz^{n+1}\right)\right|-\beta\left|\mu(1-\alpha)-\sum_{n=1}^{\infty}(n\mu+\mu\alpha)D_n(\lambda)a_nz^{n+1}\right| \qquad (6)$$

$$\leq\sum_{n=1}^{\infty}[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)a_n-\beta\mu(1-\alpha)\leq 0.$$

Hence by the principle of maximum modulus , f(z)$\in H$ $(\alpha,\mu,\beta,\lambda)$.

Conversely, suppose that f(z) defined by (2) is in the class H $(\alpha,\mu,\beta,\lambda)$, then from (4), we have

$$\left|\frac{\alpha\left(z^2(D^\lambda f(z))\right)' + zD^\lambda f(z))}{\mu z^2(D^\lambda f(z))' + \mu\alpha z D^\lambda f(z)}\right|$$

$$=\left|\frac{\alpha\left(\sum_{n=1}^{\infty}(n+1)D_n(\lambda)a_n z^{n+1}\right)}{\mu(1-\alpha)-\sum_{n=1}^{\infty}(n\mu+\mu\alpha)D_n(\lambda)a_n z^{n+1}}\right| < \beta.$$

Since $|\mathrm{Re}(z)| \le |z|$ for all z , we have

$$\mathrm{Re}\left\{\frac{\alpha\left(\sum_{n=1}^{\infty}(n+1)D_n(\lambda)a_n z^{n+1}\right)}{\mu(1-\alpha)-\sum_{n=1}^{\infty}(n\mu+\mu\alpha)D_n(\lambda)a_n z^{n+1}}\right\} < \beta.$$

Choose the value of z on the real axis so that $\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}$ is real. Upon clearing the denominator of (6) and letting $z \to 1$ through real values , we get

$$\sum_{n=1}^{\infty}\alpha(n+1)D_n(\lambda)a_n \le \beta\mu(1-\alpha) - \sum_{n=1}^{\infty}\beta\mu(n+\alpha)D_n(\lambda)\,a_n\ ,$$

which implies the inequality(5). Sharpness of the result follows by setting

$$f(z) = \frac{1}{z} + \frac{\beta\mu(1-\alpha)}{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)} z^n, (n \ge 1). \tag{7}$$

<u>Corollary 1</u> : Let f(z) $\in H(\alpha,\mu,\beta,\lambda)$. Then

$$a_n \le \frac{\beta\mu(1-\alpha)}{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)},$$

where $0 \le \alpha < 1, 0 < \beta \le 1, 0 \le \mu \le 1\, and\, \lambda > -1.$

3. Distortion and Growth Theorems

In the following theorems, we prove distortion and growth bounds associated with the class introduced in (3).

Theorem 2: Let the function f(z) defined by (2) be in the class $H(\alpha,\mu,\beta,\lambda)$. Then

$$\frac{1}{r}-\frac{2\beta\mu(1-\alpha)}{(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)}r \le |f(z)| \le \frac{1}{r}+\frac{2\beta\mu(1-\alpha)}{(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)}r, 0<|z|=r<1. \quad (8)$$

The equality in (8) is attained by the function f(z) given by

$$f(z)=\frac{1}{z}+\frac{2\beta\mu(1-\alpha)}{(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)}z.$$

Proof: Since the function f(z) defined by (2) in the class $H(\alpha,\mu,\beta,\lambda)$, we have from Theorem 1,

$$\sum_{n=1}^{\infty}a_n \le \frac{2\beta\mu(1-\alpha)}{(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)}.$$

Thus $|f(z)| \le \frac{1}{z}+\sum_{n=1}^{\infty}a_n|z|^n \le \frac{1}{r}+r\sum_{n=1}^{\infty}a_n$

$$\le \frac{1}{r}-\frac{2\beta\mu(1-\alpha)}{\left(\beta\mu(1+\alpha)\right)+2\alpha)(\lambda+1)(\lambda+2)}r.$$

Similarly ,

$$|f(z)| \ge \frac{1}{|z|}-\sum_{n=1}^{\infty}a_n|z|^n \ge \frac{1}{r}-r\sum_{n=1}^{\infty}a_n$$

$$\ge \frac{1}{r}-\frac{2\beta\mu(1-\alpha)}{(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)}r.$$

Theorem 3 : Let the function f(z) defined by (2)be in the class

$H(\alpha,\mu,\beta,\lambda)$ and

$$\frac{(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)}{2} \le \frac{(\beta\mu(n+\alpha)+\alpha(n+1))D_n(\lambda)}{n}.$$

$$\frac{1}{r^2} - \frac{2\beta\mu(1-\alpha)}{(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)} \le |f'(z)| \le$$

$$\frac{1}{r^2} + \frac{2\beta\mu(1-\alpha)}{(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)}, 0<|z|=r<1,$$

with equality for

$$f(z) = \frac{1}{z} + \frac{2\beta\mu(1-\alpha)}{(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)} z.$$

Proof: Theorem 3 can be proved easily by following lines similar to Theorem 2.

4.**Closure Theorems**

In the next theorem, we obtain extreme points for our class $H(\alpha,\mu,\beta,\lambda)$.

Theorem 4 : Let $f_0(z)=\frac{1}{z}$ and

$$f_n(z) = \frac{1}{z} + \frac{\beta\mu(1-\alpha)}{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)} z^n, where$$

$n \ge 1, n \in \mathbb{N}, 0 \le \alpha < 1, 0 < \beta \le 1, 0 \le \mu \le 1, \lambda > -1 \, and \, D_n(\lambda)$

is given by ($*$). Then f(z) is in the class $H(\alpha,\mu,\beta,\lambda)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \sigma_n f_n(z), where(\sigma_n \ge 0 \, and \sum_{n=0}^{\infty} \sigma_n = 1 \text{ or } 1 = \sigma_0 + \sum_{n=1}^{\infty} \sigma_n).$$

Proof: Let

$$f(z)=\sum_{n=0}^{\infty}\sigma_n f_n(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{\beta\mu(1-\alpha)\sigma_n}{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}z^n.$$

Then $\sum_{n=1}^{\infty}\left[\frac{(\beta\mu(n+\alpha)+\alpha(n+1))D_n(\lambda)}{\beta\mu(1-\alpha)}\right]\sigma_n\frac{\beta\mu(1-\alpha)}{(\beta\mu(n+\alpha)+\alpha(n+1))D_n(\lambda)}$

$$=\sum_{n=1}^{\infty}\sigma_n=1-\sigma_0\le 1.$$

Using Theorem 1, we easily obtain f(z)$\in H(\alpha,\mu,\beta,\lambda)$.

Conversely , let f(z) $\in H(\alpha,\mu,\beta,\lambda)$ is of the form (2).

Then

$$a_n\le\frac{\beta\mu(1-\alpha)}{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)},(n\ge 1,n\in \mathbb{N}).$$

Setting

$$\sigma_n=\frac{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{\beta\mu(1-\alpha)}\ a_n\text{ , for n=1,2, …}$$

and $\sigma_0=1-\sum_{n=1}^{\infty}\sigma_n.$ *Then*

$$f(z)=\sum_{n=0}^{\infty}\sigma_n f_n(z)=\sigma_0 f_0(z)+\sum_{n=1}^{\infty}\sigma_n f_n(z).$$

Now , we shall prove that the class $H(\alpha,\mu,\beta,\lambda)$ is closed under arithmetic mean and convex linear combinations.

Let the function $f_k(z)(k=1,2,\ldots,m)$ be defined by

$$f_k=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,k}z^n,(a_{n,k}\ge 0,n\in\mathbb{N}\text{ , n}\ge 1)\ . \qquad (9)$$

Theorem 5 : Let the functions $f_k(z)$ defined by (9) be in the class $H(\alpha,\mu,\beta,\lambda)$ for every k=1,2,...,m.

Then the arithmetic mean of $f_k(z)$ (k=1,...,m) is defined by

$g(z)=\frac{1}{z}+\sum_{n=1}^{\infty} b_n z^n, (b_n \geq 0, n \geq 1, n \in \mathbb{N})$, also belongs to the class $H(\alpha,\mu,\beta,\lambda)$, where

$$b_n = \frac{1}{m}\sum_{k=1}^{m} a_{n,k}.$$

Proof : Since $f_k(z) \in H(\alpha,\mu,\beta,\lambda)$, therefore from Theorem 1, we get

$$\sum_{n=1}^{\infty}\left[\beta\mu(n+\alpha)+\alpha(n+1)\right]D_n(\lambda)a_{n,k} \leq \beta\mu(1-\alpha). \tag{10}$$

Hence $\sum_{n=1}^{\infty}\left[\beta\mu(n+\alpha)+\alpha(n+1)\right]D_n(\lambda)b_n$

$$= \sum_{n=1}^{\infty}\left[\beta\mu(n+\alpha)+\alpha(n+1)\right]D_n(\lambda)\left[\frac{1}{m}\sum_{k=1}^{m} a_{n,k}\right] \leq \beta\mu(1-\lambda)$$

(by (10)) which shows that $g(z) \in H(\alpha,\mu,\beta,\lambda)$ and this completes the proof .

Theorem 6 : The class $H(\alpha,\mu,\beta,\lambda)$ is closed under convex linear combination.

Proof : Let the function $f_k(z)$ (k=1,2) defined by (9) be in the class $H(\alpha,\mu,\beta,\lambda)$. We show the function.

$$g(z) = \sigma\, f_1(z) + (1-\sigma) f_2(z), (0 \leq \sigma \leq 1)$$

is also in the class $H(\alpha,\mu,\beta,\lambda)$. Since for $0 \leq \sigma \leq 1$,

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left[\sigma\, a_{n,1} + (1-\sigma)a_{n,2}\right] z^n.$$

Therefore by Theorem 1, we have

$$\sum_{n=1}^{\infty}\left[\beta\mu(n+\alpha)+\alpha(n+1)\right]D_n(\lambda)\left[\sigma\, a_{n,1}+(1-\sigma)a_{n,2}\right]$$
$$=\sigma\sum_{n=1}^{\infty}\left[\beta\mu(n+\alpha)+\alpha(n+1)\right]D_n(\lambda)a_{n,1}$$
$$+(1-\sigma)\sum_{n=1}^{\infty}\left[\beta\mu(n+\alpha)+\alpha(n+1)\right]D_n(\lambda)a_{n,2}$$
$$\leq \beta\mu(1-\alpha).$$

Hence by Theorem 1 , we get g(z) $\in H(\alpha,\mu,\beta,\lambda)$.

<u>Theorem 7</u> : Let the function f_k(z) defined by (9) be in the class $H(\alpha_k,\mu\ ,\beta,\lambda)$ $(0\leq\alpha_k<1, 0<\beta\leq 1, 0\leq\mu\leq 1$ and $n\geq 1, n\in\mathbb{N}$ for each k=1,2,...,m. Then the function g(z) defined by

$g(z)=\frac{1}{z}+\frac{1}{m}\sum_{n=1}^{\infty}\left[\sum_{k=1}^{m}a_{n,k}\right]z^n$ is in the class $H(\alpha,\mu,\beta,\lambda)$, where

$$\alpha=\min_{1\leq k\leq m}\{\alpha_k\}. \tag{11}$$

<u>Proof</u> : Since f_k(z) $\in H(\alpha_k\ ,\mu,\beta,\lambda)$ for each k=1,2,…,m, we note that

$$\sum_{n=1}^{\infty}\left[\beta\mu(n+\alpha_k)+\alpha_k(n+1)\right]D_n(\lambda)a_{n,k}\leq\beta\mu(1-\alpha_n).$$

Therefore

$$\sum_{n=1}^{\infty}\left[\beta\mu(n+\alpha_k)+\alpha_k(1+n)\right]D_n(\lambda)\left[\frac{1}{m}\sum_{k=1}^{m}a_{n,k}\right]$$
$$=\frac{1}{m}\sum_{k=1}^{m}\sum_{n=1}^{\infty}\left[\beta\mu(n+\alpha_k)+\alpha_k(1+n)\right]D_n(\lambda)a_{n,k}$$
$$\leq\frac{1}{m}\sum_{k=1}^{m}\beta\mu(1-\alpha_k)\leq\beta\mu(1-\alpha).$$

Thus , we get

$$\sum_{n=1}^{\infty}[\beta\mu(n+\alpha_k)+\alpha_K(1+n)]\left[\frac{1}{m}\sum_{k=1}^{m}a_{n,k}\right]\leq\beta\mu(1-\alpha).$$

Hence, by Theorem 1 , we have g(z) $\in H(\alpha,\mu,\beta,\lambda)$, where α is given by (11). This completes the proof of Theorem 7.

Theorem 8: Let the function f_k(z) defined by (9) be in the class $H(\alpha,\mu,\beta,\lambda)$ for every k=1,2,...,m. Then the function g(z) defined by

$$g(z)=\sum_{k=1}^{m}d_k f_k(z)\; and \sum_{k=1}^{m}d_k=1,(d_k\geq 0)$$ in the class $H(\alpha,\mu,\beta,\lambda)$.

Proof : By definition of g(z) , we have

$$g(z)=\left[\sum_{k=1}^{m}d_k\right]\frac{1}{z}+\sum_{n=1}^{\infty}\left[\sum_{k=1}^{m}d_k a_{n,k}\right]z^n.$$

Since f_k(z) are in $H(\alpha,\mu,\beta,\lambda)$ for every k=1,2,…,m, we get

$$\sum_{n=1}^{\infty}[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)a_{n,k}\leq\beta\mu(1-\alpha)$$

for every k=1,2,...,m. Hence we can see that

$$\sum_{n=1}^{\infty}[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)\left[\sum_{k=1}^{m}d_k a_{n,k}\right]$$

$$=\sum_{k=1}^{m}d_k\left[\sum_{n=1}^{\infty}[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)a_{n,k}\right]$$

$$\leq\beta\mu(1-\alpha)\sum_{k=1}^{m}d_k\leq\beta\mu(1-\alpha).$$

$Thus\, g(z)\in H(\alpha,\mu,\beta,\lambda).$

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