FUGLEDE – PUTNAM THEOREM FOR CLASS $A(K^*)$ OPERATORS

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Abstract **: We say that the operators A,B on a Hilbert space satisfy Fuglede – Putnam theorem if AX=XB for some X implies that $A^*X = XB^*$. In this paper , the hypotheses on A and B can be relaxed by using a Hilbert-Schmidt operator X : Let A belong to class $A(K^*)$ and let B^* be invertible operator belong to class $A(K^*)$ such that AX=XB for a Hilbert-Schmidt operator X ,then $A^*X = XB^*$.**

المستخلص **:**

في هذا البحث سنبرهن بأن نظرية Fuglede-Putnam متحققة عندما تكون المؤثرا ت $A\,B^*$, القابل للانعكاس تنتمي إلى الصنف $A(K^*)$.

1) *Introduction*

Let H be a separable complex Hilbert space and let L(H) denote the algebra of all bounded linear operators on H . An operator $T \in L(H)$ is called normal if $T^*T = TT^*$, hyponormal if $TT^* \leq T^*T$,p-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$ for $p > 0$ .We say that an operator $T \in L(H)$ belongs to the class $A(K^*)$ if $|T^*|^2 \leq (T^*|T|^{2k}T)^{1/(k+1)}$ for each k > 0. Where $|T|$ is a positive square root of T^*T . Class $A(K^*)$ was first introduced by S. Panayappan [5] as a subclass of absolute K *-paranormal operators. The following Theorem A is one of the results associated with class $A(K^*)$.

<u>Theorem</u> A ([5]) .

For each K>0 every class $A(K^*)$ operator is an absolute K *-paranormal operator.

The familiar Fuglede-Putnam theorem is as follows(see[3]and[4]) :

<u>Theorem</u> B. If A and B are normal operators and if X is an operator such that AX=XB , then $A^*X = XB^*$.

S.K.Berberian [1] has extended the result by assuming A and B^* are hyponormal and X is a Hilbert – Schmidt operator . Recently , Chō &Huruya [2] have extended the result by assuming A , B^* and X to be p-hyponormal , invertible p-hyponormal and Hilbert-Schmidt respectively .
In this paper , we extended the result in theorem B by assuming A ,B^* and X to be belong to class $A(K^*)$, invertible operator belong to class $A(K^*)$ and Hilbert-Schmidt respectively.

<u>Theorem</u> 1.1 . If T is an invertible operator belong to class $A(K^*)$, then so T^{-1}.
***<u>Proof</u>* :**
Since

$$(|T^*|^2 = T.T^* \text{ and } (T^*|T|^{2k}T)^{1/(k+1)} = \left(T^{*(k+1)}.T^{(k+1)}\right)^{1/(k+1)}), \text{then}$$

$$\left(T.T^*\right)^{(k+1)} \le \left(T^*.T\right)^{(k+1)}, \text{ we have}$$

$$\left(T^*.T\right)^{\frac{-(k+1)}{2}}\left(\left(T^*T\right)^{(k+1)} - \left(TT^*\right)^{(k+1)}\right)\left(T^*T\right)^{\frac{-(k+1)}{2}} \ge 0.$$

This is equivalent to

$$I \ge \left(T^*T\right)^{\frac{-(k+1)}{2}}\left(TT^*\right)^{(k+1)}\left(T^*T\right)^{\frac{-(k+1)}{2}}.$$

It is well known that $A \ge I$ implies $A^{-1} \le I$. Thus

$$0 \le \left(T^*T\right)^{\frac{(k+1)}{2}}\left(TT^*\right)^{-(k+1)}\left(T^*T\right)^{\frac{(k+1)}{2}} - I$$

$$= \left(T^*T\right)^{\frac{(k+1)}{2}}\left(\left(TT^*\right)^{-(k+1)} - \left(T^*T\right)^{-(k+1)}\right)\left(T^*T\right)^{\frac{(k+1)}{2}}.$$

This is equivalent to

$$0 \le \left(TT^*\right)^{-(k+1)} - \left(T^*T\right)^{-(k+1)}$$

$$= \left(\left(T^{-1}\right)^* T^{-1}\right)^{(k+1)} - \left(T^{-1}\left(T^{-1}\right)^*\right)^{(k+1)}.$$

So, T^{-1} is belong to class $A(k^*)$.

<u>Theorem</u>1.2. **Let T belong to class A(K^*).If Tx=$\lambda x, \lambda \neq 0,$ then $T^*x = \bar{\lambda}x$.**

Proof:

We may assume $x \neq 0$. Since $\left\langle \left|T^*\right|^2 x, x \right\rangle \leq \left\langle \left(T^*|T|^{2k}T\right)^{1/(k+1)} x, x \right\rangle$

and $\left\langle \left(T^*|T|^{2k}T\right)^{1/(k+1)} x, x \right\rangle = \left\langle T^*Tx, x \right\rangle = |\lambda|^2 \|x\|^2.$

Thus $\left|T^*\right|^2 x \leq |\lambda|^2 x$ **and**

$$\left\|T^*x - \bar{\lambda}x\right\|^2 = \left\langle T^*x - \bar{\lambda}x, T^*x - \bar{\lambda}x \right\rangle$$
$$= \left\langle T^*x, T^*x \right\rangle - \bar{\lambda}\left\langle x, T^*x \right\rangle - \bar{\lambda}\left\langle T^*x, x \right\rangle + |\lambda|^2$$
$$= \left\langle \left|T^*\right|^2 x, x \right\rangle - \bar{\lambda}\left\langle Tx, x \right\rangle - \bar{\lambda}\left\langle x, Tx \right\rangle + |\lambda|^2$$
$$\leq |\lambda|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 = 0$$

Hence $T^*x = \bar{\lambda}x.$

2) ***<u>Main Results</u>***

Let T be an operator in L(H)and let $\{e_n\}$ be an orthonormal basis for H . We define the Hilbert-Schmidt norm of T to be

$$\|T\|_2 = \left(\sum_{n=1}^{\infty} \|Te_n\|^2 \right)^{\frac{1}{2}}.$$

This definition is independent of the choice of basis (*see* [3]). If $\|T\|_2 < \infty$, then T is said to be a Hilbert-Schmidt operator and we denote the set of all Hilbert-Schmdit operators on H by $B_2(H)$.

Let $B_1(H)$ be the set $\{C = AB \mid A, B \in B_2(H)\}$. Then operators belonging to $B_1(H)$ are called trace class operators .We define a linear functional $tr : B_1(H) \to C$ by $tr(C) = \sum_{n=1}^{\infty} \langle Ce_n, e_n \rangle$ for an orthonormal basis $\{e_n\}$ for H .

In this case , the definition of tr(C) dose not depend on the choice of an orthonormal basis and tr(C) is called the trace of C. Then we know the followings:

<u>Theorem</u> 2.1 [3].

1)The set $B_2(H)$ is self-adjoint ideal of L(H) .

2) If $\langle A, B \rangle = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle = tr(B^* A) = tr(AB^*)$ for A and B in $B_2(H)$,

then $\langle .,. \rangle$ is an inner product on $B_2(H)$ and $B_2(H)$ is a Hilbert space with respect to this inner product.

<u>Theorem</u>2.2[3].If $T \in L(H)$ and $A \in B_2(H)$,then $\|A\| \le \|A\|_2 = \|A^*\|_2$,

$\|TA\|_2 \le \|T\| \|A\|_2$ and $\|AT\|_2 \le \|A\|_2 \|T\|$.

For each pair of operators A and B in L(H) , an operator J in L($B_2(H)$) is defined by JX=AXB, which is due to Berberian [1] .

Evidently , by the above Theorem 2.1 and Theorem 2.2 , $\|J\| \le \|A\|.\|B\|$. And the adjoint of J is given by the formula $J^* X = A^* XB^*$, as one sees from the calculation

$$\langle J^* X, Y \rangle = \langle X, JY \rangle = \langle X, AYB \rangle = tr((AYB)^* X) = tr(XB^* Y^* A^*) = tr(A^* XB^* Y^*)$$

$$= \langle A^* XB^*, Y \rangle.$$

If $A \geq 0$ and $B \geq 0$, then also $J \geq 0$ and $J^{\frac{1}{2}}X = A^{\frac{1}{2}}XB^{\frac{1}{2}}$ because of

$$\langle JX, X\rangle = tr\left(AXBX^*\right) = tr\left(A^{\frac{1}{2}}XBX^*A^{\frac{1}{2}}\right)$$

$$= tr\left(\left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)\left(A^{\frac{1}{2}}XB^{\frac{1}{2}}\right)^*\right) \geq 0.$$

<u>Lemma</u> 2.3 . If A and B^* are belongs to class $A(K^*)$, then the operator J in $L(B_2(H))$ defined by JX=AXB is also belong to class $A(K^*)$.

***<u>Proof</u>* :**

Since $J^*JX = A^*AXBB^*$ and $JJ^*X = AA^*XB^*B$ for any operator X in $B_2(H)$,

we get $|J|X = |A|X|B^*|$ and $|J^*|X = |A^*|X|B|$

and so $|J|^2 X = |A|^2 X|B^*|^2$ and $|J^*|^2 X = |A^*|^2 X|B|^2$,

we have $|J|^{2k} X = |A|^{2k} X|B^*|^{2k}$ and $|J^*|^{2k} X = |A^*|^{2k} X|B|^{2k}$ for each $k\rangle 0$.

Thus , we have $\left(J^*|J|^{2k} J\right)X = \left(A^*|A|^{2k} A\right)X\left(B^*|B^*|^{2k} B\right)$

$$\geq \left(|A^*|^2\right)^{k+1} X\left(|B|^2\right)^{k+1} \left(\text{since } A \text{ and } B^* \text{ belong to class } A\left(K^*\right)\right)$$

$$= \left(|J^*|^2\right)^{k+1} X$$

Which completes the proof.

<u>Theorem</u> 2.4. If A is belong to class $A(K^*)$ and B^* is invertible belong to class $A(K^*)$ such that AX=XB for any operator X in $B_2(H)$, then $A^*X = XB^*$.

Proof **: Let J be the operator on $B_2(H)$ defined by $JX = AXB^{-1}$. Since $\left(B^*\right)^{-1} = \left(B^{-1}\right)^*$ is belong to class $A\left(K^*\right)$ by theorem 1.1 , by lemma 2.3 , J is also belong to class $A\left(K^*\right)$. The hypothesis AX=XB implies $JX = AXB^{-1}$ =X and so , by Theorem 1.2 $J^*X = X$. Hence we have $A^*X\left(B^{-1}\right)^* = J^*X = X$. Therefore , $A^*X = XB^*$ which is the desired relation.**

3) ***REFRENCES***

1) **S.K.Berberian : Extensions of a theorem of fuglede and Putnam. Proc.Amer. Math. Soc. 71(1978), no.1,113-114.**
2) **M.Ch$\bar{o}$ &T.Huruya: P-hyponormal operators for 0<p<$\frac{1}{2}$. Comment.Math.Prace Mat. 33(1993),23-29.**
3) **J.B.Conway:Subnormal operators, Research Notes in Mathematics,51.Pitman Advanced Pub . Program, 1981.**
4) **P.R.Halmos, A Hilbert space problem book , Springer-Verlage, New York, 1974.**
5) **S.Panayappan: A note on p *-paranormal and absolute k *-paranormal operators . Int.Journal of Math. Analysis, vol.2,2008,no.26,1257-1261.**