Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.17.(1) 2025,pp.Math 1–8

JQCM

University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

Weakly Nearly Primary Submodules

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A R T I C L E I N F O

Article history:
Received: 10 /12/2024
Rrevised form: 23 /12/2024
Accepted : 13 /1/2025
Available online: 30 /3/2025

Keywords:

Weakly nearly primary submodules
Weakly prime submodules
Compressible module
Zero divisor
Secondary module

A B S T R A C T

The concept of weakly nearly primary submodule in this paper is a new generalization of weakly prime submodule. Several basic properties and examples of this concept were established. Furthermore, many characterizations of weakly nearly primary submodules are given. Moreover, we show that by example the residual of a weakly nearly primary submodule need not be a weakly nearly primary ideal, but the converse holds under sufficient conditions. Finally, in this paper, we stand for a commutative ring with identity and M is a unitary left R-module.

https://doi.org/10.29304/jqcsm.2025.17.11982

1. Introduction

Throughout this note, all rings are commutative with identity and all modules are unitary left R-modules. A proper submodule T of an R-module M is weakly prime (respectively primary) submodule if whenever $0 \neq rm \in T$, where $r \in R$, and $m \in M$, implies that $m \in T + J(M)$ or $r \in (T:_R M)$ (respectively $r \in \sqrt{(T:_R M)}$) [1, 2]. Many authors studied weakly prime submodules see for example [3, 4]. We introduce in this paper new generalizations of weakly prime submodules which we called weakly nearly primary submodules, many characterizations, examples, and basic properties of this concept given. Also we prove that every weakly prime submodules weakly nearly primary submodule but not conversely. So under certain condition we prove that the two concepts weakly prime and weakly nearly primary submodules are equivalent. An R-module M is a semi-simple, if every submodule of M is a direct summand of M[5], and a non-zero R-module M is compressible if every non-zero submodule of M contains an isomorphic copy of M[6]. And an R-module M is multiplication (for short multi.) if every submodule T in M is in the form IM for some ideal I in R [7]. An R-module M is faithful if $ann_R(M) = (0)$, where $ann_R(M) = \{r \in R: rM = (0)\}$ [5]. If M be a faithful multi. R-module, then $soc(R)M = soc(M)$ [7, coro.(2.14)(i)] A submodule T of an R-module M is small if $T + K = M$, then $K = M$ for any proper submodule K of M [8]. A zero divisor on M is an element $r \in R$ for

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which there exists a non-zero element $m \in M$ such that $rm = 0$, the set of all zero divisors of M is denoted by $Zdr_M(M)$[9]. An R-module M is secondary, provided that for every element $r \in R$ either $rM = M$ or $r^n M = 0$ for some positive-integer n [10]. We use the notation $T \hookrightarrow M$ for T is a proper submodule of M.

2. Weakly Nearly Primary Submodules

This section is dedicated to introducing the definition of weakly nearly primary submodules and explaining some of their characteristics, examples, and properties.

Definition 2.1 A proper submodule T of an R-module M is a weakly nearly primary submodule (for short WNprimary), if whenever $0 \neq rm \in T$, where $r \in R$, and $m \in M$, implies that $m \in T + J(M)$ or $r \in \sqrt{(T + J(M):_R M)}$, that is $r^n M \subseteq T + J(M)$ for some positive-integer n.

And an ideal I of a ring R is a WNprimary ideal of R, if I is WNprimary R-submodule of an R-module R.

Remark and example 2.2

1. It is obvious that every primary submodule is weakly primary, but contrariwise isn't true, since (0) is always a weakly primary (by definition) but not primary [2].

2. It is obvious that every weakly primary submodule of an R-module M is WNprimary, but contrariwise isn't true as in example:
The submodule Z of the Z-module Q is WNprimary since $J(Q) = Q$. But Z is not weakly primary submodule of Q.

3. Every weakly prime submodule of an R-module M is WNprimary, but contrariwise isn't true.

 Proof Let T be weakly prime submodule of M, then by [2] T is weakly primary, it follows that by(2) T is WNprimary. For the converse example:
 The submodule $\langle\bar{4}\rangle$ of the Z-module Z_{36} is WNprimary, since whenever $0 \neq rm \in \langle\bar{4}\rangle$, where $r \in Z$, and $m \in Z_{36}$, implies $m \in \langle\bar{4}\rangle + J(Z_{36}) = \langle\bar{4}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$ or $r \in \sqrt{(\langle\bar{4}\rangle + J(Z_{36}):_Z Z_{36})} = \sqrt{(\langle\bar{4}\rangle + \langle\bar{6}\rangle:_Z Z_{36})} = \sqrt{(\langle\bar{2}\rangle:_Z Z_{36})} = 2Z$. That is $2.\bar{2} \in \langle\bar{4}\rangle$, for $2 \in Z$, $\bar{2} \in Z_{36}$, it follows $\bar{2} \in \langle\bar{4}\rangle + J(Z_{36}) = \langle\bar{2}\rangle$ and $2 \in \sqrt{(\langle\bar{4}\rangle + J(Z_{36}):_Z Z_{36})} = 2Z$.. But $\langle\bar{4}\rangle$ is not prime submodule of Z_{36}.

4. The submodules $\langle\bar{2}\rangle$ and $\langle\bar{3}\rangle$ are weakly prime of Z-module Z_{36} hence by (3) they are WNprimary submodules of Z_{36}.

5. The intersection of two WNprimary submodules of R-module M need not to be WNprimary submodule, the following example shows that:
The submodules $\langle\bar{2}\rangle$ and $\langle\bar{3}\rangle$ are WNprimary of Z-module Z_{36}. But $\langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$ is not WNprimary submodule, since $0 \neq 2 \cdot \bar{3} \in \langle\bar{6}\rangle$, for $2 \in Z$, $\bar{3} \in Z_{36}$ but $\bar{3} \notin \langle\bar{6}\rangle + J(Z_{36}) = \langle\bar{6}\rangle + \langle\bar{6}\rangle = \langle\bar{6}\rangle$ and $2 \notin \sqrt{(\langle\bar{6}\rangle + J(Z_{36}):_Z Z_{36})} = \sqrt{(\langle\bar{6}\rangle:_Z Z_{36})} = 6Z$.

6. The residual of WNprimary submodule need not to be WNprimary ideal, the following example shows that:
In the Z-module Z_6 the submodule $\langle\bar{0}\rangle$ is always WNprimary. But $(\langle\bar{0}\rangle:_Z Z_6)$=6Z is not WNprimary ideal of Z, because $0 \neq 2.3 \in 6Z$ for $2, 3 \in Z$ but $3 \notin 6Z + J(Z) = 6Z + (0) = 6Z$ and $2 \notin \sqrt{(6Z + J(Z):_Z Z)} = 6Z$.

The following results are characterizations of WNprimary submodule.

Proposition 2.3 Let M be an R-module, and $T \hookrightarrow M$. Then T is a WNprimary submodule of M if and only if $(0) \neq IK \subseteq T$ for I is an ideal in R and K is a submodule in M, implies that $K \subseteq T + J(M)$ or $I \subseteq \sqrt{(T + J(M):_R M)}$.

Proof $\mapsto$ Assume $(0) \neq I\,K \subseteq T$ with $K \nsubseteq T + J(M)$, it follows that there exists $0 \neq k \in K$ and $k \notin T + J(M)$. We must show that $I \subseteq \sqrt{(T + J(M):_R M)}$. Now, let $a \in I$, if $0 \neq ak \in T$ and T is a WNprimary submodule, then $a \in \sqrt{(T + J(M):_R M)}$, that is $I \subseteq \sqrt{(T + J(M):_R M)}$. So, we assume that$ak = 0$. Again suppose that $aK \neq (0)$, that is $0 \neq a\,s \in T$ for some $s \in K$, if $s \notin T + J(M)$ such that $s \notin T$, and T is a WNprimary submodule, it follows that $a \in \sqrt{(T + J(M):_R M)}$, so $I \subseteq \sqrt{(T + J(M):_R M)}$. If $c \in T \subseteq T + J(M)$, then $0 \neq as = a(s + k) \in T$ and T is a

WNprimary submodule, so either $s+k \in T+J(M)$ or $a \in \sqrt{(T+J(M):_R M)}$. Thus $I \subseteq \sqrt{(T+J(M):_R M)}$. Again we can assume that $aK=(0)$. Now, suppose that $Ik \neq (0)$, that is $0 \neq ck \in T$ for some $c \in I$, and since T is WNprimary submodule of M, then $c \in \sqrt{(T+J(M):_R M)}$. Since $0 \neq ck=(a+c)k \in T$ and T is a WNprimary, we get $a+c \in \sqrt{(T+J(M):_R M)}$, that is $a \in \sqrt{(T+J(M):_R M)}$, hence $I \subseteq \sqrt{(T+J(M):_R M)}$. Thus we can assume that $Ik=(0)$. But $IK \neq (0)$, implies that there exists $s_1 \in K, b \in I$ such that $0 \neq bs_1 = b(s_1+k) \in T$, so we have two cases:

Case I: If $b \in \sqrt{(T+J(M):_R M)}$ and $s_1+k \notin T+J(M)$. But $0 \neq (a+b)(s_1+k)=bs_1 \in T$ and T is WNprimary submodule of M, then $(a+b) \in \sqrt{(T+J(M):_R M)}$ so $a \in \sqrt{(T+J(M):_R M)}$. Thus $I \subseteq \sqrt{(T+J(M):_R M)}$.

Case II: If $b \notin \sqrt{(T+J(M):_R M)}$ and $s_1+k \in T+J(M)$. Since $0 \neq bs_1 \in T$ and N is a WNprimary, we have $s_1 \in T+J(M)$, so $k \in T+J(M)$ which is a contradiction. Hence $I \subseteq \sqrt{(T+J(M):_R M)}$.

↤ Assume that $0 \neq rm \in T$ for $r \in R$, $m \in M$, then $(0) \neq <r><m> \subseteq T$, it follows by hypothesis either $<m> \subseteq T+J(M)$ or $<r> \subseteq \sqrt{(T+J(M):_R M)}$. Hence either $m \in T+J(M)$ or $r \in \sqrt{(T+J(M):_R M)}$. That is T is a WNprimary submodule of M.

The proposition 2.3 directly leads to the following corollaries.

Corollary 2.4 Let M be an R-module, and $T \hookrightarrow M$. Then T is a WNprimary submodule of M if and only if $(0) \neq rK \subseteq T$ for $r \in R$ and K is a submodule in M, implies that $K \subseteq T+J(M)$ or $r \in \sqrt{(T+J(M):_R M)}$.

Corollary 2.5 Let M be an R-module, and $T \hookrightarrow M$. Then T is a WNprimary submodule of M if and only if $(0) \neq Im \subseteq T$ for I is an ideal in R and $m \in M$, implies that $m \in T+J(M)$ or $I \subseteq \sqrt{(T+J(M):_R M)}$.

Proposition 2.6 Let M be an R-module, and $T \hookrightarrow M$. Then T is a WNprimary submodule of M if and only if for $m \in M \setminus (T+J(M))$, $(T:_R m) \subseteq \sqrt{(T+J(M):_R M)} \cup (0:_R m)$.

Proof ↦ Let $a \in (T:_R m)$, where $m \in M \setminus (T+J(M))$, then $am \in T$. If $am=0$, then $a \in (0:_R m)$, it follows that $a \in \sqrt{(T+J(M):_R M)} \cup (0:_R m)$. If $0 \neq am \in T$ and N is a WNprimary with $m \notin T+J(M)$ then $a \in \sqrt{(T+J(M):_R M)}$. Hence $a \in \sqrt{(T+J(M):_R M)} \cup (0:_R m)$. Thus $(T:_R m) \subseteq \sqrt{(T+J(M):_R M)} \cup (0:_R m)$.

↤ Suppose that $(T:_R m) \subseteq \sqrt{(T+J(M):_R M)} \cup (0:_R m)$ with $m \in M \setminus (T+J(M))$. Let $0 \neq rm \in T$ for $r \in R, m \in M$, $m \notin T+J(M)$, it follows that $r \in (T:_R m)$, implies that $r \in \sqrt{(T+J(M):_R M)} \cup (0:_R m)$. But $0 \neq rm$, then $r \notin (0:_R m)$, hence $r \in \sqrt{(T+J(M):_R M)}$. So T is a WNprimary submodule of M.

Proposition 2.6 and corollary 2.4 directly lead to the following corollary.

Corollary 2.7 Let M be an R-module, and $T \hookrightarrow M$. Then T is a WNprimary submodule of M if and only if for $K \subseteq M \setminus (T+J(M))$, $(T:_R K) \subseteq \sqrt{(T+J(M):_R M)} \cup (0:_R K)$.

Proposition 2.8 Let M be an R-module, and $T \hookrightarrow M$. Then T is a WNprimary if and only if $(T:_M r) \subseteq (T+J(M):_M r^n) \cup (o:_M r)$, for $r \in R$, and n is a positive-integer.

Proof ↦ Let $m \in (T:_M r)$ with $m \notin T+J(M)$, it follows that $rm \in T$. If $rm=0$, implies that $m \in (o:_M r)$, so $m \in (T+J(M):_M r^n) \cup (o:_M r)$. If $0 \neq rm \in T$ and T is a WNprimary, and $m \notin T+J(M)$, then $r^n \in (T+J(M):_M M)$ for some positive-integer n. That is $r^n M \subseteq T+J(M)$, so $r^n m \in T+J(M)$ for all $m \in M$, it follows that $m \in (T+J(M):_M r^n)$ and hence so $m \in (T+J(M):_M r^n) \cup (o:_M r)$. Thus $(T:_M r) \subseteq (T+J(M):_M r^n) \cup (o:_M r)$.

↤ Let $0 \neq rm \in T$ where $r \in R$, and $m \in M$, with $m \notin T+J(M)$. Then $m \in (T:_M r)$ therefor $m \in (T:_M r) \subseteq (T+J(M):_M r^n) \cup (o:_M r)$, it follows that $m \in (T+J(M):_M r^n)$, that is $r^n m \in T+J(M)$ for all $m \in M \setminus (T+J(M))$, it follows that $r^n M \subseteq T+J(M)$, that is $r^n \in (T+J(M):_R M)$, so $r \in \sqrt{(T+J(M):_R M)}$. Thus T is a WNprimary submodule of M.

Before we present the following proposition, we must review the following lemma that appears in [11, Th.(5-1)].

Lemma 2.9 "If I is an ideal of a ring R. Then I is a maximal if and only if $I + < a >= R$ for all $a \notin I$."

Proposition 2.10 Let M be an R-module, and $T \hookrightarrow M$ with $(T + J(M):_R M)$ is a maximal semi-prime ideal of R. Then T is a WNprimary submodule of$M$.

Proof Assume $(T + J(M):_R M)$ is semi-prime ideal of R, that is $\sqrt{(T + J(M):_R M)} = (T + J(M):_R M)$. Let $0 \neq rm \in T$ where $r \in R, m \in M$ with $r \notin \sqrt{(T + J(M):_R M)} = (T + J(M):_R M)$. Since $(T + J(M):_R M)$ is maximal, it follows that $\sqrt{(T + J(M):_R M)}$ is maximal, so by lemma 2.9 $R =< r > + \sqrt{(T + J(M):_R M)}$ where $< r >$ is an ideal in R, that is $R =< r > + (T + J(M):_R M)$, implies that $1 = ar + b$ for some $a \in R, b \in (T + J(M):_R M)$. Thus, $m = arm + bm \in T + J(M)$, so T is a WNprimary submodule.

Proposition 2.11 Let M be an R-module, and I is a maximal semi-prime ideal in R with $IM + J(M)$ is a proper submodule of M.Then IM is a WNprimary submodule of M.

Proof Since $IM \subseteq IM + J(M)$, it follows that $I \subseteq (IM + J(M):_R M)$, means that there exists $c \in (IM + J(M):_R M)$ and $c \notin I$. But I is a maximal, so be lemma 2.9. We have $R = I + < c >$, it follows that $1 = a + bc$ for some $b \in R, a \in I$, hence $m = am + bcm$ for each $m \in M$, it follows that $m \in IM + J(M)$ for each $m \in M$. Thus $M \subseteq IM + J(M)$. But $IM + J(M) \subseteq M$, it follows that $IM + J(M) = M$, this is a contradiction since $IM + J(M)$ is a proper submodule of M. Therefore $c \in I$, and so that we have $(IM + J(M):_R M) \subseteq I$, hence $I = (IM + J(M):_R M)$ which is maximal semi-prime ideal in R. Thus by proposition2.10 we have IM is a WNprimary submodule of$M$.

Proposition 2.12 Let M be an R-module and $T \hookrightarrow M$ such that $(T + J(M):_R M)$= $(T + J(M):_R K)$ with $T + J(M) \subsetneq K$ for each submodule K of M. Then T is a WNprimary submodule of M.

Proof Let $0 \neq rm \in T$, where $r \in R, m \in M$such that $m \notin T + J(M)$. Let $K = (T + J(M)) + < m >$, so $T + J(M) \subsetneq K$, then $m \in K$. Since $0 \neq rm \in T$ and $m \in K$, implies that $r \in (T:_R K)$ and $T \subseteq T + J(M)$, it follows that $(T:_R K) \subseteq (T + J(M):_R K)$. But by hypothesis we have $(T + J(M):_R K)$= $(T + J(M):_R M)$, it follows that $(T:_R K) \subseteq (T + J(M):_R M)$, implies that $r \in (T + J(M):_R M) \subseteq \sqrt{(T + J(M):_R M)}$. Thus $r \in \sqrt{(T + J(M):_R M)}$, that is T is a WNprimary submodule of M.

Proposition 2.13 Let M be torsion free R-module and $0 \neq T \hookrightarrow M$ with $J(M) \subseteq T$. Then, the following statements are equivalent:

i. T is a WNprimary submodule of M.

ii. $(T:_M I)$ is a WNprimary submodule of M $\forall$ ideal I in R.

iii. $(T:_M r)$ is a WNprimary submodule of M $\forall r \in R$.

Proof i $\mapsto$ ii Let $0 \neq rm \in (T:_M I)$ for $r \in R, m \in M$, then $r(mI) \subseteq T$. If $(0) \neq r(mI) \subseteq T$ and T is a WNprimary submodule of M, it follows that $mI \subseteq T + J(M)$ or $r \in \sqrt{(T + J(M):_R M)}$. But $J(M) \subseteq T$, then $T + J(M) = T$, that is $mI \subseteq T$ or $r \in \sqrt{(T:_R M)}$. Thus $m \in (T:_M I)$ or $r \in \sqrt{(T:_R M)}$, that is $m \in (T:_M I)$ or $r \in \sqrt{(T:_R M)} \subseteq \sqrt{((T:_M I):_R M)}$. It follows that $m \in (T:_M I) \subseteq (T:_M I) + J(M)$ or $r^n \in (T:_R M) \subseteq (T:_R M) + J(M)$ for n is a positive-integer. So $m \in (T:_M I) + J(M)$ or $r \in \sqrt{((T:_R M) + J(M):_R M)}$. That is $(T:_M I)$ is a WNprimary submodule of M. If $(0) = r(mI)$, then $r(mb) = 0$ for some $0 \neq b \in I$, implies that $rm \in \tau(M)$. But M is torsion free, then $\tau(M) = 0$. Hence $rm = 0$, this a contradiction.

ii $\mapsto$ iii Clear.

iii $\mapsto$ I It follows easily by taking $r = 1$.

Proposition 2.14 Let M be an R-module, and $K, L \hookrightarrow M$ with $L \subseteq K$. If K is a WNprimary submodule of M, then $\frac{K}{L}$ is a WNprimary submodule of $\frac{M}{L}$.

Proof Let $0 \neq r(m + L) = rm + L \in \frac{K}{L}$ where $m + L \in \frac{M}{L}, m \in M$, and $r \in R$, implies that $rm \in K$. If $rm = 0$, then $r(m + L) = 0$ this is a contradiction, so $0 \neq rm \in K$. But K is a WNprimary submodule of M, it follows that

$m \in K + J(M)$ or $r^n M \subseteq K + J(M)$. Thus $m + L \in \frac{K+J(M)}{L}$ or $r^n \frac{M}{L} \subseteq \frac{K+J(M)}{L}$, it follows that $m + L \in \frac{K}{L} + \frac{K+J(M)}{L} \subseteq \frac{K}{L} + J(\frac{M}{L})$ or $r^n \frac{M}{L} \subseteq \frac{K}{L} + \frac{K+J(M)}{L} \subseteq \frac{K}{L} + J(\frac{M}{L})$. That is $m + L \in \frac{K}{L} + J(\frac{M}{L})$ or $r \in \sqrt{\left(\frac{K}{L} + J\left(\frac{M}{L}\right) :_R \frac{M}{L}\right)}$. Hence $\frac{K}{L}$ is a WNprimary submodule of $\frac{M}{L}$.

Proposition 2.15 Let M be a semi-simple R-module, and $K, L \hookrightarrow M$ with $L \subseteq K$. If L and $\frac{K}{L}$ is a WNprimary submodule of M and $\frac{M}{L}$ respectively, then K is WNprimary submodule of M.

Proof Let $0 \neq rm \in K$, where $r \in R, m \in M$, implies that $0 \neq r(m + L) = rm + L \in \frac{K}{L}$. If $0 \neq rm \in L$ and it is given that L is a WNprimary submodule of M, then $m \in L + J(M) \subseteq K + J(M)$ or $r^n M \subseteq L + J(M) \subseteq K + J(M)$ for some positive-integer n. Thus K is a WNprimary submodule of M. Suppose that $rm \notin L$, then $0 \neq r(m + L) \in \frac{K}{L}$, but $\frac{K}{L}$ is a WNprimary submodule of $\frac{M}{L}$, implies that $m + L \in \frac{K}{L} + J(\frac{M}{L})$ or $r^n \frac{M}{L} \subseteq \frac{K}{L} + J(\frac{M}{L})$. But M is semi-simple then $J\left(\frac{M}{L}\right) = \frac{L+J(M)}{L}$ [5, p.239], hence $m + L \in \frac{K}{L} + \frac{L+J(M)}{L}$ or $r^n \frac{M}{L} \subseteq \frac{K}{L} + \frac{L+J(M)}{L}$. But $L \subseteq K$, then $L + J(M) \subseteq K + J(M)$, hence $\frac{K}{L} + \frac{L+J(M)}{L} \subseteq \frac{K}{L} + \frac{K+J(M)}{L}$, and since $\frac{K}{L} \subseteq \frac{K+J(M)}{L}$, implies that $\frac{K}{L} + \frac{L+J(M)}{L} = \frac{K+J(M)}{L}$. Thus we have either $m + L \in \frac{K+J(M)}{L}$ or $r^n \frac{M}{L} \subseteq \frac{K+J(M)}{L}$, implies that either $m \in K + J(M)$ or $r^n M \subseteq K + J(M)$, that is $m \in K + J(M)$ or $r \in \sqrt{(K + J(M) :_R M)}$. K is a WNprimary submodule of$M$.

Proposition 2.16 Let M be an R-module and $T \hookrightarrow M$ with $\frac{M}{T}$ is a compressible module. Then T is a WNprimary submodule of M.

Proof Let $0 \neq rL \subseteq T$ where $r \in R$ and L is a submodule in M with $T \subseteq L$ such that $L \nsubseteq T + J(M)$. So $\frac{L}{T}$ is a submodule of $\frac{M}{T}$. Since $\frac{M}{T}$ is compressible, then there exists a monomorphism $\varphi: \frac{M}{T} \to \frac{L}{T}$ such that $r\varphi(\frac{M}{T}) = (0)$, that is $\varphi(r\frac{M}{T}) = (0)$, implies that $r\frac{M}{T} = (0)$, that is $rM \subseteq T \subseteq T + J(M)$. Thus $r \in (T + J(M) :_R M) \subseteq \sqrt{\in (T + J(M) :_R M)}$, that is $r \in \sqrt{\in (T + J(M) :_R M)}$. Hence T is a WNprimary submodule of M.

Proposition 2.17 Let M be an R-module and $T \hookrightarrow M$ with $J(M) \subseteq T$. Then $(T :_R M)$ is a WNprimary ideal of R.

Proof Let $0 \neq rs \in (T :_R M)$, where $r, s \in R$, that is $0 \neq r(sM) \subseteq T$. But T is WNprimary submodule of M, then by corollary 2.4 $sM \subseteq T + J(M)$ or $r^n M \subseteq T + J(M)$ for some positive-integer n. Since $J(M) \subseteq T$, then $T + J(M) = T$, it follows that $sM \subseteq T$ or $r^n M \subseteq T$, that is $s \in (T :_R M) \subseteq (T :_R M) + J(R)$ or $r^n \in (T :_R M) \subseteq (T :_R M) + J(R) = \left((T :_R M) + J(R) :_R R\right)$. So $s \in (T :_R M) + J(R)$ or $r \in \sqrt{((T :_R M) + J(R) :_R R)}$. Hence $(T :_R M)$ is a WNprimary ideal of$R$.

The proposition 2.17's opposite is generally untrue, the example that follows clarifies.

Example 2.18 Let $T = (0)\oplus 5Z$ be a submodule of the Z-module $Z \oplus Z$, then $\left((0)\oplus 5Z :_Z Z \oplus Z\right) = (0)$ is a WNprimary ideal of Z, (since (0) is a weakly primary ideal of Z). But $T = (0)\oplus 5Z$ is not WNprimary of the Z-module $Z \oplus Z$, because $(0,0) \neq 5(0,3) = (0,15) \in (0)\oplus 5Z$ for $5 \in Z$ and $(0,3) \in Z \oplus Z$, but $(0,3) \notin (0)\oplus 5Z + J(Z \oplus Z)$, and $2 \notin \sqrt{((0) \oplus 5Z + J(Z \oplus Z) :_Z Z \oplus Z)} = \sqrt{((0) \oplus 5Z :_Z Z \oplus Z)} = \sqrt{(0)} = (0)$.

Proposition 2.19 Let M be a faithful multi. R-module, and $T \hookrightarrow M$. Then T is a WNprimary if and only if $(T :_R M)$ is WNprimary ideal in R.

Proof $\mapsto$ Let $(0) \neq aI \in (T :_R M)$, where $a \in R$, I is an ideal in R, implies that $(0) \neq a(IM) \subseteq T$. Since T is a WNprimary, then by corollary 2.4 we have $IM \subseteq T + J(M)$ or $a^n M \subseteq T + J(M)$. But M is faithful multi. implies that $J(M) = J(R)M$. Thus, we have $IM \subseteq (T :_R M)M + J(R)M$ or $a^n M \subseteq (T :_R M)M + J(R)M$. That is $I \subseteq (T :_R M) + J(R)$ or $a^n \in (T :_R M) + J(R)$, it follows that $I \subseteq (T :_R M) + J(R)$ or $a^n \in (T :_R M) + J(R) = \left((T :_R M) + J(R) :_R R\right)$. Therefore by corollary 2.4 $(T :_R M)$ is a WNprimary ideal of R.

$\leftarrow$ Let $(0) \neq mK \subseteq T$, where $m \in M$, K is a submodule of M. Since M is a multi. then $m = Rm = JM$, $K = IM$ for some ideals I, J in R, that is $(0) \neq IJM \subseteq T$, so $(0) \neq IJ \subseteq (T :_R M)$. But $(T :_R M)$ is a WNprimary ideal in R, then by proposition2.3 we have $I \subseteq (T :_R M) + J(R)$ or $J^n \subseteq ((T :_R M) + J(R) :_R R) = (T :_R M) + J(R)$, it follows that

$I \subseteq (T:_R M) + J(R)$ or $J^n \subseteq (T:_R M) + J(R)$. But M is faithful multi. hence $J(R)M = J(M)$, thus, $IM \subseteq T + J(M)$ or $J^n M \subseteq (T:_R M) + J(M)$, that is $K \subseteq T + J(M)$ or $m^n \in (T:_R M) + J(M)$. Hence by corollary 2.4 T is a WNprimary submodule of M.

Before we introduce the behavior of WNprimary submodules under R-homoorphism, we need to recall the following lemma that appear in [5, Cor.(9.1.5)(a)].

Lemma 2.20 Let $\varphi: M \xrightarrow{\text{epimorphism}} M^{\circ}$, and $\ker\varphi$ is a small submodule of M then $\varphi\big(J(M)\big) = J(M^{\circ})$ and $\varphi^{-1}\big(J(M^{\circ})\big) = J(M)$.

Proposition 2.21 Let $\varphi: M \xrightarrow{\text{epimorphism}} M^{\circ}$, and $\ker\varphi$ is a small submodule of M. If $\varphi^{-1}(T)$ is a WNprimary submodule of M. Then T is a WNprimary submodule of M°.

Proof Since $\varphi^{-1}(T) \hookrightarrow M$, then $\varphi^{-1}(T) \neq M$, that is $\exists m \in M$ s.t $m \notin \varphi^{-1}(T)$, so $\varphi(m) \notin T$, thus $T \neq M^{\circ}$. Let $0 \neq rx \in T$ and $x \notin T + J(M^{\circ})$, where $r \in R$, $x \in M^{\circ}$. Since φ is epimorphism, then $\exists e \in M$ s.t $\varphi(e) = x$, it follows that $0 \neq rx = r\,\varphi(e) = \varphi(re) \in T$. That is $0 \neq re \in \varphi^{-1}(T)$ with $e \notin \varphi^{-1}(T) + J(M)$. But $\varphi^{-1}(T)$ is a WNprimary submodule of M, implies that $r \in \sqrt{(\varphi^{-1}(T) + J(M):_R M)}$, that is $r^n M \subseteq \varphi^{-1}(T) + J(M)$. To show that $r^n M^{\circ} \subseteq T + J(M^{\circ})$. Let $x_1 \in M^{\circ}$, but φ is an epimorphism, then there exists $m_1 \in M$ such that $\varphi(m_1) = x_1$. Thus $r^n m_1 \in \varphi^{-1}(T) + J(M)$, it follows that $\varphi(r^n m_1) = r^n \varphi(m_1) = r^n x_1 \in \varphi\big(\varphi^{-1}(T)\big) + \varphi\big(J(M)\big) = T + J(M^{\circ})$ by lemma 2.20. That is $r \in \sqrt{(T + J(M^{\circ}):_R M^{\circ})}$. Hence T is WNprimary submodule of M°.

Proposition 2.22 Let $\varphi: M \xrightarrow{\text{epimorphism}} M^{\circ}$, and $\ker\varphi$ is a small submodule of M, T is a WNprimary submodule of M°, then $\varphi^{-1}(T)$ is a WNprimary submodule of M.

Proof Let $0 \neq rm \in \varphi^{-1}(T)$, where $r \in R$, $m \in M$ with $m \notin \varphi^{-1}(T) + J(M)$, then $\varphi(m) \notin \varphi\big(\varphi^{-1}(T)\big) + \varphi\big(J(M)\big) = T + J(M^{\circ})$ by lemma 2.20. Now, since $0 \neq rm \in \varphi^{-1}(T)$, so $r\varphi(m) \in T$. But T is a WNprimary submodule of M°, it follows that $r^n M^{\circ} \subseteq T + J(M^{\circ})$ for some positive-integer n. To show that $r^n \varphi^{-1}(M^{\circ}) \subseteq \varphi^{-1}(T) + J(M)$, if $m \in \varphi^{-1}(M^{\circ})$, then $\varphi(m) \in M^{\circ}$. Thus $r^n \varphi(m) \in \varphi(r^n m) \in T + J(M^{\circ})$, implies that $r^n m \in \varphi^{-1}(T) + \varphi\Big(J(M^{\circ})\Big) = \varphi^{-1}(T) + J(M)$, that is $r^n \varphi^{-1}(M^{\circ}) \subseteq \varphi^{-1}(T) + J(M)$, so $\varphi^{-1}(T)$ is a WNprimary submodule of M.

Proposition 2.23 Let M be an R-module, and $T \hookrightarrow M$ with $J(M) \subseteq T$. Then T is a WNprimary if and only if $\frac{M}{T} \neq 0$ and for every zero divisor s of $\frac{M}{T}$ there exists $m \in M$ and $m \notin T$ such that $s \in (0:_R m) \cup \sqrt{(0:_R \frac{M}{T})}$.

Proof $\mapsto$ Let s be a zero divisor of an R-module $\frac{M}{T}$, then there exists a non-zero element $m + T \in \frac{M}{T}$ such that $s(m + T) = T$, that is $sm + T = T$, it follows that $sm \in T$ and $m \notin T = T + J(M)$. If $sm = 0$, then $s \in (0:_R m)$, implies that $s \in (0:_R m) \cup \sqrt{(0:_R \frac{M}{T})}$. If $0 \neq sm \in T$ and T is WNprimary submodule of M, then $s^n M \subseteq T + J(M)$ for some positive-integer n. But $J(M) \subseteq T$, then $T + J(M) = T$, that is $s^n M \subseteq T$ implies that $s^n \frac{M}{T} \subseteq 0$, it follows, $s^n \in (0:_R \frac{M}{T})$, that $s \in \sqrt{(0:_R \frac{M}{T})}$. Hence $s \in (0:_R m) \cup \sqrt{(0:_R \frac{M}{T})}$.

$\mapsfrom$ Suppose that $\frac{M}{T}$ is a non-zero R-module it follows that $T \hookrightarrow M$. Let $0 \neq sm \in T$ with $m \notin T = T + J(M)$, implies that $s(m + T) = 0 = T$ for $0 \neq m + T \in \frac{M}{T}$. That is s is a zero divisor on $\frac{M}{T}$, so $s \in (0:_R m) \cup \sqrt{(0:_R \frac{M}{T})}$. But $0 \neq sm$, implies that $s \notin (0:_R m)$. Thus $s \in \sqrt{(0:_R \frac{M}{T})}$, it follows that $s^n \frac{M}{T} = 0$ for some positive-integer n. That is $s^n M \subseteq T \subseteq T + J(M)$, so $s \in \sqrt{(T + J(M):_R M)}$. Hence T is a WNprimary submodule of M.

Proposition 2.24 Let M be an R-module, and $T \hookrightarrow M$ such that $J(M) \subseteq T$. If T is WNprimary submodule, then $T[x]$ (the set of all polynomial whose coefficients in T) is WNprimary submodule of $M[x]$.

Proof Let $\varphi: M[x] \to \frac{M}{T}[x]$ defined by $\varphi(m_1x + m_2x^2 + \cdots + m_nx^n) = (m_1 + T)x + (m_2 + T)x^2 + \cdots + (m_n + T)x^n$ is an R-epimorphism, where $m_1, m_2, \ldots, m_n \in M$, and $m_1 + T, m_2 + T, \ldots, m_n + T \in \frac{M}{T}$. The kernel of φ is obtain by reducing coefficients in$T$, it follows that $M[x]T[x] \cong \frac{M}{T}[x]$. Since $\frac{M}{T} \neq (0) = T$, implies that $M[x]T[x] \neq (0) = T[x]$. Now, let s be a zero divisor of$\frac{M}{T}[x]$, so there exists a polynomial $(m_1 + T)x + (m_2 + T)x^2 + \cdots + (m_n + T)x^n$ in $\frac{M}{T}[x]$ such that $s((m_1 + T)x + (m_2 + T)x^2 + \cdots + (m_n + T)x^n) = (0)$, thus there exists $1 \leq j \leq n$ with $s(m_j + T) = (0) = T$ and $m_j + T \neq (0) = T$, it follows that $sm_j \in T$ and $m_j \notin T + J(M)$. If $sm_j = 0$ then $s \in (0:_R m_j)$. If $0 \neq sm_j \in T$ with $m_j \notin T + J(M)$ and T is a WNprimary submodule of M, implies that $s \in \sqrt{(T + J(M):_R M)}$, that is $s^nM \subseteq T + J(M) = T$, thus $s^nM \subseteq T$, so $s^n\frac{M}{T} = (0)$, therefore $s \in \sqrt{(0:_R \frac{M}{T})}$. Hence $s \in (0:_R m_j) \cup \sqrt{(0:_R \frac{M}{T})}$. But $\frac{M}{T} \subseteq \frac{M}{T}[x]$ so, $s \in (0:_R m_j) \cup \sqrt{(0:_R \frac{M}{T}[x])}$. Thus by proposition 2.23 $T[x]$ is a WNprimary submodule of $M[x]$.

Proposition 2.25 Let M be an R-module, and $K, L \hookrightarrow M$ such that K containing L and $J(M) \subseteq J(K)$. If L is a WNprimary submodule of M, then L is a WNprimary submodule of K.

Proof Let $0 \neq rm \in L$ for $r \in R, m \in K, L$ it follows that $m \in M$. Since L is a WNprimary submodule of M, then $m \in L + J(M)$ or $r \in \sqrt{(L + J(M):_R M)}$. But $J(M) \subseteq J(K)$. Thus $m \in L + J(K)$ or $r \in \sqrt{(L + J(K):_R M)} \subseteq \sqrt{(L + J(K):_R K)}$. Hence L is a WNprimary submodule of K.

Proposition 2.26 Let L and K be WNprimary submodules of an R-module M with $K \nsubseteq L$ and $J(M) \subseteq L$ or $J(M) \subseteq K$. Then $L \cap K$ is a WNprimary submodule of M.

Proof Clearly that $L \cap K$ is a proper submodule of M. Let $0 \neq rm \in L \cap K$, where $r \in R, m \in M$, with $r^n \notin ((L \cap K) + J(M):_R M)$ for some positive-integer n, that is $r^nM \nsubseteq (L \cap K) + J(M)$, implies that $r^nM \nsubseteq L + J(M)$ and $r^nM \nsubseteq K + J(M)$. So, $0 \neq rm \in L$ and $0 \neq rm \in K$. Since both L and K be WNprimary submodules of M, it follows that $m \in L + J(M)$ and $m \in K + J(M)$, it follows that $m \in (L + J(M)) \cap (K + J(M))$. Let $J(M) \subseteq L$, then $L + J(M) = L$. Thus $m \in L \cap (K + J(M))$, it follows by modular law $m \in (L \cap K) + J(M)$. That is $L \cap K$ is a WNprimary submodule of M.

Proposition 2.27 Let M be a secondary R-module, and K is a non-zero WNprimary submodule of M with $J(M) \subseteq K$. Then K is a secondary.

Proof Let $r \in R$ and if $r^nM = (0)$ for some positive-integer n, then $r^nK \subseteq r^nM = (0)$, implies that $r^nK = (0)$. Suppose that $rM = M$ we show that $rK = K$. It is clear that $rK \subseteq K$. Now, let $0 \neq a \in K$, it follows that $a = rm$ for some $m \in M$. That is $0 \neq rm \in K$ and $r^nM \nsubseteq K + J(M) = K$, that is $r^nM \nsubseteq K$, it follows that $m \in K + J(M) = K$, it follows that $m \in K$. So, $a = rm \in rK$. Thus $rK = K$. Hence K is a secondary.

We prove in the following proposition that, under certain conditions, the concepts of weakly prime and WNprimary submodules are equivalent.

Proposition 2.28 Let M be an R-module and $T \hookrightarrow M$ with $J(M) \subseteq T$. Then T is weakly prime if and only if T is WNprimary submodule of M.

Proof $\mapsto$ By remarks and examples 2.2 (1).

$\leftarrow$ Suppose that T is WNprimary submodule of an R-module M. Let $0 \neq rm \in T$ for $r \in R, m \in M$, and $r \notin (T:_R M) \subseteq \sqrt{(T:_R M)} \subseteq \sqrt{(T + J(M):_R M)}$. Since T is WNprimary submodule of$M$, and $r \notin \sqrt{(T + J(M):_R M)}$ then $m \in T + J(M)$. But $J(M) \subseteq T$, thus $m \in T$. Hence T is weakly prime submodule of$M$.

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