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# *Pure Essential-Coessential and Pure Essential-Coclosed Submodules*

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ABSTRACT

During this work, we introduce concepts named pure-essential coessential sub-modules and pure- essential coclosed submodules. Various properties related with these concepts are considered. Let $K$, $N$ be sub-modules of an $T$-module $D$ in which case $K \leq N \leq D$, subsequently $K$ is called pure essential-coessential sub-module of $D$, if $\frac{N}{K} \ll_{pre} \frac{D}{K}$, and $K$ is said to be pure essential-coclosed sub-module, if $\frac{N}{K} \ll_{pre} \frac{D}{K}$ implies that $K = N$.

MSC.

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## 1. Introduction

$T$ is a commutative ring with identity, and $D$ is an $T$-module (shortly, $T$-mod). A proper sub-module (shortly, sub-mod) $N$ of $D$ namely small ($N \ll M$), if for any sub-mod $K$ of $D$ such that $N + K = D$ indicates $K = D$ [1]. A sub-mod $N$ of $D$ is said to be an essential (or $D$ namely an essential extension of a sub-mod $N$) ($N \leq_e D$), if $N \cap W \neq \{0\}$, for each nonzero sub-mod $W$ of $D$ [2-3]. The sub-mod $N$ of a an $T$- mod $D$ to be pure if $ID \cap N = IN$ for each ideal $I$ of $T$ [4]. Ibrahim and Al-Mothafar in [5] defined a generalization of essential sub-mod, which is called pure- essential ( $Pr$-essential), as follows, a sub-mod $N$ is said to be $Pr$-essential in $D$,(denoted by $N \trianglelefteq_{pr} D$), if $N \cap W = \{0\}$ , indicates $W$ is a pure sub-mod of $D$. For $K \leq N \leq D$ , $K$ is called coessential sub-mod of $N$ in $D$ ( $K \leq_{ce} N$) if $\frac{N}{K} \ll \frac{D}{K}$, and $K$ is called coclosed in $D$ denoted by($K \leq_{cc} D$ ), if $K$ has no proper coessential sub-mod of $D$. Equivalently, if $\frac{K}{L} \ll \frac{D}{L}$ for any sub-mod $L$ of $D$ implies that $D = L$, see [2] and [6-8]. Many authors present generalization of a small sub-mod such as [9-12]. In [13] the generalization of a small sub-mod known as pure essential small sub-mod, briefly ($Pre$-small) was introduced by Ibrahim and Almothafar, where a sub-mod $N$ of an $T$-mod $D$ is called $Pre$-small, and denoted by ($N \ll_{pre} D$ ), if $D = N + L$, for any $Pr$- essential sub-mod $L$ of $D$, indicates$L = D$. Equivalent, if $N + L \neq D$, for any proper $Pr$- essential sub-mod $L$ of $D$. In section 2 of this work we study first a new concept, named pure-essential coessential sub-mod ($Pre$- coessential), which is a generalization of the coessential sub-mod and it is more powerful than the notion of pure-essential sub-mod, such as a sub-mod $K$ namely pure- essential coessential sub-mod of $N$ in $D$, briefly($Pre$-coessential sub-mod), denoted by ($K \trianglelefteq_{pr.co} N$ ), if $\frac{N}{K} \ll_{pre} \frac{D}{K}$. In section 3

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we study another term which is called pure essential coclosed sub-mod ( $Pre$- coclosed), as a generalization of coclosed sub-mod, if $K$ is $Pre$-coessential of $N$ in $D$, (i.e. if $\frac{N}{K} \ll_{pre} \frac{D}{K}$), implies $N = K$. We give some results analogue to the known results on coclosed submodules.

**Main Results:**

**Lemma 1.1 [13]:** For any $T$-mod $D$:

1. Every small submodule is $Pre$-small. But the converse is not true in general.
2. Let $f: D \longrightarrow M$ be an epimorphisim, where $D, M$ are modules, if $N \ll_{pre} D$ then $f(N) \ll_{pre} M$.
3. If $N_1 \oplus N_2 \ll_{pre} H_1 \oplus H_2$, then $N_1 \ll_{pre} H_1$, also $N_2 \ll_{pre} H_2$, where $D = H_1 \oplus H_2$ and $N_1 \leq H_1$ , $N_2 \leq H_2$,
4. Suppose that $N, H$ are sub-mod of $T$-mod $D$, with $N \leq H \leq D$. If $H \ll_{pre} D$, then $N \ll_{pre} D$.
5. Let $N, K$ and $H$ be submodules of an $T$-mod $D$, with $N \leq K \leq H \leq D$. If $\frac{H}{N} \ll_{pre} \frac{D}{N}$, then $\frac{H}{K} \ll_{pre} \frac{D}{K}$.
5. Suppose that $N, K$ are sub-mod of $T$-mod $D$, with $N \leq K \leq D$. If $N \ll_{pre} K$, then $N \ll_{pre} D$.
6. Suppose that $D$ be an $T$-mod, also let $N \leq D$. If $D$ be a $Pre$-holow mod, hence $\frac{D}{N}$ is $Pre$-holow mod.

**Lemma 1.2 [5]:**

For any $T$-mod $D$ and $N, K$ are sub-mod of $D$, with $K \leq N$ . If $\frac{N}{K} \trianglelefteq_{pr} \frac{D}{K}$ and $K \leq_p D$, then $N \trianglelefteq_{pr} D$ .

**2. Pure Essential Coessential Submodules.**

In this part we introduce a new type of submodule called pure essential coessential submodule, we start with a few characteristics of this category of submodule.

**Definition 2.1**:

Let $D$ be an $T$-mod, and let $N, K$ are submodules of $D$, in which case $K \leq N \leq D$. Then $K$ is called pure essential coessential submodule of $N$ in $D$, briefly($Pre$-coessential submodule), denoted by ($K \trianglelefteq_{pr.co} N$ ), if $\frac{N}{K} \ll_{pre} \frac{D}{K}$.

**Remarks and Examples 2.2**

1. Clearly that each coesential sub-mod is $Pre$- coesential. Since every small submodule is $Pre$-small submodule, see Lemma (1.1).

2. In general, the opposite of (1) may not be correct. Consider in $Z_{24}$ *as* $Z$ – module: $\frac{4Z_{24}}{(\bar{0})} \ll_{pre} \frac{Z_{24}}{(\bar{0})}$, hence $(\bar{0}) \trianglelefteq_{pr.co} 4Z_{24}$. But $4Z_{24}$ is not small sub-mod of $Z_{24}$ , since $4Z_{24} + 3Z_{24} = Z_{24}$, while $3Z_{24} \neq Z_{24}$, implies $(\bar{0})$ is not coessential submodules in $Z_{24}$.

3. Let $D$ be an $T$-mod , and let $N$ be a sub-mod of $D$. Then $N \ll_{pre} D$ if and only if $(\bar{0}) \trianglelefteq_{pr.co} N$ in $D$.

4. In $Z_6$ as a $Z$-module, $(\bar{0})$ it cannot be $Pre$- coessential of $(\bar{3})$ from $Z_6$. Since $\frac{(\bar{3})}{(\bar{0})} \simeq (\bar{3})$ is not $Pre$-small in $\frac{Z_6}{(\bar{0})} \simeq Z_6$.

**Proposition 2.3:**

Let $N, K$ and $H$ are sub-modules of an $T$-mod $D$, with $N \leq K \leq H \leq D$. Then $K \trianglelefteq_{pr.co} H$ in $D$ if and only if $\frac{K}{N} \trianglelefteq_{pr.co} \frac{H}{N}$ in $\frac{D}{N}$.

**Proof:**

$\Longrightarrow$) Since $K \trianglelefteq_{pr.co} H$, implies $\frac{H}{K} \ll_{pre} \frac{D}{K}$. Then by second isomorphism theorem: $\frac{D}{K} \simeq \frac{D/N}{K/N}$ and $\frac{H}{K} \simeq \frac{H/N}{K/N}$, hence $\frac{H/N}{K/N} \ll_{pre} \frac{D/N}{K/N}$. Thus $\frac{K}{N} \trianglelefteq_{pr.co} \frac{H}{N}$ in $\frac{D}{N}$.

$\Longleftarrow$) Suppose that $\frac{K}{N} \trianglelefteq_{pr.co} \frac{H}{N}$ in $\frac{D}{N}$, then by second isomorphism theorem: $\frac{D}{K} \simeq \frac{D/N}{K/N}$ and $\frac{H}{K} \simeq \frac{H/N}{K/N}$, hence $\frac{H/N}{K/N} \ll_{pre} \frac{D/N}{K/N}$. Thus $\frac{H}{K} \ll_{pre} \frac{D}{K}$, so $K \trianglelefteq_{pr.co} H$ in $D$.

**Corollary 2.4**:

Let $A, B$ and $C$ be submodules of an $T$-mod $D$, with $A \leq B \leq C \leq D$. If $B \trianglelefteq_{pr.co} C$, then $\frac{B}{A\cap B} \trianglelefteq_{pr.co} \frac{C}{A\cap B}$.

**Proof:**

Since $A \cap B \leq B \leq C$ in $D$ and $B \trianglelefteq_{pr.co} C$, then by Proposition (2.3) $\frac{B}{A\cap B} \trianglelefteq_{pr.co} \frac{C}{A\cap B}$.

**Corollary 2.5:**

Let $A, B$ and $C$ be submodules of an $T$-mod $D$, with $A \leq B \leq C \leq D$. If $A + B \trianglelefteq_{pr.co} C$, then $\frac{A+B}{A} \trianglelefteq_{pr.co} \frac{C}{A}$.

**Proof:**

Since $A \leq A + B \leq C$ and $A + B \trianglelefteq_{pr.co} C$, then by Proposition (2.3) $\frac{A+B}{A} \trianglelefteq_{pr.co} \frac{C}{A}$.

**Proposition 2.6:**

Every non-zero epimorphic image of $Pre$- coessential is $Pre$- coessential.

**Proof:**

Suppose that $f : D \rightarrow M$ is an epimorpism, and let $A, B$ be submodules of $D$, such that $A \leq B \leq D$. If $A \trianglelefteq_{pr.co} B$ in $D$, implies that $\frac{B}{A} \ll_{pre} \frac{D}{A}$, but $f$ is an epimorphism, then by Lemma (1.1, 2) $f(\frac{B}{A}) \ll_{pre} f(\frac{D}{A})$, hence $\frac{f(B)}{f(A)} \ll_{pre} \frac{f(D)}{f(A)}$, thus $\frac{f(B)}{f(A)} \ll_{pre} \frac{M}{f(A)}$. Therefore $f(A) \trianglelefteq_{pr.co} f(B)$ in $f(D) = M$.

Now, we will present some properties of $Pre$- coessential submodules.

**Proposition 2.7:**

Let $A$, $B$, $N$ and $K$ are sub-modules of an $T$- mod $D$. If $A \trianglelefteq_{pr.co} N$ and $B \trianglelefteq_{pr.co} K$ in $D$, then $A + B \trianglelefteq_{pr.co} N + K$.

**Proof:**

Since $A \trianglelefteq_{pr.co} N$ and $B \trianglelefteq_{pr.co} K$ in $D$, so $\frac{N}{A} \ll_{pre} \frac{D}{A}$ and $\frac{K}{B} \ll_{pre} \frac{D}{B}$. Now, let $f$: $\frac{D}{A} \rightarrow \frac{D}{A+B}$ be a map defined by $f(x + A) = x + (A + B)$, for each $x \in D$, and $g$: $\frac{D}{B} \rightarrow \frac{D}{A+B}$ be a map defined by $g(x + B) = x + (A + B)$, for each $x \in D$, clearly each $f$ and $g$ are epimorphisms, then by Lemma (1.1)$f\left(\frac{N}{A}\right) = \frac{N+B}{A+B} \ll_{pre} \frac{D}{A+B}$ and $g(\frac{K}{B}) = \frac{A+K}{A+B} \ll_{pre} \frac{D}{A+B}$, hence by Lemma (1.1) $\frac{N+B}{A+B} + \frac{A+K}{A+B} \ll_{pre} \frac{D}{A+B}$, thus $\frac{N+K}{A+B} \ll_{pre} \frac{D}{A+B}$. Therefore $A + B \trianglelefteq_{pr.co} N + K$.

**Corollary 2.8:**

Let $A$, $B$ and $C$ are submodules of $D$ , if $B \trianglelefteq_{pr.co} A$ then $B + C \trianglelefteq_{pr.co} A + C$ in $D$ .

**Proof:**
Since $C \trianglelefteq_{pr.co} C$, then by Proposition (2.7) $B + C \trianglelefteq_{pr.co} A + C$ in $D$ .

**Proposition 2.9:**

Let $N, L, K$ be sub-modules of $D$, such that $L \leq N \leq D$, with $L$ is pure in $D$. If $N = L + K$ and $K \ll_{pre} D$, then $L \trianglelefteq_{pr.co} N$ .

**Proof:**
To prove $\frac{N}{L} \ll_{pre} \frac{D}{L}$. Let $\frac{U}{L} \trianglelefteq_{pr} \frac{D}{L}$ such that $\frac{D}{L} = \frac{U}{L} + \frac{N}{L}$, hence $D = U + N = U + L + K$, then $D = U + K$. Since $\frac{U}{L} \trianglelefteq_{pr} \frac{D}{L}$ and $L \leq_{Pr} D$, then by lemma (1.2) $U \trianglelefteq_{pr} D$. But $K \ll_{pre} D$, implies that $D = U$. Thus $L \trianglelefteq_{pr.co} N$ .

In the following propositions results:

**Proposition 2.10:**

Let $A, B$ and $C$ are sub-modules of an $T$-mod $D$, such that $A \leq B \leq C \leq D$. If $A \trianglelefteq_{pr.co} C$ in $D$, then $A \trianglelefteq_{pr.co} B$ and $B \trianglelefteq_{pr.co} C$ in $D$.

**Proof:**
Suppose that $A \trianglelefteq_{pr.co} C$, then $\frac{C}{A} \ll_{pre} \frac{D}{A}$, since $\frac{B}{A} \leq \frac{C}{A} \leq \frac{D}{A}$ then by Lemma (1.1) $\frac{B}{A} \ll_{pre} \frac{D}{A}$, hence $A \trianglelefteq_{pr.co} B$. Since $\frac{C}{A} \ll_{pre} \frac{D}{A}$, then by Lemma (1.1) $\frac{C}{B} \ll_{pre} \frac{D}{B}$, thus $B \trianglelefteq_{pr.co} C$ in $D$.

**Proposition 2.11:**

Let $N, L, K$ and $A$ be submodules of $D$ , then the following are equivalent:

1. If $N \trianglelefteq_{pr.co} N + L$ in $D$, then $N \cap L \trianglelefteq_{pr.co} L$ in $D$.

2. If $N \trianglelefteq_{pr.co} L$ in $D$, then $N \cap A \trianglelefteq_{pr.co} L \cap A$ in $D$.

3. If $N \trianglelefteq_{pr.co} L$ and $A \trianglelefteq_{pr.co} K$, then $N \cap A \trianglelefteq_{pr.co} L \cap K$ in $D$.

**Proof:**
($1 \Longrightarrow 2$) Let $N \trianglelefteq_{pr.co} L$ in $D$. Since $N \leq L$ and $L \cap A \leq L$, then $N + (L \cap A) \leq L$. Since $N \leq N + (L \cap A)$ in $D$, then by Proposition (2.10) $N \trianglelefteq_{pr.co} N + (L \cap A)$. By (1) $N \cap (L \cap A) \trianglelefteq_{pr.co} (L \cap A)$, but $N \cap L \leq N$, hence $N \cap A \trianglelefteq_{pr.co} L \cap A$.

($2 \Longrightarrow 3$) Suppose that $N \trianglelefteq_{pr.co} L$ and $A \trianglelefteq_{pr.co} K$. Since $A \leq D$, then by (2) $N \cap A \trianglelefteq_{pr.co} L \cap A$. Also $A \trianglelefteq_{pr.co} K$ and $L \leq D$, then by (2) $L \cap A \trianglelefteq_{pr.co} L \cap K$. Then by Proposition (2.10) $N \cap A \trianglelefteq_{pr.co} L \cap K$ in $D$.

($3 \Longrightarrow 1$) Suppose that $N \trianglelefteq_{pr.co} N + L$. Since $L \trianglelefteq_{pr.co} L$ in $D$, then by (3) $N \cap L \trianglelefteq_{pr.co} (N + L) \cap L$. Thus $N \cap L \trianglelefteq_{pr.co} L$.

## 3. Pure Essential Coclosed submodules:

Within this part we present a new idea, pure essential coclosed submodules, some properties related to this concept will be discussed. First we have the following definition:

**Definition 3.1:**

A submodule $N$ of an $T$-mod $D$ is called pure essential coclosed submodule ($Pre$-coclosed), if $\frac{N}{K} \ll_{pre} \frac{D}{K}$, implies that $N = K$, for each $K$ contained in $N$ and denoted by ( $N \trianglelefteq_{pr.cc} D$).

Equivalently, a submodule$N$ of an $T$-mod $D$ is called pure essential coclosed submodule of $D$, if $N$ has no proper $Pre$-coessential submodule in $D$.

**Remarks and Examples 3.2:**

1. Clearly that each $Pre$-coclosed sub-mod is coclosed.

Proof: Let $\frac{N}{K} \ll \frac{D}{K}$, then by lemma (1.1) $\frac{N}{K} \ll_{pre} \frac{D}{K}$. But $N$ is $Pre$-coclosed, hence $N = K$. Thus $N$ is coclosed.

2. The converse of (1) in general may not be correct, for example. In $Z_6$ as $Z$-module, since $(\bar{2}) \simeq \frac{(\bar{2})}{(\bar{0})} \ll_{pre} \frac{(\bar{6})}{(\bar{0})} \simeq (\bar{6})$, hence $(\bar{2})$ is coclosed of $Z_6$, but not $Pre$-coclosed in $Z_6$. Since $(\bar{0})$ is $Pre$-coessential sub-mod in $(\bar{2})$ of $Z_6$.

3. Every simple module is not $Pre$-coclosed.

4. $(\bar{2})$ in $Z_4$ as $Z$-module is not $Pre$-coclosed. Since $(\bar{2}) \simeq \frac{(\bar{2})}{(\bar{0})}$ and $\frac{Z_4}{(\bar{0})} \simeq Z_4$, by Lemma (1.1) $(\bar{2}) \ll_{pre} Z_4$ and $(\bar{2}) \neq (\bar{0})$ .

5. Let $D = Z \oplus Z_4$ as $Z$-module. $(\bar{0})$ be a proper sub-mod of $Z_4$ in $D$, but $Z_4 \simeq \frac{Z_4}{(\bar{0})}$, $D \simeq \frac{D}{(\bar{0})}$ so $Z_4$ is not $Pre$-small of $M$ and $Z_4 \neq (\bar{0})$. Therefore $Z_4$ is $Pre$-coclosed of $D$.

**Proposition 3.3:**

Let $N$ be a non- zero sub-mod of an $T$-mod, if $N \trianglelefteq_{pr.cc} D$, then $N$ is not $Pre$-small in $D$.

**Proof:**

Assume that $0 \neq N \ll_{pre} D$, and $N \trianglelefteq_{pr.cc} D$. Since $(\bar{0}) \leq N$ and $N \simeq \frac{N}{(\bar{0})} \ll_{pre} \frac{M}{(\bar{0})} \simeq D$, but $N \trianglelefteq_{pr.cc} D$, implies $N = (\bar{0})$ contradiction. Therefore $N$ is not $Pre$-small in $D$.

The following propositions gives basic properties for $Pre$-coclosed submodules:

**Proposition 3.4:**

Let $A$,$B$ are submodules of an $T$-mod $D$, such that $A \leq B \leq D$. If $B \trianglelefteq_{pr.cc} D$, then $\frac{B}{A} \trianglelefteq_{pr.cc} \frac{D}{A}$.

**Proof:**

Suppose that $B$ is $Pre$-coclosed in $D$, and let $\frac{L}{A} \leq \frac{D}{A}$ such that $\frac{B/A}{L/A} \ll_{pre} \frac{D/A}{L/A}$, then by third isomorphism theorem [1] $\frac{B/A}{L/A} \simeq \frac{B}{L}$ and $\frac{D/A}{L/A} \simeq \frac{D}{L}$, so $\frac{B}{L} \ll_{pre} \frac{D}{L}$. Since $B \trianglelefteq_{pr.cc} D$, implies $B = L$ and $\frac{L}{A} = \frac{B}{A}$. Therefore $\frac{B}{A} \trianglelefteq_{pr.cc} \frac{D}{A}$.

**Proposition 3.5:**

Let $D$ be an $T$-mod, and $A$, $B$ be submodules of $D$ such that $A \leq B$. If $A \trianglelefteq_{pr.cc} D$ , then $A \trianglelefteq_{pr.cc} B$.

**Proof:**

Let $L < A$ such that $\frac{A}{L} \ll_{pre} \frac{B}{L} \leq \frac{D}{L}$, hence by Lemma (1.1) $\frac{A}{L} \ll_{pre} \frac{D}{L}$. Since $A \trianglelefteq_{pr.cc} D$, hence $A = L$. Therefore $A \trianglelefteq_{pr.cc} B$.

**Proposition 3.6:**

Let $D$ be an $T$-mod, and let $A \leq B \leq D$. If $A \ll B$ and $\frac{B}{A} \trianglelefteq_{pr.cc} \frac{D}{A}$, then $B \trianglelefteq_{pr.cc} D$.

**Proof:**

Assume that $L \leq B$ such that $\frac{B}{L} \ll_{pre} \frac{D}{L}$, so by third isomorphism theorem [1] $\frac{B/A}{L+A/A} \simeq \frac{B}{L+A}$ and $\frac{D/A}{L+A/A} \simeq \frac{D}{L+A}$. Let $\pi: D \rightarrow \frac{D}{A}$ be the natural epimorphism, define as $\pi(L)= L + A$. Hence by Proposition (2.6) $\frac{L+A}{A} \trianglelefteq_{pr.cc} \frac{B}{A}$, then $\frac{B/A}{L+A/A} \ll_{pre} \frac{D/A}{L+A/A}$. Since $\frac{B}{A} \trianglelefteq_{pr.cc} \frac{D}{A}$, so $\frac{L+A}{A} = \frac{B}{A}$, then $B = L + A$. But $A \ll B$, implies $B = L$. Therefore $B \trianglelefteq_{pr.cc} D$.

**Corollary 3.7:**

Let $A$ is $Pre$-essential sub-mod of an $T$-mod $D$ and $A \leq B \leq D$. If $A \ll_{Pre} B$ and $\frac{B}{A} \trianglelefteq_{pr.cc} \frac{D}{A}$, then $B \trianglelefteq_{pr.cc} D$.

**Proof:**
By the same argument of proof proposition (3.6) until, then $B = L + A$. But $A \ll_{Pre} B$ implies $B = L$. Therefore $B \trianglelefteq_{pr.cc} D$.

Recall that a non-zero $T$-mod $D$ is called pure-uniform, briefly ($Pr$-uniform), if every non-zero sub-mod of $D$ is $Pr$-essential. [5]

**Corollary 3.8:**

Let $D$ be a $Pr$- uniform $R$-mod, and $N \leq K \leq D$. If $N \ll_{pre} K$ and $\frac{K}{N} \trianglelefteq_{pr.cc} \frac{D}{N}$, then $K \trianglelefteq_{pr.cc} D$.

**Proof:**
Clear by Lemma (1.1) and by Proposition (3.6).

In the following results we have a $Pre$-coclosed submodule of direct summand is also $Pre$-coclosed:

**Proposition 3.9:**

Let $D = D_1 \oplus D_2$ be $T$-mod, and $N \trianglelefteq_{pr.cc} D_1$, then $N \trianglelefteq_{pr.cc} D$.

**Proof:**
Let $L < N$, such that $\frac{N}{L} \ll_{pre} \frac{D}{L} = \frac{D_1 \oplus D_2}{L}$. Hence $\frac{N}{L} \ll_{pre} \frac{D_1}{L} \oplus \frac{L+D_2}{L}$, then by lemma (1.1) $\frac{N}{L} \ll_{pre} \frac{D_1}{L}$. Since $N \trianglelefteq_{pr.cc} D_1$, implies $N = L$. Thus $N \trianglelefteq_{pr.cc} D$.

**Proposition 3.10:**

Let $D$ be $T$-mod, and let $A \trianglelefteq_{pr.cc} D$. If $B \ll_{pre} D$, then $B \ll_{pre} A$, for every $B \leq A$ and for some $A \leq D$.

**Proof:**
Let $L \trianglelefteq_{pr} A$ such that $A = L + B$ to prove that $A = L$. Since $A \trianglelefteq_{pr.cc} D$, then by Proposition (3.4) $\frac{A}{L} \trianglelefteq_{pr.cc} \frac{D}{L}$, hence $\frac{A/B}{L/B} \ll_{pre} \frac{D/B}{L/B}$, where $B \leq L$, then $\frac{A}{B} = \frac{L}{B}$. Thus $A = L$, therefore $B \ll_{pre} A$.

Recall that, a non-zero $T$-mod $D$ namely pure essential hollow ($Pre$-holow), if each proper submodule of $D$ is $Pre$-small submodule of $D$.[13]

**Proposition 3.11:**

Every $Pre$-coclosed sub-mod of $Pre$-hollow module is also $Pre$-coclosed.

**Proof:**
Assume that $D$ is $Pre$-hollow module, and $A$ be $Pre$-coclosed in $D$. Let $L$ be a proper sub-mod of $A$ and $H$ be a $Pre$-essential sub-mod of $A$ such that $A = L + H$. Since $D$ is $Pre$-hollow by Lemma (1.1, 7) $\frac{D}{H}$ is $Pre$-hollow, then $\frac{A}{H}$ is a

$Pre$-small submodule of $\frac{D}{H}$. Since $A$ is $Pre$-coclosed, hence $A = D$ and $H$ is $Pre$-small submodule of $A$. Thus $A$ is $Pre$-hollow.

**Conclusion:**

In this work, pure-essential coessential and pure- essential coclosed submodules, which are generalization of coessential submodules and pure coclosed submodules respectively. We also show some of the following results:

- $K \trianglelefteq_{pr.co} H$ in $D$ if and only if $\frac{K}{N} \trianglelefteq_{pr.co} \frac{H}{N}$ in $\frac{D}{N}$, with $N \leq K \leq H \leq D$.
- Every non-zero epimorphic image of $Pre$- coessential is $Pre$- coessential.
- If $N = L + K$ and $K \ll_{pre} D$, then $L \trianglelefteq_{pr.co} N$, such that $L \leq N \leq D$, and $L$ is pure in $D$.
- If $A \trianglelefteq_{pr.co} C$ in $D$, then $A \trianglelefteq_{pr.co} B$ and $B \trianglelefteq_{pr.co} C$ in $D$, where $A \leq B \leq C \leq D$.
- If $N \trianglelefteq_{pr.cc} D$, then $N$ is not $Pre$-small in $D$.
- If $B \trianglelefteq_{pr.cc} D$, then $\frac{B}{A} \trianglelefteq_{pr.cc} \frac{D}{A}$.
- If $A \trianglelefteq_{pr.cc} D$ , then $A \trianglelefteq_{pr.cc} B$, such that $A \leq B \leq D$.

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