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Coefficient Estimates for a Subclass of Bi-Univalent Function Associated with Borel Distributions Using the Subordination Principle

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ABSTRACT

In the present paper, we obtain some new subclasses of bi-univalent functions associated with Borel distributions using the subordination principle. Also, we obtain the bounds for the modulus of initial coefficients of the function in these classes.

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1-Introduction

Consider A as the class of function defined by the equation

$$f(z) = z + \sum_{v=2}^{\infty} a_v z^v \ , \tag{1}$$

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Which functions are analytic in the disk U defined as $\{z \in \mathbb{C}: |z| < 1\}$ and satisfy the normalization condition $f'(0) - 1 = 0 = f(0)$. Furthermore, we denote by $\mathcal{T}$ the subclass that consists of functions satisfying equation (1) and are also univalent in U.

It is widely recognized that for any function $f \in \mathcal{T}$, there exists an inverse function f^{-1}, which is defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in U)$$

and

$$w = f\big(f^{-1}(w)\big) \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots, (w \in U) \quad (2)$$

If both f and f^{-1} are univalent in U, then a function $f \in A$ is classified as bi-univalent in U. We denote by Σ the class of all analytic bi-univalent functions in U, as defined by the extension of the Taylor-Maclaurin series (1). The examination of several categories of bi-univalent functions was revived in 2010 by Srivastava et al. [10]. The class Σ is not devoid of elements, to the best of our knowledge. For example, the functions

$$f_1(z) = \frac{z}{1-z}, \qquad f_2(z) = -\log(1-z), \qquad f_3(z) = \frac{1}{2}\log\frac{1+z}{1-z},$$

with their respective inverses

$$f_1^{-1}(w) = \frac{w}{1+w}, \qquad f_2^{-1}(w) = \frac{e^w - 1}{e^w}, \qquad f_3^{-1}(w) = \frac{e^{3w} - 1}{e^{3w} + 1}.$$

Orthogonal polynomials have garnered extensive study attention in recent years due to their importance in mathematical physics, probability theory, engineering, and statistical mathematics.

If f and g are two analytic functions in U, we denote that f is subordinate to g, or equivalently, that g is superordinate to f, using the following symbolic representation (See [7]):

$$f \prec g \qquad or \qquad f(z) \prec g(z), \qquad (z \in U),$$

If the there exists a Schwarz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$

$(z \in U)$, such that $f(z) = g\big(w(z)\big), (z \in U)$.

A discrete random variable x is said to have a Borel distribution if it takes the values $1, 2, 3, \cdots$ with the probabilities $\frac{e^{-\mu}}{1!}, \frac{2\mu e^{-2\mu}}{2!}, \frac{9\mu^2 e^{-3\mu}}{3!}, \cdots$ respectively, where μ is called the parameter.

$$P(x=\mathfrak{r})=\frac{(\mu\mathfrak{r})^{\mathfrak{r}-1}e^{-\mu\mathfrak{r}}}{\mathfrak{r}!},\qquad \mathfrak{r}=1,2,3,\cdots$$

We now present a power series whose coefficients correspond to the probability of the Borel distribution.

$$\mathcal{M}(\mu,z)=z+\sum_{v=2}^{\infty}\frac{\big(\mu(v-1)\big)^{v-2}e^{-\mu(v-1)}}{(v-1)!}z^{v},\qquad z\in U \tag{3}$$

where $0\le\mu\le 1$. We see that, by employing the ratio test, we ascertain that the radius of the aforementioned power series is infinite.

We now examine the linear operator $B_{\mu}(z): A\to \mathrm{A}$, characterised by the convolution or Hadamard product.

$$B_{\mu}(z)=\mathcal{M}(\mu,z)*f(z)=z+\sum_{v=2}^{\infty}\frac{(\mu(v-1))^{v-2}e^{-\mu(v-1)}}{(v-1)!}a_{v}z^{v},\qquad z\in U\,.$$

In 1983, Askey and Ismail [5] identified a family of polynomials that can be regarded as q-analogues of the Gaussian polynomial. They can be characterised as polynomials $\mathfrak{B}_{q}^{(\lambda)}(\ell,z)$.

$$\mathfrak{B}_{q}^{(\lambda)}(\ell,z)=\sum_{v=0}^{\infty}\mathrm{C}_{v}^{(\lambda)}(\ell;q)z^{v}. \tag{4}$$

The 2006 study by Chakrabarti et al. [6] identified a category of polynomials that can be regarded as q-analogues of the geometric progression (GP). The subsequent recurrence relations can be employed to comprehend this category of polynomials:

$$\begin{gathered}\mathrm{C}_{0}^{(\lambda)}(\ell;q)=1,\mathrm{C}_{1}^{(\lambda)}(\ell;q)=[\lambda]_{q}\mathrm{C}_{1}^{1}(\ell)=2[\lambda]_{q}\ell\,,\\ \mathrm{C}_{2}^{(\lambda)}(\ell;q)=[\lambda]_{q^{2}}\mathrm{C}_{2}^{1}(\ell)-\tfrac{1}{2}\big([\lambda]_{q^{2}}-[\lambda]_{q}^{2}\big)\mathrm{C}_{1}^{2}(\ell)=2\big([\lambda]_{q^{2}}+[\lambda]_{q}^{2}\big)\ell^{2}-[\lambda]_{q^{2}}\end{gathered} \tag{5}$$

where $0<q<1$, and $\lambda\in\mathbb{N}=\{1,2,3,\cdots\}$.

$\mathfrak{B}^{(\lambda)}(\ell,z)$, with $z\in U$ and $\ell\in[-1,1]$, represents the traditional GP examined by Amourah et al. [1, 2] in 2021. The function $\mathfrak{B}^{(\lambda)}$ is analytic in U given a constant x, permitting its expansion in a Taylor series as

$$\mathfrak{B}^{(\lambda)}(\ell,z)=\sum_{v=0}^{\infty}\mathrm{C}_{v}^{\alpha}(\ell)\,z^{v},$$

where $\mathrm{C}_{v}^{\alpha}(\ell)$ is the classical GP of degree v.

Amourah et al. [3] illustrated the use of q-GP over three separate subclasses of analytic and bi-univalent functions. Alsoboh et al. [4] utilized the q-GP in conjunction with a generalisation of the Neutrosophic Poisson distribution series to demonstrate the existence of a novel subclass of bi-univalent functions. It is possible to derive Fekete-Szegö inequalities for functions associated with these subclasses, together with the initial coefficient bounds $|a_2|$ and $|a_3|$.

However, there are only a few works determining the general coefficient bounds $|a_2|$ and $|a_3|$ for the analytic bi-univalent functions in the literature. ([8,911,12]).

Definition 1: A bi-univalent function f represented by equation (1), is classified as belonging to the class $J_\Sigma\left(\alpha, \mathfrak{B}_q^{(\lambda)}(\ell,z)\right)$, if it satisfies the following subordination conditions:

$$\left(\frac{zB'_\mu(z)}{B_\mu(z)}\right)^{1-\alpha}\left(1+\frac{zB''_\mu(z)}{B'_\mu(z)}\right)^{\alpha} \prec \mathfrak{B}_q^{(\lambda)}(\ell,z), \tag{6}$$

and

$$\left(\frac{wB'_\mu(w)}{B_\mu(w)}\right)^{1-\alpha}\left(1+\frac{wB''_\mu(w)}{B'_\mu(w)}\right)^{\alpha} \prec \mathfrak{B}_q^{(\lambda)}(\ell,w), \tag{7}$$

where $\lambda > 0, \ell \in \left(\frac{1}{2},1\right], (0 \le \alpha \le 1), z,w \in U$ and $g = f^{-1}$ and the function g is given by (2).

Remark 1: If we take $\alpha = 0$ in Definition (1), the class $J_\Sigma\left(\alpha, \mathfrak{B}_q^{(\lambda)}(\ell,z)\right)$ reduce to the class $S_\Sigma(x)$ which was studied recently by Wanas see [13].

Remark 2: If we take $\alpha = 1$ in Definition (1), the class $J_\Sigma\left(\alpha, \mathfrak{B}_q^{(\lambda)}(\ell,z)\right)$ reduce to the class $C_\Sigma(x)$ which was studied recently by Wanas see [13].

Theorem 1: Let $f(z)$ defined by (1) belong to the class $J_\Sigma\left(\alpha, \mathfrak{B}_q^{(\lambda)}(\ell,z)\right)$, then

$$|a_2| \le \frac{2\left|[\lambda]_q\right|\ell\sqrt{2\left|[\lambda]_q\right|\ell}}{\sqrt{\left|4[\lambda]_q^2\ell^2(\alpha^2-7\alpha-2+8\mu\alpha+4\mu)e^{-2\mu}-2(1+\alpha)^2e^{-2\mu}\ 2\left([\lambda]_{q^2}+[\lambda]_q^2\right)\ell^2-[\lambda]_{q^2}\right|}}$$

and

$$|a_3| \le \frac{2[\lambda]_q^2\ell^2}{(1+\alpha)^2e^{-2\mu}}+\frac{\left|[\lambda]_q\right|\ell}{2\mu(1+2\alpha)e^{-2\mu}}$$

Proof: Let $f(z) \in J_\Sigma\left(\alpha, \mathfrak{B}_q^{(\lambda)}(\ell,z)\right)$ and $g = f^{-1}$. Then, there are two analytic functions $\theta, \Phi: U \to U$ given by

$$\Phi(z) = c_1z + c_2z^2 + c_3z^3 + \cdots, \qquad (z \in U) \tag{8}$$

and

$$\theta(w) = d_1w + d_2w^2 + d_3w^3 + \cdots, \qquad (w \in U) \tag{9}$$

With $\Phi(0) = 0$ and $\theta(0) = 0, |\Phi(z)| < 1, |\theta(w)| < 1$, such that

$$\left(\frac{zB'_\mu(z)}{B_\mu(z)}\right)^{1-\alpha}\left(1+\frac{zB''_\mu(z)}{B'_\mu(z)}\right)^{\alpha}=\mathfrak{B}_q^{(\lambda)}(\ell,\Phi(z)), \tag{10}$$

and

$$\left(\frac{wB'_\mu(w)}{B_\mu(w)}\right)^{1-\alpha}\left(1+\frac{wB''_\mu(w)}{B'_\mu(w)}\right)^{\alpha}=\mathfrak{B}_q^{(\lambda)}(\ell,\theta(w)), \tag{11}$$

from the equation (9) and (10), we obtain that

$$\left(\frac{zB'_\mu(z)}{B_\mu(z)}\right)^{1-\alpha}\left(1+\frac{zB''_\mu(z)}{B'_\mu(z)}\right)^{\alpha}=1+C_1^{(\lambda)}(\ell;q)\,c_1z+\left[C_1^{(\lambda)}(\ell;q)c_2+C_2^{(\lambda)}(\ell;q)c_1^2\right]z^2+\cdots, \tag{12}$$

and

$$\left(\frac{wB'_\mu(w)}{B_\mu(w)}\right)^{1-\alpha}\left(1+\frac{wB''_\mu(w)}{B'_\mu(w)}\right)^{\alpha}=1+C_1^{(\lambda)}(\ell;q)\,d_1w+\left[C_1^{(\lambda)}(\ell;q)d_2+C_2^{(\lambda)}(\ell;q)d_1^2\right]w^2+\cdots, \tag{13}$$

it is quite well-known that if $|\Phi(z)|<1, |\theta(w)|<1, z,w\in U$ we get $|c_i|<1$ and $|d_i|<1 (\forall i\in\mathbb{N})$.

The equivalent coefficients in (11) and (12) are so compared, and the result is

$$(1+\alpha)e^{-\mu}a_2=C_1^{(\lambda)}(\ell;q)\,c_1, \tag{14}$$

$$2(1+2\alpha)\mu e^{-2\mu}a_3+\left(\frac{\alpha^2-7\alpha-2}{2}\right)e^{-2\mu}a_2^2=C_1^{(\lambda)}(\ell;q)\,c_2+C_2^{(\lambda)}(\ell;q)c_1^2, \tag{15}$$

$$-(1+\alpha)e^{-\mu}a_2=C_1^{(\lambda)}(\ell;q)\,d_1, \tag{16}$$

and

$$\left(\frac{\alpha^2+16\mu\alpha+8\mu-7\alpha-2}{2}\right)e^{-\mu}a_2^2-2(1+2\alpha)\mu e^{-2\mu}a_3=C_1^{(\lambda)}(\ell;q)\,d_2+C_2^{(\lambda)}(\ell;q)d_1^2, \tag{17}$$

It follows from (14) and (16) that

$$c_1=-d_1 \tag{18}$$

and

$$2\,(1+\alpha)^2e^{-2\mu}a_2^2=\left[C_1^{(\lambda)}(\ell;q)\right]^2(c_1^2+d_1^2). \tag{19}$$

If we add (15) and (17), we obtain

$$(\alpha^2-7\alpha-2+8\mu\alpha+4\mu)e^{-2\mu}a_2^2=C_1^{(\lambda)}(\ell;q)(c_2+d_2)+C_2^{(\lambda)}(\ell;q)(c_1^2+d_1^2). \tag{20}$$

By substituting the value of $(c_1^2 + d_1^2)$ from (18) in the right hand side of (19), we deduce that

$$\left(\frac{\left[C_1^{(\lambda)}(\ell;q)\right]^2(\alpha^2 - 7\alpha - 2 + 8\mu\alpha + 4\mu)e^{-2\mu} - C_2^{(\lambda)}(\ell;q)2\,(1+\alpha)^2e^{-2\mu}}{\left[C_1^{(\lambda)}(\ell;q)\right]^2}\right) \times a_2^2 = C_1^{(\lambda)}(\ell;q)(c_2 + d_2). \qquad (21)$$

Additionally, after performing certain computations using (5) we conclude that

$$|a_2| \le \frac{2\left|[\lambda]_q\right|\ell\sqrt{2\left|[\lambda]_q\right|\ell}}{\sqrt{\left|4[\lambda]_q^2\ell^2(\alpha^2 - 7\alpha - 2 + 8\mu\alpha + 4\mu)e^{-2\mu} - 2(1+\alpha)^2e^{-2\mu}\ 2\left([\lambda]_{q^2} + [\lambda]_q^2\right)\ell^2 - [\lambda]_{q^2}\right|}}$$

Now, find $|a_3|$, by subtracting (17) from (15), we get

$$4\mu(1 + 2\alpha)e^{-2\mu}(a_3 - a_2^2) = C_1^{(\lambda)}(\ell;q)(c_2 - d_2) + C_2^{(\lambda)}(\ell;q)(c_1^2 - d_1^2) \qquad (22)$$

Then, by using (18) and (19) in (22), we get

$$a_3 = \frac{\left[C_1^{(\lambda)}(\ell;q)\right]^2}{2\,(1+\alpha)^2e^{-2\mu}}(c_1^2 + d_1^2) + \frac{C_1^{(\lambda)}(\ell;q)}{4\mu(1+2\alpha)e^{-2\mu}}(c_2 - d_2) \qquad (23)$$

Using (5), we have

$$|a_3| \le \frac{2[\lambda]_q^2\ell^2}{(1+\alpha)^2e^{-2\mu}} + \frac{\left|[\lambda]_q\right|\ell}{2\mu(1+2\alpha)e^{-2\mu}}$$

By substituting $\alpha = 0$ into theorem (1), we obtain the following outcome:

Corollary 1: If $f \in A$ defined by (1) be in the class $J_\Sigma\left(\alpha, \mathfrak{B}_q^{(\lambda)}(\ell, z)\right)$.

$$|a_2| \le \frac{2\left|[\lambda]_q\right|\ell\sqrt{2\left|[\lambda]_q\right|\ell}}{\sqrt{\left|4[\lambda]_q^2\ell^2(4\mu - 2)e^{-2\mu} - 2e^{-2\mu}\,(\,2\left([\lambda]_{q^2} + [\lambda]_q^2\right)\ell^2 - [\lambda]_{q^2})\right|}}$$

and

$$|a_3| \le \frac{2[\lambda]_q^2\ell^2}{e^{-2\mu}} + \frac{\left|[\lambda]_q\right|\ell}{2\mu e^{-2\mu}}$$

By substituting $\alpha = 1$ into theorem (1), we obtain the following outcome:

Corollary 2: If $f \in A$ defined by (1) be in the class $J_\Sigma\left(1, \mathfrak{B}_q^{(\lambda)}(\ell, z)\right)$

$$|a_2| \leq \frac{2\left|[\lambda]_q\right|\ell\sqrt{2\left|[\lambda]_q\right|\ell}}{\sqrt{\left|4[\lambda]_q^2\ell^2(12\mu-8)e^{-2\mu}-8e^{-2\mu}\left(2\left([\lambda]_{q^2}+[\lambda]_q^2\right)\ell^2-[\lambda]_{q^2}\right)\right|}}$$

and

$$|a_3| \leq \frac{[\lambda]_q^2\ell^2}{2e^{-2\mu}}+\frac{\left|[\lambda]_q\right|\ell}{6\mu e^{-2\mu}}$$

In the next theorem, we present the Fekete-szegö inequality for $f \in J_\Sigma\left(\alpha, \mathfrak{B}_q^{(\lambda)}(\ell, z)\right)$. Then

Theorem 2: For $0 \leq \alpha \leq 1$ and $\tau \in \mathbb{R}$, let $f \in A$ be in the class $J_\Sigma\left(\alpha, \mathfrak{B}_q^{(\lambda)}(\ell, z)\right)$. Then

$$|a_3-\tau a_2^2| \leq \begin{cases} \dfrac{\left|[\lambda]_q\right|\ell}{\mu(1+2\alpha)e^{-2\mu}}; & \left(|1-\tau| \leq \dfrac{1}{4\mu(1+2\alpha)e^{-2\mu}} \times \left|(\alpha^2-7\alpha-2+8\mu\alpha+4\mu)e^{-2\mu}-\dfrac{(1+\alpha)^2e^{-2\mu}G}{2[\lambda]_q^2\ell^2}\right|\right). \\ \dfrac{16\left|[\lambda]_q^3\right|\ell^3(1-\tau)}{4[\lambda]_q^2\ell^2(\alpha^2-7\alpha-2+8\mu\alpha+4\mu)e^{-2\mu}-2(1+\alpha)^2e^{-2\mu}G}; & \left(|1-\tau| \geq \dfrac{1}{4\mu(1+2\alpha)e^{-2\mu}}\left|(\alpha^2-7\alpha-2+8\mu\alpha+4\mu)e^{-2\mu}-\dfrac{(1+\alpha)^2e^{-2\mu}G}{2[\lambda]_q^2\ell^2}\right|\right). \end{cases}$$

Where $G=2\left([\lambda]_{q^2}+[\lambda]_q^2\right)\ell^2-[\lambda]_{q^2}$

Proof: From (21) and (22), we get

$$a_3-\tau a_2^2 = \frac{C_1^{(\lambda)}(\ell;q)}{4\mu(1+2\alpha)e^{-2\mu}}(c_2-d_2)+(1-\tau)\frac{\left[C_1^{(\lambda)}(\ell;q)\right]^3}{\left[C_1^{(\lambda)}(\ell;q)\right]^2} \times \frac{1}{(\alpha^2-7\alpha-2+8\mu\alpha+4\mu)e^{-2\mu}-2(1+\alpha)^2e^{-2\mu}C_2^{(\lambda)}(\ell;q)}(c_2+d_2)$$

$$= C_1^{(\lambda)}(\ell;q)\left[\left(L(\tau)+\frac{1}{4\mu(1+2\alpha)e^{-2\mu}}\right)c_2+\left(L(\tau)-\frac{1}{4\mu(1+2\alpha)e^{-2\mu}}\right)d_2\right],$$

where

$$L(\tau) = \frac{(1-\tau)}{\left[C_1^{(\lambda)}(\ell;q)\right]^2(\alpha^2-7\alpha-2+8\mu\alpha+4\mu)e^{-2\mu}-2(1+\alpha)^2} \times \frac{1}{e^{-2\mu}C_2^{(\lambda)}(\ell;q)}$$

Thus, according to (5), we have

$$|a_3-\tau a_2^2| = \begin{cases} \dfrac{\left|[\lambda]_q\right|\ell}{\mu(1+2\alpha)e^{-2\mu}}; & \left(0 \le |L(\tau)| \le \dfrac{1}{4\mu(1+2\alpha)e^{-2\mu}}\right), \\ 4\left|[\lambda]_q\right|\ell|L(\tau)|; & \left(|L(\tau)| \ge \dfrac{1}{4\mu(1+2\alpha)e^{-2\mu}}\right), \end{cases}$$

Hence, after some calculations, gives

$$|a_3-\tau a_2^2| \le \begin{cases} \dfrac{\left|[\lambda]_q\right|\ell}{\mu(1+2\alpha)e^{-2\mu}}; \quad \left(|1-\tau| \le \dfrac{1}{4\mu(1+2\alpha)e^{-2\mu}} \times \left|(\alpha^2-7\alpha-2+8\mu\alpha+4\mu)e^{-2\mu} - \dfrac{(1+\alpha)^2e^{-2\mu}G}{2[\lambda]_q^2\ell^2}\right|\right). \\ \dfrac{16\left|[\lambda]_q^3\right|\ell^3(1-\tau)}{4[\lambda]_q^2\ell^2(\alpha^2-7\alpha-2+8\mu\alpha+4\mu)e^{-2\mu}-2(1+\alpha)^2e^{-2\mu}\,G}; \\ \left(|1-\tau| \ge \dfrac{1}{4\mu(1+2\alpha)e^{-2\mu}}\left|(\alpha^2-7\alpha-2+8\mu\alpha+4\mu)e^{-2\mu} - \dfrac{(1+\alpha)^2e^{-2\mu}G}{2[\lambda]_q^2\ell^2}\right|\right). \end{cases}$$

By substituting $\alpha = 0$ into theorem (2), we obtain the following outcome:

Corollary 3: If $f \in A$ defined by (1) be in the class $J_\Sigma\left(\alpha, \mathfrak{B}_q^{(\lambda)}(\ell, z)\right)$.

$$|a_3-\tau a_2^2| \le \begin{cases} \dfrac{\left|[\lambda]_q\right|\ell}{\mu e^{-2\mu}}; \quad \left(|1-\tau| \le \dfrac{1}{4\mu e^{-2\mu}}\left|(4\mu-2)e^{-2\mu} - \dfrac{e^{-2\mu}G}{2[\lambda]_q^2\ell^2}\right|\right), \\ \dfrac{16\left|[\lambda]_q^3\right|\ell^3(1-\tau)}{4[\lambda]_q^2\ell^2(4\mu-2)e^{-2\mu}-2e^{-2\mu}\,G}; \\ \left(|1-\tau| \ge \dfrac{1}{4\mu e^{-2\mu}}\left|(4\mu-2)e^{-2\mu} - \dfrac{e^{-2\mu}G}{2[\lambda]_q^2\ell^2}\right|\right). \end{cases}$$

By substituting $\alpha = 1$ into theorem (2), we obtain the following outcome:

Corollary 4: If $f \in A$ defined by (1) be in the class $J_\Sigma\left(\alpha, \mathfrak{B}_q^{(\lambda)}(\ell, z)\right)$.

$$|a_3-\tau a_2^2| \le \begin{cases} \dfrac{\left|[\lambda]_q\right|\ell}{3\mu e^{-2\mu}}; \quad \left(|1-\tau| \le \dfrac{1}{3\mu}\left|(3\mu-2) - \dfrac{G}{2[\lambda]_q^2\ell^2}\right|\right), \\ \dfrac{2\left|[\lambda]_q^3\right|\ell^3(1-\tau)}{2[\lambda]_q^2\ell^2(3\mu-2)e^{-2\mu}-e^{-2\mu}\,G}; \\ \left(|1-\tau| \ge \dfrac{1}{3\mu}\left|(3\mu-2) - \dfrac{G}{2[\lambda]_q^2\ell^2}\right|\right). \end{cases}$$

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