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Synchronization State in the Coupled Van Der Pol Oscillators within a Small Invariant Subspace

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ABSTRACT

This study aims to analyze the dynamics of trajectories in a system of Van der Pol with the coupled oscillator. The control parameters considered are damping and coupling strength. We focus on study the behaviour of this system within a specific small invariant subspace. It is used phase difference to reduce the dimension of this system into two dimensions. Then, the Jacobian matrix is computed eigenvalues in order to determine the stability of equilibrium point. this work specifically discovers the effects the damping and coupling strength parameters to emerge synchronization states. additionally, it investigates how change the value of damping parameter influence on the system's energy.

MSC..

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1. Introduction:

Coupled oscillators are essential models for studying complex dynamical systems across diverse sciences, including physics [1], biology [2], chemical [3] and electrical oscillators [4]. The Van der Pol oscillator, a typical example of a nonlinear oscillator, is used to investigate self-sustained oscillatory dynamics through the interaction of energy generation and dissipation [5]. When Van der Pol oscillators are coupled, their interactions show many phenomena such as synchronization, energy exchange, and bifurcation patterns. The model of the Van der Pol system is an ideal model for studying environment and engineered systems, for example, cardiac rhythms, neural networks, and power grids [6, 7].

The important parameters that affect the behaviour of coupled van der Pol oscillators include the damping parameters K and the coupled strength μ. The damping parameter influences nonlinearity term and generation or

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dissipation energy, while the coupled strength determines the degree of interaction or coherence between oscillators.

Both of these parameters have significant effects on the behaviour of trajectory, such as stability, phase portraits or bifurcation. when damping has a small value (which means reduced nonlinear dissipation) combined with strong coupling is large, the system exhibits distinct patterns, for example, it allows oscillators to sustain harmonic that is similar to motion with minimal energy loss. On the other hand, Strong coupling refers to the energy exchange between oscillators to promote coherent collective behaviour [8]. Conversely, When damping is strong, and coupling is weak, the behaviour of coupled van der Pol oscillators leads to show individual dynamics rather than collective dynamics. Also, strong damping refers to the nonlinear dissipation that implicates robust self-sustain oscillations with each oscillator and displays a limit cycle and poor synchronization [9, 10]. The study of the Van der Pol system reveals the mechanisms governing synchronization, stable limit cycle, exchange energy and bifurcation as parameters vary [11, 12].

This work is arranged as follows: In section two, it produces the Van der Pol equation with coupled oscillators and a small invariant subspace. Also, it is transformed the second-order differential equation into a system with first-order equations, which is easier to study mathematically. Section three investigates the influences of varying damping effects and coupling strength on appearing coherent states, stable/unstable equilibrium points and the behaviour of trajectories towards a limit cycle.

2. Coupled oscillators in the Van der Pol equation

We consider the model Van der Pol equation with coupled oscillators to study collective behaviour in networks of nonlinear oscillators [5],

$$\ddot{x}_i - \mu(1 - x_i^2)\dot{x}_i + x_i = K \sum_{j=1}^{N} g\big(x_j - x_i\big), \qquad (1)$$

where x_i state of $i-th$ oscillators, K is coupled strength and g is a coupled function, where

$$g(\theta) = \sin(\theta - \lambda) + d\sin(2\theta).$$

We will concentrate on the model (1) analysis within one of the exotic balanced polydiagonals, which is an invariant subspace A, from the table at [13, 14].

$$A = (a_1, a_2, a_3, a_4, a_5, a_6) = (\varphi_1, \varphi_1 + \pi, \varphi_2, \varphi_1, \varphi_1 + \pi, \varphi_2 + \pi).$$

Networks often display patterns of frequency synchrony, in which two or more oscillators have the same behave. In a mathematical sense, The solutions of equation(1) which are φ_i and φ_j are frequency synchronized on the same trajectory, if

$$\Delta_{ij} = \lim_{T\to\infty} \frac{1}{T}\big[\varphi_i - \varphi_j\big] = 0,$$

and partial synchrony if $\Delta_{ij} \neq 0$.

Now, we write invariant subspace A with equation (1) and coupled function(2), we get

$$\ddot{\varphi}_1 - \mu(1 - \varphi_1^2)\dot{\varphi}_1 + \varphi_1 = K \sum_{j=1}^{2} g(\varphi_2 - \varphi_1),$$

$$\ddot{\varphi}_2 - \mu(1-\varphi_2^2)\dot{\varphi}_2 + \varphi_2 = K\sum_{i=1}^{2} g(\varphi_1 - \varphi_2). \tag{3}$$

Then, we can write the equitations (3) as a system by:

$$\dot{\varphi}_1 = \xi_1,$$

$$\dot{\xi}_1 = \mu(1-\varphi_1^2)\xi_1 - \varphi_1 + K\sum_{j=1}^{2} g(\varphi_2 - \varphi_1),$$

$$\dot{\varphi}_2 = \xi_2,$$

$$\dot{\xi}_2 = \mu(1-\varphi_2^2)\xi_2 - \varphi_2 + K\sum_{i=1}^{2} g(\varphi_1 - \varphi_2). \tag{4}$$

Let us consider $\lambda = 0$ and $d = 0$ in the interacted function g, we get

$$\dot{\varphi}_1 = \xi_1,$$

$$\dot{\xi}_1 = \mu(1-\varphi_1^2)\xi_1 - \varphi_1 + K[\sin(\varphi_2 - \varphi_1) + \sin(\varphi_1 + \pi - \varphi_2)],$$

$$\dot{\varphi}_2 = \xi_2,$$

$$\dot{\xi}_2 = \mu(1-\varphi_2^2)\xi_2 - \varphi_2 + K[si\,n(\varphi_1 - \varphi_2) + si\,n(\varphi_1 + \pi - \varphi_2 - \pi)]. \tag{5}$$

After simplify, the system becomes,

$$\dot{\varphi}_1 = \xi_1,$$

$$\dot{\xi}_1 = \mu(1-\varphi_1^2)\xi_1 - \varphi_1 + 2K\sin(\varphi_2 - \varphi_1),$$

$$\dot{\varphi}_2 = \xi_2,$$

$$\dot{\xi}_2 = \mu(1-\varphi_2^2)\xi_2 - \varphi_2 + 2K\sin(\varphi_1 - \varphi_2). \tag{6}$$

The phase difference between the two oscillators is assumed as:

$$\Delta\varphi = \varphi_1 - \varphi_2.$$

Then, it is substituted $\Delta\varphi$ in (6) to get,

$$\Delta\dot{\varphi} = \xi_1 - \xi_2 = f(\Delta\varphi, \Delta\xi)$$

$$\Delta\dot{\xi} = \dot{\xi}_1 - \dot{\xi}_2 = \mu[(1-\varphi_1^2)\xi_1 - (1-\varphi_2^2)\xi_2] - \Delta\varphi + K[2\sin(-\Delta\varphi) - 2\sin(\Delta\varphi)]$$

$$= g(\Delta\varphi, \Delta\xi). \tag{7}$$

Which is system of coupled oscillators with two parameters coupled strength K and damping μ.

3. Stability and Parameters effects

The important steps to analyze the dynamics of the van Der Pol system are to find equilibrium points and determine their stability. Fixed points indicate that the system's derivatives disappear. In other words, a fixed point can be obtained by solving the system (7), when

$$\Delta\dot{\varphi} = 0,$$

$$\Delta\dot{\xi} = 0.$$

Then the equilibrium point is (0, 0). The next step is to compute the Jacobian matrix to evaluate the local linear behaviour around this point.

$$J = \begin{bmatrix} \frac{\partial f}{\partial \Delta\varphi} & \frac{\partial f}{\partial \Delta\xi} \\ \frac{\partial g}{\partial \Delta\varphi} & \frac{\partial g}{\partial \Delta\xi} \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 \\ -1 - 4K(\cos(\Delta\varphi)) - 2\mu\Delta\xi\Delta & \mu(1 - \Delta\varphi^2) \end{bmatrix}.$$

The eigenvalues of the Jacobian that are computed at the fixed point (0, 0) determine stability. The fixed point is stable if it has all eigenvalues with negative real part. Also, the point is called unstable if any eigenvalues have a positive real part. Now, we will display different cases for the changes behaviour of the coupled Van der Pol oscillators due to varying values of the damping parameter and coupled strength.

Case 1: when k = 1, μ = 0.3.

The Jacobian matrix with these values of parameters at the fixed point (0, 0) has two complex eigenvalues with positive real parts, which means the fixed point is an unstable spiral point that is plotted as a black point in Figure 1A_1. In addition, the single limit cycle is created due to the strong coupled K. In Figure 1A_1, the trajectories travel slowly with extensive spiralling and take a long time to settle into the stable limit cycle because the damping has a small value and the system slowly dissipates energy. Moreover, These values mean there is a strong coupling, which leads to emerging synchrony between oscillators. In other words, the oscillators continuously oscillate with a constant phase difference (Figure 1A_2), such that these values show anti-synchronization as seen in Figure 1A_3. In addition, a small μ refers to the self-sustained oscillations of the oscillators, and it is weaker. the strong coupled and μ is small, encouraging synchronization and enabling the efficient exchange of energy between oscillators.

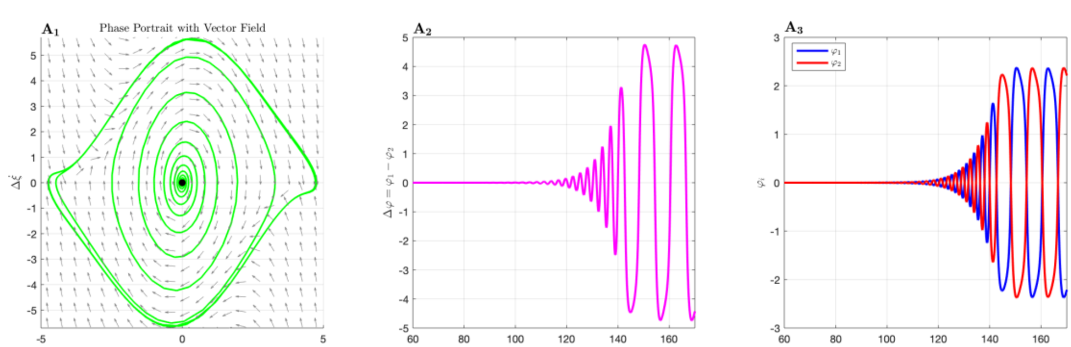
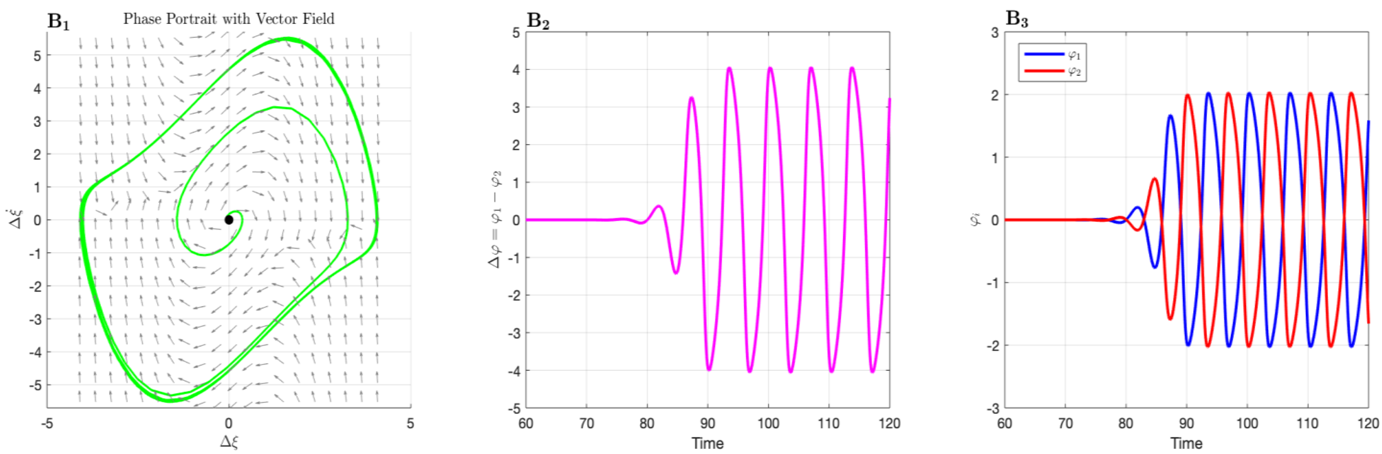


Figure 1: This figure shows phase portrait in A_1 for system (7) with μ = 0.3 and K=1. Also, the Phase difference is in panel A_2. Finally, A_3 displays anti-synchronization motion between two coupled oscillators.

Case 2: when μ = 1, K = 0.1.

At these values, there is no change in the stability of the fixed point is still an unstable spiral point. But, you can see in Figure $2B_1$, that the trajectory that starts near an unstable fixed point goes to the stable limit cycle with



minimal spiralling because the stronger damping quickly rectifies energy imbalance and directs trajectories toward the limit cycle. The values of strong damping μ and weak coupled K lead the system to show high nonlinearity which is represented by the term μ(1 − x2), which leads to strong self-sustained oscillations. The stable limit cycles are formed as a result of cyclic generation and dissipation of energy within each oscillator. In other words, K indicates a weak coupling, which means the interaction between oscillators is very little. Also, the energy exchange between oscillators is limited because of weak coupled. Moreover, in Figure $2B_2$, phase differences between the oscillators persist over time, and their individual dynamics dominate, resulting in a lack of coherent collective motion. Also, the oscillators have anti-synchronization motion which is represented in Figure $2B_3$.

Figure 2: This figure shows a phase portrait in B_1 for system (7) with μ = 1 and K = 0.1, the black point represents an unstable spiral point. The phase difference is in panel B_2. Finally, B_3 displays anti-synchronization motion between two coupled oscillators.

Case 3: when μ = 3, K = 0.1.

The equilibrium (0, 0) became an unstable node because their eigenvalues are just positive real parts. The trajectories diverge exponentially from this point (see Figure $3C_1$). In addition, μ = 3 indicates strong nonlinearity and K = 0.1 is weak coupling. In Figure $3C_1$, The phase portrait at these values displays trajectories like symmetry wings with the black unstable node point. These trajectories finally go into stable limit cycles with large amplitude oscillations due to strong nonlinear damping. The energy of dynamics at this behaviour is internal generation and dissipation within each oscillator and there is minimal energy exchange owing to weak coupling. Also, oscillators keep continual phase differences (Figure $3C_2$,). Both of oscillators have the same waveform with temporally shifted as seen in Figure $3C_3$.

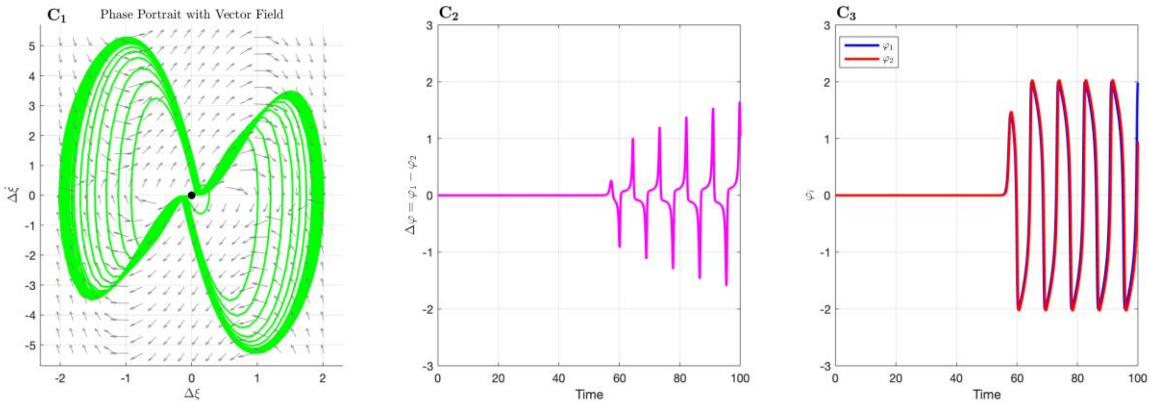


Figure 3: Phase portrait is shown in panel C_1 for system (7) with μ = 3 and K = 0.1, black point represents the unstable node point. A phase difference is in panel C_2. Finally, C_3 displays synchronization motion between two coupled oscillators with small shifts.

4. Conclusion

This paper provides an analysis of the behaviour of trajectories of the coupled Van der Pol oscillators, with effects on two essential parameters: damping and coupling strength. This system is reduced into two dimensions by using phase differences in order to simplify the analysis. Also, it enables us to find the Jacobian matrix and its eigenvalues that asset to determine the stability of fixed points. Moreover, it highlights the great influence of damping and coupling strength on the emergence of synchronization states. These results give a deeper understanding of the motion of coupled oscillators in a nonlinear system and their complex behaviour, with potential applications in different sciences where synchronization plays an essential role. This work focused on studying the behaviour of trajectories with a specific small number of oscillators. An interesting direction for future work uses different sizes of an invariant subspace, changing the symmetry of the invariant set or adding noise to the essential equation.

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