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On Special Fuzzy Differential Superordination For Univalent Functions Defined by Integral Operator

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ABSTRACT

Miller and Mocanu introduced the concept of differential superordination as the dual counterpart to differential subordination, as discussed in [3]. In [4], the notion of fuzzy subordination was introduced, while in [5], the authors extended this idea by defining fuzzy differential subordination. Furthermore, in [6], They derived conditions under which a function acts as a dominant in fuzzy differential subordination and determined the optimal dominant. This work focuses on investigating certain special cases of fuzzy differential superordination for univalent functions defined by an integral operator.

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1. Introduction and Preliminaries

Let $\mho = \{\omega \in \mathbb{C}: |\omega| < 1\}$ represent the open unit disk in the complex plane, with its closure denoted by $\overline{\mho} = \{\omega \in \mathbb{C}: |\omega| \leq 1\}$. Let $\aleph(\mathrm{U})$ Represent the set of analytic functions within $\mho$. For $\varrho \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by

$$\aleph[\varrho, n] = \{f \in \aleph(\mho): f(\omega) = \varrho + \varrho_{n+1}\omega^{n+1} + \cdots, \omega \in \mathbb{C}\},$$

Similarly, let

$$\mathcal{A}_n = \{f \in \aleph(\mho): f(\omega) = \omega + \varrho_{n+1}z^{n+1} + \cdots, \omega \in \mathbb{C}\},$$

where for $n = 1$, we denote $\mathcal{A}_1$ simply as $\mathcal{A}$. The set of functions that are both analytic and univalent in $\mho$, denoted by S, consists of functions satisfying $f(0) = 0$ and $f'(0) = 1$, and is given by the power series expansion:

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$$f(\omega)=\omega+\varrho_2\omega^2+\cdots,\omega\in\mho. \tag{1.1}$$

Further, we define important subclasses of $\mathcal{A}$ as follows:

The set of functions that are normalized starlike functions, denoted by

$$\mathcal{S}^*=\left\{f\in\mathcal{A}:Re\left(\frac{\omega f'(\omega)}{f(\omega)}\right)>0,\omega\in\mho\right\},$$

The set of functions that are normalized convex functions, denoted by

$$\mathcal{C}=\left\{f\in\mathcal{A}:Re\left(\frac{\omega f''(\omega)}{f'(\omega)}+1\right)>0,\omega\in\mho\right\},$$

The set of functions that are normalized close-to-convex functions, denoted by

$$K=\left\{f\in\mathcal{A}:Re\left\{\frac{f'(\omega)}{g'(\omega)}\right\}>0,g(\omega)\in\mathcal{C},\omega\in\mho\right\},$$

Differential Superordination Method

The general framework of the differential superordination method can be expressed as follows. Let Ω and Δ be subsets of $\mathbb{C}$, and let p be an analytic function within $\mho$. Consider a function $\varphi(r,s,t;\omega):\mathbb{C}^3\times\mho\longrightarrow\mathbb{C}$, which is defined in terms of p and its derivatives. The primary objective is to investigate the implications of the following relationships:

$$\Omega\subset\{\varphi(p(\omega),\omega p'(\omega),\omega^2p''(\omega);\omega):\omega\in\mho\}\Longrightarrow\Delta\subset p(\mho). \tag{1.2}$$

If Δ is a simply connected domain that includes a point ϱ, and $\Delta\neq\mathbb{C}$, then there exists a conformal mapping $q:\mho\to\Delta$ such that $q(0)=\varrho$. Under this condition, the given implication can be reformulated as follows:

$$\Omega\subset\{\varphi(p(\omega),\omega p'(\omega),\omega^2p''(\omega);\omega):\omega\in\mho\}\Longrightarrow q(\omega)\prec p(\omega). \tag{1.3}$$

If Ω is also simply connected and $\Omega\neq\mathbb{C}$, then there exists a conformal mapping $\rho:\mho\to\Omega$ such that $\rho(0)=\phi(\varrho,0,0;0)$. Moreover, if the function $\phi(p(\omega),\omega p'(\omega),\omega^2p''(\omega);\omega)$ is univalent in U, the relation can be expressed as:

$$\rho(\omega)\prec\varphi(p(\omega),\omega p'(\omega),\omega^2p''(\omega);\omega)\Longrightarrow\rho(\omega)\prec p(\omega). \tag{1.4}$$

For a more detailed discussion on the differential superordination method, refer to the monographs [1,8,9,10].

Definition (1.1) [3] We denote by $\hat{Q}$ the set of functions q that are analytic and injective on $\overline{\mho}\backslash E(q)$, were

$$E(q)=\left\{\zeta\in\partial\mho:\lim_{\omega\to\zeta}q(\omega)=\infty\right\},$$

and are such that $q'(\zeta)\neq0$ for $\zeta\in\partial\mho\backslash E(q)$. The set $E(q)$ is called the exemption set.

Definition (1.2) [11] Let Y be a non-empty set. A function $\mathrm{F}: Y \to [0,1]$ is called a fuzzy subset of Y. A more precise definition can be given as follows:

A fuzzy subset is defined as a pair $(\mathrm{A}, \mathrm{F}_A)$, $\mathrm{F}_A : Y \to [0,1]$ is a function and

$$A = \{y \in Y: 0 < \mathrm{F}_A \leq 1\} = supp(A, \mathrm{F}_A),$$

Here, A represents the support of the fuzzy set, consisting of all elements in Y with nonzero membership values. The function F_A is referred to as the membership function of the fuzzy set $(\mathrm{A}, \mathrm{F}_A)$, which assigns each element of Y a degree of membership in the fuzzy subset.

Definition (1.3) [11]: Let $(M', \mathrm{F}_{M'})$ and $(N', \mathrm{F}_{N'})$ be two fuzzy subsets of a set Y. We say that the fuzzy subsets M' and N' are equal if and only if they have the same membership function, i.e.,

$$\mathrm{F}_{M'}(y) = \mathrm{F}_{N'}(y), \forall\, y \in Y$$

This is denoted by:

$$(M', \mathrm{F}_{M'}) = (N', \mathrm{F}_{N'}).$$

A fuzzy subset $(M', \mathrm{F}_{M'})$ is contained in another fuzzy subset $(N', \mathrm{F}_{N'})$ if and only if its membership function does not exceed that of $(N', \mathrm{F}_{N'})$ for all elements in Y, i.e.,

$$\mathrm{F}_{M'}(y) \leq \mathrm{F}_{N'}(y), \forall\, y \in Y$$

This inclusion relation is denoted by:

$$(M', \mathrm{F}_{M'}) \subseteq (N', \mathrm{F}_{N'}).$$

Definition (1.4) [4] Let $D \subseteq \mathbb{C}$, and let $\omega_0 \in D$ be a fixed point. Suppose that $f, g \in \aleph(\mho)$ are analytic functions. A function f is considered to be fuzzy subordinate to g, denoted as $f \prec_F g$ or $f(\omega) \prec_F g(\omega)$, if the following conditions are satisfied:

1- $f(\omega_0) = g(\omega_0)$,

2- $F_{f(D)}\big(f(\omega)\big) \leq F_{g(D)}\big(g(\omega)\big), \omega \in \mho$.

Remark [4] Let $f, \hbar \in \aleph(\mho)$ are functions and $\hbar$ is a univalent function then $f \prec \hbar$ if

$f(0) = \hbar(0)$ *and* $f(\mho) \subset \hbar(\mho)$. Thus, if $\hbar$ is a univalent function, then $f \prec_F \hbar$ if and only if $f \prec \hbar$.

Let $f, \hbar \in \aleph(\mho)$, and assume that $\hbar$ is a univalent function. Then, f is said to be fuzzy subordinate to $\hbar$, denoted as $f \prec \hbar$, Provided that the following criteria are met

i. $f(0) = \hbar(0)$,

ii. $f(\mho) \subseteq \hbar(\mho)$.

From this, if $\hbar$ is a univalent function, it follows that $f \prec_F \hbar \Leftrightarrow f \prec \hbar$.

Definition (1.5) [5] A function $L(\omega, t)$ where $\omega \in \mho$ and $t \geq 0$, is called a fuzzy subordination chain as long as it meets the following conditions:

i. For each fixed $t \geq 0$, the function $L(\cdot, t)$ is analytic and univalent in $\mho$.

ii. The function $L(\omega, t)$ is continuously differentiable with respect to t on $[0, \infty)$ for all $\omega \in \mho$.

iii. The fuzzy membership function satisfies the monotonicity condition:

$$F_{L[\mho\times[0,\infty)]}\big(L(\omega, t_1)\big) \leq F_{L[\mho\times[0,\infty)]}\big(L(\omega, t_2)\big), t_1 \leq t_2.$$

Theorem (1.6) [1] Let ℓ_1 be convex in $\mho$, with $\ell_1(0) = a\,\gamma \neq 0$, with $Re(\gamma) \geq 0$, and $p \in \aleph[a, 1] \cap Q$ if $\wp(\omega) + \frac{1}{\gamma}\omega\wp'(\omega)$ is univalent in $\mho$.

$$\ell_1(\omega) \prec_F \wp(\omega) + \frac{1}{\gamma}\omega\wp'(\omega), \tag{1.5}$$

and

$$q_1(\omega) = \frac{\gamma}{z^\gamma}\int_0^\omega \ell_1(t)\, t^{\gamma-1} dt. \tag{1.6}$$

Then

$$\ell_1(\omega) \prec_F \wp(\omega),$$

then function q_1 is convex and is the fuzzy best subordinate.

Theorem (1.7) [1] Let q be convex in $\mho$ and let ℓ be defined by

$\ell(\omega) = q(\omega) + \frac{1}{\gamma}\omega q'(\omega)$, with $Re\gamma > 0$. If $\wp \in \aleph[a, 1] \cap Q, \wp(\omega) + \frac{1}{\gamma}\omega\wp'(\omega)$ is univalent in $\mho$ and

$$F_{\ell(\mho)}\big(\ell(\mho)\big) \leq F_{(\mathbb{C}^3\times\mho)}\left(\wp(\omega) + \frac{1}{\gamma}\omega\wp'(\omega)\right),$$

then

$$q(\omega) \prec_F \wp(\omega),$$

where
$$q(\omega) = \frac{\gamma}{\omega^\gamma}\int_0^\omega \ell(t)\, t^{\gamma-1} dt.$$

The function q is the fuzzy best subordinant.

Theorem (1.8) [1] Let ℓ be starlike in $\mho$, with $\ell(0) = 0$. If $p \in \aleph[0,1] \cap Q$ and $z\wp'(z)$ is univalent in $\mho$, then

$$F_{\ell(\mho)}\big(\ell(\omega)\big) \le F_{(\mathbb{C}^3\times\mho)}\big(\omega p'(\omega)\big), \tag{1.7}$$

implies

$$F_{q(\mho)}q(v) \le F_{p(\mho)}p(\omega), \qquad \omega \in \mho,$$

where

$$q(\omega) = \int_0^{\omega} \ell(t)\, t^{-1} dt. \tag{1.8}$$

The function q is convex and is the fuzzy best subordinant.

Let $f_j \in \mathcal{A}, j = 1,2$ as given in form (1.1), where

$$f_j(\omega) = \omega + \sum_{n=2}^{\infty} a_{n,j}\, \omega^n, \qquad (j = 1,2),$$

the Hadamard product (or convolution) of $f_1(\omega)$ and $f_2(\omega)$ is defined by

$$(f_1 * f_2)(\omega) = \omega + \sum_{n=2}^{\infty} a_{n,1} a_{n,2}\, \omega^n = (f_2 * f_1)(\omega).$$

Definition 1.9 Let $f \in \mathcal{A}, m \in \mathbb{N}$ and $\alpha > 0,\ \beta \ge 0$. We define the differential operator $\mathrm{I}_{\beta,\alpha}^m : \mathcal{A} \longrightarrow \mathcal{A}$, where

$$\mathrm{I}_{\beta,\alpha}^0 f(\omega) = f(\omega),$$

$$\mathrm{I}_{\beta,\alpha}^1 f(\omega) = \frac{\alpha+\beta+2}{\omega^{\alpha+\beta+1}} \int_0^{\omega} t^{\alpha+\beta} f(t) dt,$$

$$\mathrm{I}_{\beta,\alpha}^2 f(\omega) = \frac{\alpha+\beta+2}{\omega^{\alpha+\beta+1}} \int_0^{\omega} t^{\alpha+\beta} \left(\mathrm{I}_{\beta,\alpha}^1 f(\omega)\right) dt, \ldots$$

$$\mathrm{I}_{\beta,\alpha}^m f(\omega) = \frac{\alpha+\beta+2}{\omega^{\alpha+\beta+1}} \int_0^{\omega} t^{\alpha+\beta} \left(\mathrm{I}_{\beta,\alpha}^{m-1} f(\omega)\right) dt$$

$$= \omega + \sum_{n=2}^{\infty} \left(\frac{\alpha+\beta+2}{\alpha+\beta+n+1}\right)^m a_n\, \omega^n. \tag{1.9}$$

The general Hurwitz-lerch Zeta function

$$\emptyset(\omega, s, \rho) = \sum_{n=0}^{\infty} \frac{\omega^n}{(\rho+n)^s},\ \rho \in \mathbb{C}\backslash\mathbb{Z}_0^- = \{0. -1, -2, \ldots\}, s \in \mathbb{C}, when\ |\omega| < 1.$$

Definition 1.10 Let $f \in \mathcal{A}, m \in \mathbb{N}$, $\alpha > 0, \beta \ge 0, \rho \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $s \in \mathbb{C}$,we define the operator ${}_{\rho}^{s}\Upsilon_{\beta,\alpha}^m : \mathcal{A} \longrightarrow \mathcal{A}$, were

$${}_{\rho}^{s}\Upsilon_{\beta,\alpha}^m f(\omega) = (\rho+1)^s [\emptyset(\omega, s, \rho) - \frac{1}{a^s}] * \mathrm{I}_{\beta,\alpha}^m f(\omega)$$

$$= \omega + \sum_{n=2}^{\infty} \left(\frac{\rho+1}{\rho+n}\right)^s \left(\frac{\alpha+\beta+2}{\alpha+\beta+n+1}\right)^m a_n\, \omega^n. \tag{1.10}$$

From (1.10), it can be easily seen that

$$\omega \left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega) \right)' = (\alpha+\beta+2)\, {}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega) - (\alpha+\beta+1)\, {}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega), \tag{1.11}$$

and, we have

i) ${}_{\rho}^{0}\Upsilon_{\beta,\alpha}^{m} f(\omega) = \mathrm{I}_{\beta,\alpha}^{m} f(\omega);$

ii) ${}_{\rho}^{0}\Upsilon_{\beta,\alpha}^{0} f(\omega) = f(\omega).$

Using the operator ${}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)$ defined in Definition (1.10), we study fuzzy superordination

2. Main Results

Theorem 2.1: Let $\ell(\omega)$ be convex function with $\ell(0) = 1$. Let $f \in \mathcal{A}, m \in \mathbb{N}, \alpha > 0, \beta \geq 0, \rho \in \mathbb{C}\backslash\mathbb{Z}_0^-, s \in \mathbb{C}$ and suppose that $\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega) \right)'$ is univalent and $\frac{\mathfrak{F}_{\alpha,\lambda}^{s,m} f(\omega)}{\omega} \in \aleph[1,1] \cap Q$. If

$$F_{\ell(\mho)}\big(\ell(\omega)\big) \leq F_{\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f\right)'_{(\mho)}} \left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega) \right)', \tag{2.1}$$

then

$$F_{q(\mho)}\big(q(\omega)\big) \leq F_{\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f\right)_{(\mho)}} \left(\frac{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}{\omega} \right), \omega \in \mho,$$

where $q(\omega) = \frac{1}{\omega}\int_0^z \hbar(t)\, dt$. The function q is convex and it is the fuzzy best subordinant.

Proof. Consider

$$p(\omega) = \frac{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}{\omega} = \frac{\omega + \sum_{n=2}^{\infty} \left(\frac{\rho+1}{\rho+n}\right)^s \left(\frac{\beta+\alpha+2}{\beta+\alpha+n+1}\right)^m a_n\, \omega^n}{\omega}$$

$$= 1 + \sum_{n=2}^{\infty} \left(\frac{\rho+1}{\rho+n}\right)^s \left(\frac{\beta+\alpha+2}{\beta+\alpha+n+1}\right)^m a_n\, \omega^{n-1}.$$

Evidently $p \in \aleph[1,1]$. We have,

$$p(\omega) + \omega p'(\omega) = \left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega) \right)', \qquad \omega \in \mho.$$

Then the fuzzy differential superordination (2.1) becomes

$$F_{\ell(\mho)}\big(\ell(\omega)\big) \leq F_{(\mathbb{C}^2 \times \mho)}\big(p(\omega) + \omega p'(\omega)\big), \qquad \omega \in \mho.$$

By using Theorem (1.6) for $\gamma = 1$, we have

$$F_{q(\mho)}\big(q(\omega)\big) \le F_{p(\mho)}\big(p(\omega)\big), i.e. F_{q(\mho)}\big(q(\omega)\big) \le F_{\left({}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f\right)_{(\mho)}}\left(\frac{{}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)}{\omega}\right),$$

Where $q(\omega) = \frac{1}{\omega}\int_0^{\omega} \ell(t)\,dt$. The function q is convex and it is the fuzzy best subordinant.

Corollary 2.2: Let $\ell(\omega) = \frac{1+(2\gamma-1)\omega}{1+\omega}$ a convex function in $\mho, 0 \le \gamma < 1$. If $m \in \mathbb{N}, \alpha > 0, \beta \ge 0, \rho \in \mathbb{C}\backslash\mathbb{Z}_0^-, s \in \mathbb{C}$ $f \in \mathcal{A}$ and suppose that $\left({}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)\right)'$ is univalent and $\frac{{}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)}{\omega} \in \aleph[1,1] \cap Q$. If

$$\frac{1+(2\gamma-1)\omega}{1+\omega} \prec_F \left({}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)\right)', \tag{2.2}$$

then

$$F_{q(\mho)}\big(q(\omega)\big) \le F_{\left({}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f\right)_{(\mho)}}\left(\frac{{}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)}{\omega}\right), \qquad \omega \in \mho.$$

where q is given by

$$q(\omega) = 2\gamma - 1 + \frac{2(1-\gamma)}{2}\ln(1+\omega).\ \omega \in \mho,$$

The function q is convex and it is the fuzzy best subordinant.

Proof. We have

$$\ell(\omega) = \frac{1+(2\gamma-1)\omega}{1+\omega}, \ell(0) = 1, \ell'(\omega) = \frac{-2(1-\gamma)}{(1+\omega)^2} \text{ and } \ell''(\omega) = \frac{4(1-\gamma)}{(1+\omega)^3}, \text{therefore}$$
$$Re\left(\frac{z\ell''(\omega)}{\ell'(\omega)} + 1\right) = Re\left(\frac{1-\omega}{1+\omega}\right) = Re\left(\frac{1-r(\cos\theta + i\sin\theta)}{1+r(\cos\theta + i\sin\theta}\right)$$

$$= \frac{1-r^2}{1+2r\cos\theta + r^2} > 0, \text{where } r = |\omega| < 1, \theta \in \mathbb{R}.$$

Following the same steps as in the proof of Theorem (2.1) and considering

$$p(\omega) = \frac{{}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)}{\omega}.$$

The fuzzy differential superordination (2.2) becomes

$$\frac{1+(2\gamma-1)\omega}{1+\omega} \prec_F p(\omega) + \omega p'(\omega), \qquad \omega \in \mho.$$

By using Theorem (1.6) for $\gamma = 1$, we have

$$F_{q(\mho)}\big(q(\omega)\big) \le F_{p(\mho)}\big(p(\omega)\big), i.e. F_{q(\mho)}\big(q(\omega)\big) \le F_{\left({}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f\right)_{(\mho)}}\left(\frac{{}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)}{\omega}\right),$$

and

$$q(\omega) = \frac{1}{\omega}\int_0^z \ell(t)\,dt = \frac{1}{\omega}\int_0^z \frac{1+(2\gamma-1)t}{1+t}\,dt = 2\gamma - 1 + \frac{2(1-\gamma)}{2}\ \ln(1+\omega).$$

Theorem (2.3): Let q convex function in U sach that $q(0) = 1$ and let ℓ be defined by $\ell(\omega) = q(\omega) + \frac{1}{\alpha+\beta+2}\omega q'(\omega)$. If $m \in \mathbb{N}$, $\alpha > 0, \beta \geq 0, \rho \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $s \in \mathbb{C}$, $f \in \mathcal{A}$ and suppose that $\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)\right)'$ is univalent and $\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)' \in \aleph[1,1] \cap Q$ and satisfies the fuzzy differential superordination

$$\ell(\omega) = q(\omega) + \frac{1}{\alpha+\beta+2}\omega q'(\omega) \prec_F \left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)\right)', \omega \in \mho, \tag{2.3}$$

then

$$F_{q(\mho)}\big(q(z)\big) \leq F_{\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f\right)'_{(\mho)}} \left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)',$$

where $q(\omega) = \frac{1}{z}\int_0^z \ell(t)\,dt$. The function q is the optimal fuzzy subordinant.

Proof. By utilizing the properties of the operator ${}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}$ and differentiating with respect to ω, we obtain

$$\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)' = \frac{\omega}{\alpha+\beta+2}\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)'' + \left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)', \qquad \omega \in \mho,$$

if we set, $p(\omega) = \left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)'$, then the fuzzy differential superordination (2.3) becomes

$$F_{\ell(\mho)}\big(\ell(\omega)\big) \leq F_{(\mathbb{C}^2\times\mho)}\left(p(\omega) + \frac{1}{\alpha+\beta+2}\omega p'(\omega)\right), \qquad \omega \in \mho.$$

By using Theorem (1.7) for $\gamma = \alpha + \beta + 2$, we have

$$F_{q(\mho)}\big(q(z)\big) \leq F_{p(\mho)}\big(p(z)\big), i.e.\ q(z) = \frac{1}{\omega}\int_0^\omega \ell(t)\,dt \prec_F \left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)',$$

and q is the fuzzy best subordinant.

Theorem (2.4): Let ℓ be starlike in $\mho$ with $\ell(0) = 0$. Let $m \in \mathbb{N}$, $\alpha > 0, \beta \geq 0, \rho \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $s \in \mathbb{C}, f \in \mathcal{A}$ and suppose that $\frac{\omega^2}{\alpha+\beta+2}\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)'' + \omega\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)'$ is univalent and ${}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(z) \in \aleph[0,1] \cap Q$, then

$$\ell(\omega) \prec_F \frac{\omega^2}{\alpha+\beta+2}\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)'' + z\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)', \tag{2.4}$$

implies

$$F_{q(\mho)}\big(q(\omega)\big) \leq F_{\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f\right)_{(\mho)}} \left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)\right), \qquad \omega \in \mho,$$

where $q(\omega) = \int_0^\omega \ell(t)t^{-1}\,dt$, the function q is convex and it is the fuzzy best subordinant.

Proof. Using the properties of operator ${}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}$ Additionally, by differentiating with respect to ω, we obtain

$$\omega p'(\omega) = \frac{\omega^2}{\alpha+\beta+2}\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)'' + \omega\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)\right)',$$

then the fuzzy differential superordination (2.4) becomes

$$\ell(\omega) \prec_F \omega \wp'(\omega), \qquad \omega \in \mho.$$

By using Theorem (1.8), we have

$$F_{q(\mho)}\big(q(\omega)\big) \le F_{\wp(\mho)}\big(\wp(\omega)\big), i.e. F_{q(\mho)}\big(q(\omega)\big) \le F_{\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f\right)_{(\mho)}}\left({}_{\rho}^{s}\Upsilon_{\alpha}^{m-1} f(\omega)\right),$$

where $q(\omega) = \int_0^{\omega} \ell(t)t^{-1}\,dt$,The function q is convex and represents the optimal fuzzy subordinant.

Example 2.5: Let $\ell(\omega) = \frac{\omega}{(1-\omega)^2}$, with $h(0) = 0$ and $\ell'(\omega) = \frac{1+\omega}{(1-\omega)^3}$.

Since

$$Re\left(\frac{\omega\ell'(\omega)}{\ell(\omega)}\right) = Re\left(\frac{1+\omega}{1-\omega}\right) = Re(\left(\frac{1+r(\cos\theta + i\sin\theta)}{1-r(\cos\theta+i\sin\theta}\right)$$

$$= \frac{1-r^2}{1-2r\cos\theta + r^2} > 0, \text{where } r = |\omega| < 1, \theta \in \mathbb{R},$$

the function ℓ is starlike.

Let $f(\omega) = \omega + 2\omega^2, s = \lambda = 0, m = 1, n = 2$, we obtain

$${}_{\rho}^{0}\Upsilon_{0,0}^{1} f(\omega) = \omega + \frac{4}{3}\omega^2 \text{ and } {}_{\rho}^{0}\Upsilon_{0,0}^{2} f(z) = \omega + \frac{8}{9}\omega^2.$$

Using Theorem (2.1), we obtain

$$\frac{\omega}{(1-\omega)^2} \prec_F \omega + \frac{16}{9}\omega^2 \Rightarrow \frac{\omega}{(1-\omega)} \prec_F \omega + \frac{8}{9}\omega^2, \qquad \omega \in \mho.$$

Theorem 2.6: Let $\ell(\omega)$ be convex function with $\hbar(0) = 1$.Let $f \in \mathcal{A}$, $m \in \mathbb{N}$, $\alpha > 0, \beta \ge 0$, $\rho \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $s \in \mathbb{C}$ and suppose that

$$\left(\frac{\omega\,{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}\right)' \text{ is univalent and } \frac{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)} \in \aleph[1,1] \cap Q. \text{If}$$

$$\ell(\omega) \prec_F \left(\frac{\omega\,{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}\right)', \tag{2.5}$$

then

$$q(\omega) \prec_F \left(\frac{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}\right), \quad \omega \in \mho,$$

where $q(\omega) = \frac{1}{\omega}\int_0^{\omega} \ell(t)\, dt$. The function q is convex and it is the fuzzy best subordinant.

Proof. Define the function by

$$p(\omega) = \frac{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}, \qquad \omega \in \mho. \tag{2.6}$$

Then the function p is analytic in $\mho$ and $p(0) = 1$. Differentiating (2.6) with respect to ω, we have

$$p(\omega) + zp'(\omega) = \left(\frac{z\,{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}\right)', \qquad \omega \in \mho.$$

Consequently, the fuzzy differential superordination given in equation (2.5) transforms into:

$$F_{\ell(\mho)}\big(\hbar(\omega)\big) \le F_{(\mathbb{C}^2\times\mho)}\big(p(\omega) + \omega p'(\omega)\big), \qquad \omega \in \mho.$$

By using the theorem (1.6) for $\gamma = 1$, we obtain

$$F_{q(\mho)}\big(q(\omega)\big) \le F_{p(\mho)}\big(p(\omega)\big), i.e. F_{q(\mho)}\big(q(\omega)\big) \le F_{\left(\frac{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}\right)(\mho)} \left(\frac{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}\right),$$

where $q(\omega) = \frac{1}{\omega}\int_0^{\omega} \hbar(t)\, dt$. The function q is convex and serves as the optimal fuzzy subordinant.

Upon setting $\ell(\omega) = e^{\tau\omega}, |\tau| \le 1$ in Theorem (2.6), As a consequence, we obtain the following result.

Corollary 2.7: Let $f \in \mathcal{A},\ m \in \mathbb{N}, \alpha > 0,\ \beta \ge 0, \rho \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ s \in \mathbb{C}$ and suppose that

$$\left(\frac{\omega\,{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}\right)' \text{ is univalent and } \frac{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)} \in \aleph[1,1] \cap Q. \text{ If}$$

$$e^{\tau\omega} \prec_F \left(\frac{\omega\,{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}\right)', \qquad \omega \in \mho, |\tau| \le 1, \tag{2.7}$$

then

$$\frac{e^{\tau\omega} - 1}{\tau\omega} \prec_F \left(\frac{\omega\,{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m-1} f(\omega)}{{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m} f(\omega)}\right), \quad \omega \in \mho.$$

The function $\frac{e^{\tau\omega}-1}{\tau\omega}$ It is convex and serves as the optimal fuzzy subordinate.

Proof. We have $\ell(\omega) = e^{\tau\omega}$, with $\ell(0) = 1,\ \ell'(\omega) = \tau e^{\tau\omega}$, and $\ell''(\omega) = \tau^2 e^{\tau\omega}$.

Therefor

$$Re\left(\frac{\omega\ell''(\omega)}{\ell'(\omega)}+1\right)=Re(\tau\omega+1)>0,$$

the function h is convex.

By following the same steps as in the proof of Theorem (2.6) and considering

$$p(\omega)=\frac{{}^{s}_{\rho}\Upsilon^{m-1}_{\beta,\alpha}f(\omega)}{{}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)}.$$

The fuzzy differential superordination (2.7) becomes

$$e^{\tau\omega}\prec_F\ p(\omega)+\omega p'(\omega),\qquad \omega\in\mho.$$

By applying Theorem (1.6) for $\gamma=1$, we have

$$F_{q(\mho)}\big(q(\omega)\big)\le F_{p(\mho)}\big(p(\omega)\big),i.e.$$

$$F_{q(\mho)}\big(q(\omega)\big)\le F_{\left(\frac{{}^{s}_{\rho}\Upsilon^{m-1}_{\beta,\alpha}f(\omega)}{{}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)}\right)(\mho)}\left(\frac{{}^{s}_{\rho}\Upsilon^{m-1}_{\beta,\alpha}f(\omega)}{{}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)}\right),$$

and

$$q(\omega)=\frac{1}{\omega}\int_0^{\omega}e^{\tau t}\,dt=\frac{e^{\tau\omega}-1}{\tau\omega},\qquad \omega\in\mho.$$

Theorem 2.8: Let $\ell(\omega)$ be convex function with $\ell(0)=1$.Let

$$f\in\mathcal{A},\qquad m\in\mathbb{N},\ \alpha>0,\beta\ge 0,\rho\in\mathbb{C}\backslash\mathbb{Z}_0^-,\qquad s\in\mathbb{C},$$

$$F(\omega)=\frac{c+2}{\omega^{c+1}}\int_0^{\omega}t^c\ell(t)\,dt,\mathrm{Re}(c)>-2,\ \text{and}$$

suppose that $\left({}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)\right)'$ is univalent in $\mho$, $\left({}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}F(\omega)\right)'\in\aleph[1,1]\cap Q$ and

$$\ell(\omega)\prec_F\left({}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}f(\omega)\right)',\qquad \omega\in\mho,\tag{2.8}$$

then

$$q(\omega)\prec_F\left({}^{s}_{\rho}\Upsilon^{m}_{\beta,\alpha}F(\omega)\right)',\quad \omega\in\mho,$$

where $q(\omega)=\frac{c+2}{\omega^{c+2}}\int_0^{z}t^{c+1}\ell(t)\,dt$. The function q is convex and serves as the optimal fuzzy subordinant.

Proof. we have $\omega^{c+1}F(\omega)=(c+2)\int_0^{\omega}t^c\ell(t)\,dt$ and differentiating it, with respect to ω, we obtain $(c+1)F(\omega)+\omega F'(\omega)=(c+2)f(\omega)$ and

$$(c+1)\,{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}F(\omega)+\omega\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}F(\omega)\right)'=(c+2)\,{}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}f(\omega),\qquad \omega\in\mho.$$

Differentiating the last relation with respect to ω,

$$\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}F(\omega)\right)'+\frac{1}{(c+2)}\,\omega\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}F(\omega)\right)''=\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}f(\omega)\right)',\qquad \omega\in\mho. \qquad (2.9)$$

Using (2.9), the fuzzy superordination (2.8) becomes

$$\ell(\omega)\prec_{F}\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}F(\omega)\right)'+\frac{1}{(c+2)}\,\omega\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}F(\omega)\right)''.$$

Now, if we set $p(\omega)=\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}f(\omega)\right)'$, we obtain

$$\ell(\omega)\prec_{F} p(\omega)+\frac{1}{(c+2)}\omega p'(\omega),\qquad \omega\in\mho.$$

Using Theorem (1.6) for $\gamma=(c+2)$, we have

$$F_{q(\mho)}\big(q(\omega)\big)\leq F_{p(\mho)}\big(p(\omega)\big)\,, i.e.\, F_{q(\mho)}\big(q(\omega)\big)\leq F_{\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}f\right)'(\mho)}\left({}_{\rho}^{s}\Upsilon_{\beta,\alpha}^{m}f(\omega)\right)',$$

where $q(\omega)=\frac{c+2}{\omega^{c+2}}\int_0^{\omega}t^{c+1}h(t)\,dt$. The function q is convex and serves as the optimal fuzzy subordinate.

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