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On Subclasses of Bi-Univalent Functions Using Quasi-Subordination

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ABSTRACT

In this paper , we introduced new subclasses $\Re\mathbb{A}(\mathfrak{E},\rho,\delta,\pi)$ and $\Re\mathbb{A}^*(\eta,\gamma,\pi)$ of bi-univalent functions defined in the open unit disk $\mathfrak{U}$. As we get upper bounds for the first two Taylor-Maclaurin $|a_2|$ and $|a_3|$. Some new corollaries are obtained for these subclasses

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1. Introduction

Let $\mathcal{A}$ denote the class of all normalized analytic functions in the open unit disk $\mathfrak{U}=\{\mathfrak{z}:|\mathfrak{z}|<1\}$ which the shape

$$f(\mathfrak{z})=\mathfrak{z}+\sum_{n=2}^{\infty}a_n\mathfrak{z}^n \qquad (\mathfrak{z}\in\mathfrak{U}), \tag{1.1}$$

and let S be the class of all functions form $\mathcal{A}$ which are univalent in $\mathfrak{U}$. According to the Koebe One Quarter Theorem [8], for every $f\in S$ the inverse f^{-1} which satisfies: $f^{-1}(f(\mathfrak{z}))=\mathfrak{z},(\mathfrak{z}\in\mathfrak{U})$ *and* $f^{-1}(f(w))=w,\left(|w|<r_0(f),\ r_0(f)\geq\frac{1}{4}\right)$, where $q\geq\frac{1}{4}$ denoted the radius of the image $f(\mathfrak{U})$. In fact the inverse function f^{-1} is given by

$$f^{-1}(w)=g(w)=w-a_2w^2+(2a_2^2-a_3)w^2-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.2}$$

By means of subordination, Ma and Minda [12] introduced the following classes :

$$S^*(\psi)=\left\{f\in\mathcal{A}:\frac{\mathfrak{z}f'(\mathfrak{z})}{f(\mathfrak{z})}\prec\psi(\mathfrak{z})\right\},$$

where $\psi(0)=1,\psi'(0)>0$, and ψ is an analytic function having a positive real portion on $\mathfrak{U}$, mapping the unit disc $\mathfrak{U}$ onto a starlike region with respect to 1, which is symmetric about the real axis. A function $f\in S^*(\psi)$ is referred to as Ma-Minda starlike. $\mathrm{C}(\psi)$ denotes the class of convex functions $f\in\mathcal{A}$ such that

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$$1+\frac{zf''(z)}{f'(z)} \prec \psi(z).$$

As special case of the classes $S^*(\psi)$ *and* $C(\psi)$ several well- known subclasses of starlike and convex function . By Robertson [19] in 1970 the concept of subordination is generalized through introduction a new concept of quasi-subordination. For two analytic functions f and ψ, the function f is quasi-subordination to ψ written as

$$f(z) \prec_q \psi(z) \quad (z \in \mathfrak{A}), \tag{1.3}$$

if there exist analytic functions $h(z)$ and F, with $|h(z)| \le 1, \mathrm{F}(0)=0$ and $|\mathrm{F}(z)| < 1$, such that

$$\frac{f(z)}{h(z)} \prec \psi(z) \equiv f(z) = h(z)\psi(\mathrm{F}(z)) \quad (z \in \mathfrak{A}).$$

If $h(z)=1$, then $f(z)=\psi(\mathrm{F}(z))$, so that $f(z) \prec \psi(z)$in $\mathfrak{A}$, also if $\mathrm{F}(z)=z$, then $f(z)=h(z)\psi(z)$, it is said that $f(z)$ is majorized by $\psi(z)$ and written as $f(z) \ll \psi(z)$ in $\mathfrak{A}$. Therefore, it is evident that quasi-subordination is a generalisation of both conventional subordination and majorization. The research on quasi-subordination is comprehensive and encompasses current studies [1,2,11,13], while further references may be found in [20,21,24,25].

In 1967, Lewin [11] examined class $\mathcal{X}$ of bi-univalent functions and established the constraint for the second coefficient a_2. Brannan and Taha [6] examined specific subclasses of bi-univalent functions analogous to the well-known subclasses of univalent functions, which include starlike, extremely starlike, and convex functions. Obtained non sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ and they introduced the bi-starlike function, bi-convex function classes. Recently Ali et al. [2], Deniz [7], Peng et al. [16], Ramchandran et al, Tang et al. [23]. [18], Murugusundaramoorthy et al. [14] etc. have introduced and investigated Ma-Minda type subclasses of bi-univalent functions class $\mathcal{X}$. Additional generalizations of Ma-Minda type subclasses of the class have been conducted by numerous authors, including ([4,13,15,22,26]), through the use of quasi-subordination. Inspired by research in [3,5,9,10] about quasi-subordination, we present and examine specific new subclasses of class $\mathcal{X}$. It follows that $\psi(z)$ is analytic in A having $\psi(0) = 1$ and let

$$\psi(z) = \mathcal{D}_0 + \mathcal{D}_1 z + \mathcal{D}_2 z^2 + \cdots, \qquad (|\psi(z)| \le 1, z \in \mathfrak{A}) \tag{1.4}$$

and

$$\pi(z) = 1 + \mathcal{K}_1 z + \mathcal{K}_2 z^2 + \cdots, \qquad (\mathcal{K}_1 \in \mathcal{R}^+). \tag{1.5}$$

Lemma 1.1. [17] **:** Let $p \in \mathcal{P}$ be a family of all functions p analytic in $\mathfrak{A}$, for which $Re\{p(z)\} > 0$ and have the form:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \text{ for } z \in \mathfrak{A}, \text{ then } |p_n| \le n, \text{ for each } n.$$

2. Main Results

Definition2.1: Let $0 \le \rho \le 1$, $0 \le \delta \le 1$ and $\mathfrak{E} \in \mathbb{C} \setminus \{0\}$, a function $f \in \mathcal{X}$ is said to be in the class $\Re\mathbb{A}(\mathfrak{E}, \rho, \delta, \pi)$, if the following two conditions are satisfied:

$$1 + \frac{1}{\mathfrak{E}}\left[\left(\frac{zf'(z) + (1-\rho)z^2 f''(z)}{(\delta+\rho)f(z) + (1-\rho)zf'(z) - \delta z} - 1\right)\right] \prec_q (\pi(z) - 1)$$

and

$$1 + \frac{1}{\mathfrak{E}}\left[\left(\frac{wg'(w) + (1-\rho)w^2 g''(w)}{(\delta+\rho)g(w) + (1-\rho)wg'(z) - \delta w} - 1\right)\right] \prec_q (\pi(w) - 1),$$

where $g = f^{-1}$ *and* π is given by (1.5) and $z, w \in \mathfrak{A}$.

Remark(2.1): We put $\mathfrak{E} = 1, \rho = 0, \delta = 0$, if a function $f \in \mathcal{A}$ is of the kind (1.1) and belongs to the class $\mathfrak{RA}(\mathfrak{E}, \rho, \delta, \pi)$, then

$$\left[\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec_q (\pi(z) - 1)$$

and

$$\left[\left(1 + \frac{wg''(w)}{g'(z)}\right)\right] \prec_q (\pi(w) - 1).$$

Remark(2.2): We put $\rho = 0, \delta = 0$, if a function $f \in \mathcal{A}$ is of the kind (1.1) and belongs to the class $\mathfrak{RA}(\mathfrak{E}, \rho, \delta, \pi)$, then

$$1 + \frac{1}{\mathfrak{E}}\left[\left(\frac{zf''(z)}{f'(z)}\right)\right] \prec_q (\pi(z) - 1)$$

and

$$1 + \frac{1}{\mathfrak{E}}\left[\left(\frac{wg''(w)}{g'(z)}\right)\right] \prec_q (\pi(w) - 1).$$

Remark(2.3): We put $\mathfrak{E} = 1, \rho = 0 , \delta = 1$, if a function $f \in \mathcal{A}$ is of the kind (1.1) and belongs to the class $\mathfrak{RA}(\mathfrak{E}, \rho, \delta, \pi)$, then

$$\left[\left(\frac{zf'(z) + z^2 f''(z)}{f(z) + zf'(z) - z}\right)\right] \prec_q (\pi(z) - 1)$$

and

$$\left[\left(\frac{wg'(w) + w^2 g''(w)}{g(w) + wg'(z) - w}\right)\right] \prec_q (\pi(w) - 1).$$

Remark(2.4): We put $\mathfrak{E} = 1, \rho = 1 , \delta = 0$, if a function $f \in \mathcal{A}$ is of the kind (1.1) and belongs to the class $\mathfrak{RA}(\mathfrak{E}, \rho, \delta, \pi)$, then

$$\left[\left(\frac{zf'(z)}{f(z)}\right)\right] \prec_q (\pi(z) - 1)$$

and

$$\left[\left(\frac{wg'(w)}{g(w)}\right)\right] \prec_q (\pi(w) - 1).$$

Remark(2.5): We put $\rho = 1 , \delta = 0$, if a function $f \in \mathcal{A}$ is of the kind (1.1) and belongs to the class $\mathfrak{RA}(\mathfrak{E}, \rho, \delta, \pi)$, then

$$1 + \frac{1}{\mathfrak{E}}\left[\left(\frac{zf'(z)}{f(z)}\right) - 1\right] \prec_q (\pi(z) - 1)$$

and

$$1 + \frac{1}{\mathfrak{E}}\left[\left(\frac{wg'(w)}{g(w)}\right) - 1\right] \prec_q (\pi(w) - 1).$$

Theorem 2.1 : For $0 \le \rho \le 1$, $0 \le \delta \le 1$ with $\mathfrak{E} \in \mathbb{C} \setminus \{0\}$. If $f \in \mathcal{A}$ is of the kind (1.1) and belongs to the class $\mathfrak{RA}(\mathfrak{E}, \rho, \delta, \pi)$, then

$$|a_2| \le min\left\{\frac{|\mathcal{D}_0\mathfrak{E}|\mathcal{K}_1}{2(2-\rho-\delta)}, \sqrt{\frac{|\mathcal{D}_0\mathfrak{E}|(\mathcal{K}_1 + |\mathcal{K}_2 - \mathcal{K}_1|)}{2+\delta^2-\delta-\rho^2}}\right\} \quad (2.1)$$

and

$$|a_3| \leq min\left\{\frac{2|\mathcal{D}_0\mathfrak{E}|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4+2\delta^2-2\delta-2\rho^2} + \frac{\mathfrak{E}(\mathcal{D}_1\mathcal{K}_1+2\mathcal{D}_0\mathcal{K}_1}{6-4\rho-\delta}, \frac{\mathfrak{E}}{6-4\rho-\delta}\left[\frac{2-\rho+\delta}{(2-\rho-\delta)}|\mathcal{D}_0|^2{\mathcal{K}_1}^2+\mathcal{K}_1|\mathcal{D}_1+\mathcal{D}_0| + |\mathcal{D}_0||\mathcal{K}_2-\mathcal{K}_1|\right]\right\}. \tag{2.2}$$

Proof: Let $f \in \Re\mathbb{A}(\mathfrak{E},\rho,\delta,\pi)$, there exist two analytic functions $r, h: \mathfrak{U} \to \mathfrak{U}$ $with$ $r(0) = h(0) = 0, |h(z)| < 1$and $|r(z)| < 1$, the function ψ defined by (1.4) satisfies:

$$1+\frac{1}{\mathfrak{E}}\left[\left(\frac{zf'(z)+(1-\rho)z^2f''(z)}{(\delta+\rho)f(z)+(1-\rho)zf'(z)-\delta z}-1\right)\right] = \psi(z)(\pi(r(z)-1) \tag{2.3}$$

and

$$1+\frac{1}{\mathfrak{E}}\left[\left(\frac{wg'(w)+(1-\rho)w^2g''(w)}{(\delta+\rho)g(w)+(1-\rho)wg'(z)-\delta w}-1\right)\right] \prec_q \psi(w)(\pi(h(w)-1)\,. \tag{2.4}$$

Define the functions $\mathfrak{p}_1$ and $\mathfrak{p}_2$ by:

$$\mathfrak{p}_1(z) = \frac{1+r(z)}{1-r(z)} = 1+\mathcal{F}_1 z+\mathcal{F}_2 z^2+\cdots, \tag{2.5}$$

$$\mathfrak{p}_2(w) = \frac{1+h(w)}{1-h(w)} = 1+j_1 w+j_2 w^2+\cdots, \tag{2.6}$$

which are equivalently

$$r(z) = \frac{\mathfrak{p}_1(z)-1}{\mathfrak{p}_1(z)+1} = \frac{1}{2}\left[\mathcal{F}_1 z+\left(\mathcal{F}_2-\frac{{\mathcal{F}_1}^2}{2}\right)z^2+\cdots\right] \tag{2.7}$$

and

$$h(w) = \frac{\mathfrak{p}_2(w)-1}{\mathfrak{p}_2(w)+1} = \frac{1}{2}\left[j_1 w+\left(j_2-\frac{{j_1}^2}{2}\right)w^2+\cdots\right]. \tag{2.8}$$

It is evident that $\mathfrak{p}_1(z)$, $\mathfrak{p}_2(z)$ are analytic and possess positive real portions in $\mathfrak{U}$. Considering (2.3), (2.4), (2.7), and (2.8), it is evident

$$1+\frac{1}{\mathfrak{E}}\left[\left(\frac{zf'(z)+(1-\rho)z^2f''(z)}{(\delta+\rho)f(z)+(1-\rho)zf'(z)-\delta z}-1\right)\right] = \psi(z)\left(\pi\left(\left(\frac{\mathfrak{p}_1(z)-1}{\mathfrak{p}_2(z)+1}\right)-1\right)\right) \tag{2.9}$$

and

$$1+\frac{1}{\mathfrak{E}}\left[\left(\frac{wg'(w)+(1-\rho)w^2g''(w)}{(\delta+\rho)g(w)+(1-\rho)wg'(z)-\delta w}-1\right)\right] \prec_q \psi(w)\left(\pi\left(\frac{\mathfrak{p}_1(w)-1}{\mathfrak{p}_2(w)+1}\right)-1\right). \tag{2.10}$$

The series expansions for $f(z)$ and $g(w)$ as shown in (1.1) with (1.2) respectively, give us

$$1+\frac{1}{\mathfrak{E}}\left[\left(\frac{zf'(z)+(1-\rho)z^2f''(z)}{(\delta+\rho)f(z)+(1-\rho)zf'(z)-\delta z}-1\right)\right]$$
$$=1+\left(\frac{2-\rho-\delta}{\mathfrak{E}}\right)a_2z+\left[\left(\frac{6-4\rho-\delta}{\mathfrak{E}}\right)a_3-\frac{((2-\rho)^2-\delta^2)}{\mathfrak{E}}a_2^2\right]z^2+\cdots \quad (2.11)$$

and

$$1+\frac{1}{\mathfrak{E}}\left[\left(\frac{wg'(w)+(1-\rho)w^2g''(w)}{(\delta+\rho)g(w)+(1-\rho)wg'(z)-\delta w}-1\right)\right]$$
$$=\left(\frac{\rho+\delta-2}{\mathfrak{E}}\right)a_2w+\left[\left(\frac{-((2-\rho)^2-\delta^2)}{\mathfrak{E}}+\frac{12-8\rho-2\delta}{\mathfrak{E}}\right)a_2^2-\left(\frac{6-4\rho-\delta}{\mathfrak{E}}\right)a_3\right]w^2+\cdots. \quad (2.12)$$

Using (2.5) and (2.6) together with (1.4) and (1.5), we get

$$\psi(z)\left(\pi\left(\frac{p_1(z)-1}{p_2(z)+1}\right)-1\right)=\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\mathcal{F}_1z+\left[\frac{1}{2}\mathcal{D}_1\mathcal{K}_1\mathcal{F}_1+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\left(\mathcal{F}_2-\frac{\mathcal{F}_1{}^2}{2}\right)+\frac{\mathcal{D}_0\mathcal{K}_2\mathcal{F}_1{}^2}{4}\right]z^2+\cdots \quad (2.13)$$

and

$$\psi(w)\left(\pi\left(\frac{p_2(w)-1}{p_2(w)+1}\right)-1\right)$$
$$=\frac{1}{2}\mathcal{D}_0\mathcal{K}_1j_1w+\left[\frac{1}{2}\mathcal{D}_1\mathcal{K}_1j_1+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\left(j_2-\frac{j_1{}^2}{2}\right)+\frac{\mathcal{D}_0\mathcal{K}_2j_1{}^2}{4}\right]w^2+\cdots. \quad (2.14)$$

Now equating (2.11) and (2.13) and comparing the coefficients to z *and* z^2, we obtain

$$\left(\frac{2-\rho-\delta}{\mathfrak{E}}\right)a_2=\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\mathcal{F}_1 \quad (2.15)$$

and

$$\left[\left(\frac{6-4\rho-\delta}{\mathfrak{E}}\right)a_3-\frac{((2-\rho)^2-\delta^2)}{\mathfrak{E}}a_2^2\right]$$
$$=\frac{1}{2}\mathcal{D}_1\mathcal{K}_1\mathcal{F}_1+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\left(\mathcal{F}_2-\frac{\mathcal{F}_1{}^2}{2}\right)+\frac{\mathcal{D}_0\mathcal{K}_2\mathcal{F}_1{}^2}{4}. \quad (2.16)$$

Similarly (2.12) and (2.14), gives us

$$\left(\frac{\rho+\delta-2}{\mathfrak{E}}\right)a_2=\frac{1}{2}\mathcal{D}_0\mathcal{K}_1j_1 \quad (2.17)$$

and

$$\left[\left(\frac{-((2-\rho)^2-\delta^2)}{\mathfrak{E}}+\frac{12-8\rho-2\delta}{\mathfrak{E}}\right)a_2^2-\left(\frac{6-4\rho-\delta}{\mathfrak{E}}\right)a_3\right]=\frac{1}{2}\mathcal{D}_1\mathcal{K}_1j_1+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\left(j_2-\frac{j_1{}^2}{2}\right)+\frac{\mathcal{D}_0\mathcal{K}_2j_1{}^2}{4}. \quad (2.18)$$

From (2.15) and (2.17), we find that

$$a_2=\frac{\mathcal{D}_0\mathcal{K}_1\ell_1\mathfrak{E}}{2(2-\rho-\delta)}=-\left(\frac{\mathcal{D}_0\mathcal{K}_1j_1\mathfrak{E}}{2(2-\rho-\delta)}\right), \quad (2.19)$$

which implies

$$|a_2| \leq \frac{|\mathcal{D}_0\mathfrak{E}|\mathcal{K}_1}{2(2-\rho-\delta)}. \tag{2.20}$$

Adding (2.16) and (2.18), we obtain

$$\left(\frac{4+2\delta^2-2\delta-2\rho^2}{\mathfrak{E}}\right){a_2}^2 = \frac{1}{2}\mathcal{D}_0\mathcal{K}_1(\mathcal{F}_2+j_2)+\frac{\mathcal{D}_0\mathcal{K}_2}{4}\left({\mathcal{F}_1}^2+{j_1}^2\right)|\mathcal{K}_2-\mathcal{K}_1|, \tag{2.21}$$

which implies

$$|a_2|^2 \leq \frac{2|\mathcal{D}_0\mathfrak{E}|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4+2\delta^2-2\delta-2\rho^2}. \tag{2.22}$$

Subsequently, to get the upper bound for $|a_3|$, we subtract (2.16) from (2.18), yielding

$$\left(\frac{2(6-4\rho-\delta)}{\mathfrak{E}}\right)a_3 = \left(\frac{2(6-4\rho-\delta)}{\mathfrak{E}}\right)a_2^2+\frac{1}{2}\mathcal{D}_1\mathcal{K}_1(\mathcal{F}_1-j_1)+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1(\mathcal{F}_2-j_2), \tag{2.23}$$

by using Lemma 1.1 and (2.21) in (2.23), we obtain

$$|a_3| \leq \frac{2|\mathcal{D}_0\mathfrak{E}|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4+2\delta^2-2\delta-2\rho^2}+\frac{\mathfrak{E}(\mathcal{D}_1\mathcal{K}_1+2\mathcal{D}_0\mathcal{K}_1}{6-4\rho-\delta}\ . \tag{2.24}$$

Next, from (2.15) and (2.16), we have

$$\frac{6-4\rho-\delta}{\mathfrak{E}}a_3 = \frac{2-\rho+\delta}{4(2-\rho-\delta)}{\mathcal{D}_0}^2{\mathcal{K}_1}^2{\mathcal{F}_1}^2+\frac{1}{2}\mathcal{D}_1\mathcal{K}_1\mathcal{F}_1+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\mathcal{F}_2-\frac{1}{4}\mathcal{D}_0\mathcal{K}_1{\mathcal{F}_1}^2+\frac{1}{4}\mathcal{D}_0\mathcal{K}_2{\mathcal{F}_1}^2,$$

which implies

$$|a_3| \leq \frac{\mathfrak{E}}{6-4\rho-\delta}\left[\frac{2-\rho+\delta}{(2-\rho-\delta)}|\mathcal{D}_0|^2{\mathcal{K}_1}^2+\mathcal{K}_1|\mathcal{D}_1+\mathcal{D}_0|+|\mathcal{D}_0||\mathcal{K}_2-\mathcal{K}_1|\right]. \tag{2.25}$$

We have the special cases below are also previously uninhibited and they are as follows:

In virtue of Remark (2.1)-(2.5) we obtain some corollaries below

Corollary(2.1) : Assume f is given by (1.1) belonging to the class $\Re\mathbb{A}(1,0,0,\pi)$, then

$$|a_2| \leq \min\left\{\frac{|\mathcal{D}_0|\mathcal{K}_1}{4},\sqrt{\frac{2|\mathcal{D}_0|(4\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{2|\mathcal{D}_0|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4}+\frac{(\mathcal{D}_1\mathcal{K}_1+2\mathcal{D}_0\mathcal{K}_1}{6},\frac{1}{6}\left[|\mathcal{D}_0|^2{\mathcal{K}_1}^2+\mathcal{K}_1|\mathcal{D}_1+\mathcal{D}_0|+|\mathcal{D}_0||\mathcal{K}_2-\mathcal{K}_1|\right]\right\}.$$

Corollary(2.2): Assume f is given by (1.1) belonging to the class $\Re\mathbb{A}(\mathfrak{E},0,0,\pi)$, then

$$|a_2| \leq \min\left\{\frac{|\mathcal{D}_0\mathfrak{E}|\mathcal{K}_1}{4},\sqrt{\frac{2|\mathcal{D}_0\mathfrak{E}|(4\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{2|\mathcal{D}_0\mathfrak{E}|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4}+\frac{\mathfrak{E}(\mathcal{D}_1\mathcal{K}_1+2\mathcal{D}_0\mathcal{K}_1}{6},\frac{\mathfrak{E}}{6}\left[|\mathcal{D}_0|^2{\mathcal{K}_1}^2+\mathcal{K}_1|\mathcal{D}_1+\mathcal{D}_0|+|\mathcal{D}_0||\mathcal{K}_2-\mathcal{K}_1|\right]\right\}.$$

Corollary(2.3): Assume f is given by (1.1) belonging to the class $\Re\mathbb{A}(1,0,1,\pi)$, then

$$|a_2| \le \min\left\{\frac{|\mathcal{D}_0|\mathcal{K}_1}{2}, \sqrt{\frac{2|\mathcal{D}_0|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4}}\right\}$$

and

$$|a_3| \le \min\left\{\frac{|\mathcal{D}_0|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4} + \frac{(\mathcal{D}_1\mathcal{K}_1+2\mathcal{D}_0\mathcal{K}_1}{5}, \frac{1}{5}\left[3|\mathcal{D}_0|^2{\mathcal{K}_1}^2+\mathcal{K}_1|\mathcal{D}_1+\mathcal{D}_0|+|\mathcal{D}_0||\mathcal{K}_2-\mathcal{K}_1|\right]\right\}.$$

Corollary(2.4): Assume f is given by (1.1) belonging to the class $\Re\mathbb{A}(1,1,0,\pi)$, then

$$|a_2| \le \min\left\{\frac{|\mathcal{D}_0|\mathcal{K}_1}{2}, \sqrt{\frac{2|\mathcal{D}_0|(4\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{2}}\right\}$$

and

$$|a_3| \le \min\left\{\frac{2|\mathcal{D}_0|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{2} + \frac{(\mathcal{D}_1\mathcal{K}_1+2\mathcal{D}_0\mathcal{K}_1)}{2}, \frac{1}{2}\left[|\mathcal{D}_0|^2{\mathcal{K}_1}^2+\mathcal{K}_1|\mathcal{D}_1+\mathcal{D}_0|+|\mathcal{D}_0||\mathcal{K}_2-\mathcal{K}_1|\right]\right\}.$$

Corollary(2.5): Assume f is given by (1.1) belonging to the class $\Re\mathbb{A}(\mathfrak{E},1,0,\pi)$, then

$$|a_2| \le \min\left\{\frac{|\mathcal{D}_0\mathfrak{E}|\mathcal{K}_1}{2}, \sqrt{\frac{2|\mathcal{D}_0\mathfrak{E}|(4\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{2}}\right\}$$

and

$$|a_3| \le \min\left\{\frac{2|\mathcal{D}_0\mathfrak{E}|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{2} + \frac{\mathfrak{E}(\mathcal{D}_1\mathcal{K}_1+2\mathcal{D}_0\mathcal{K}_1}{2}, \frac{\mathfrak{E}}{2}\left[|\mathcal{D}_0|^2{\mathcal{K}_1}^2+\mathcal{K}_1|\mathcal{D}_1+\mathcal{D}_0|+|\mathcal{D}_0||\mathcal{K}_2-\mathcal{K}_1|\right]\right\}.$$

Definition 2.2: Let $0 \le \rho \le 1$, $0 \le \delta \le 1$ and $\mathfrak{E} \in \mathbb{C} \setminus \{0\}$, a function $f \in \mathcal{X}$ is said to be in the class $\Re\mathbb{A}^*(\eta,\gamma,\pi)$, if the following two conditions are satisfied:

$$(1-\eta)\left[\left(1+\frac{zf''(z)}{f'(z)}\right)-\gamma\left(\frac{zf'(z)}{f(z)}+\frac{f(z)}{z}-2\right)\right]+\eta\left[\frac{z^2f''(z)+(1-\gamma)zf'(z)+\gamma z}{zf'(z)}\right] \prec_q (\pi(z)-1)$$

and

$$(1-\eta)\left[\left(1+\frac{wg''(w)}{g'(w)}\right)-\gamma\left(\frac{wg'(w)}{g(w)}+\frac{g(w)}{w}-2\right)\right]+\eta\left[\frac{w^2g''(w)+(1-\gamma)zw'(z)+\gamma w}{wg'(w)}\right] \prec_q (\pi(w)-1)$$

where $g = f^{-1}$ *and* π is given by (1.5) and $z, w \in \mathfrak{U}$.

Remark(2.6): we put $\eta = 0$, if a function $f \in \mathcal{A}$ of the form (1.1) belonging to the class $\Re\mathbb{A}^*(\eta,\gamma,\pi)$, then

$$\left[\left(1+\frac{zf''(z)}{f'(z)}\right)-\gamma\left(\frac{zf'(z)}{f(z)}+\frac{f(z)}{z}-2\right)\right] \prec_q (\pi(z)-1)$$

and

$$\left[\left(1+\frac{wg''(w)}{g'(w)}\right)-\gamma\left(\frac{wg'(w)}{g(w)}+\frac{g(w)}{w}-2\right)\right] \prec_q (\pi(w)-1).$$

Remark(2.7): we put $\eta = 1$, if a function $f \in \mathcal{A}$ of the form (1.1) belonging to the class $\Re\mathbb{A}^*(\eta,\gamma,\pi)$, then

$$\left[\frac{z^2f''(z)+(1-\gamma)zf'(z)+\gamma z}{zf'(z)}\right] \prec_q (\pi(z)-1)$$

and

$$\left[\frac{w^2 g''(w) + (1-\gamma)zw'(z) + \gamma w}{wg'(w)}\right] \prec_q (\pi(w) - 1).$$

Theorem 2.2 : For $0 \le \gamma \le 1$, $\eta \le 0$. If $f \in \mathcal{A}$ of the form (1.1) belonging to the class $\Re\mathbb{A}^*(\eta, \gamma, \pi)$, then

$$|a_2| \le min\left\{\frac{|\mathcal{D}_0|\mathcal{K}_1}{2(2-2\gamma)}, \sqrt{\frac{2|\mathcal{D}_0|(\mathcal{K}_1 + |\mathcal{K}_2 - \mathcal{K}_1|)}{4 - 4\gamma + 6\eta\gamma}}\right\} \tag{2.26}$$

and

$$|a_3| \le min\left\{\frac{2|\mathcal{D}_0|(\mathcal{K}_1 + |\mathcal{K}_2 - \mathcal{K}_1|)}{4 - 4\gamma + 6\eta\gamma} + \frac{(\mathcal{D}_1\mathcal{K}_1 + 2\mathcal{D}_0\mathcal{K}_1)}{6 - 3\gamma}, \frac{1}{6-3\gamma}\left[\frac{4 - \gamma - 3\eta\gamma}{4(2-2\gamma)^2}|\mathcal{D}_0|^2\mathcal{K}_1{}^2 + \mathcal{K}_1|\mathcal{D}_1 + \mathcal{D}_0| + |\mathcal{D}_0||\mathcal{K}_2 - \mathcal{K}_1|\right]\right\}. \tag{2.27}$$

Proof: Let $f \in \Re\mathbb{A}^*(\eta, \gamma, \pi)$, there exist two analytic functions $r, h: \mathfrak{A} \to \mathfrak{A}$ $with$ $r(0) = h(0) = 0, |h(z)| < 1 and$ $|r(z)| < 1$, the function ψ defined by (1.4) satisfies:

$$(1-\eta)\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) - \gamma\left(\frac{zf'(z)}{f(z)} + \frac{f(z)}{z} - 2\right)\right] + \eta\left[\frac{z^2 f''(z) + (1-\gamma)zf'(z) + \gamma z}{zf'(z)}\right] = \psi(z)(\pi(r(z) - 1) \tag{2.28}$$

and

$$(1-\eta)\left[\left(1 + \frac{wg''(w)}{g'(w)}\right) - \gamma\left(\frac{wg'(w)}{g(w)} + \frac{g(w)}{w} - 2\right)\right] + \eta\left[\frac{w^2 g''(w) + (1-\gamma)zw'(z) + \gamma w}{wg'(w)}\right] \prec_q \psi(w)(\pi(h(w) - 1). \tag{2.29}$$

Define the functions $\mathfrak{p}_1$ and $\mathfrak{p}_2$ by:

$$\mathfrak{p}_1(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + \mathcal{F}_1 z + \mathcal{F}_2 z^2 + \cdots, \tag{2.30}$$

$$\mathfrak{p}_2(w) = \frac{1 + h(w)}{1 - h(w)} = 1 + j_1 w + j_2 w^2 + \cdots, \tag{2.31}$$

which are equivalently

$$r(z) = \frac{\mathfrak{p}_1(z) - 1}{\mathfrak{p}_1(z) + 1} = \frac{1}{2}\left[\mathcal{F}_1 z + \left(\mathcal{F}_2 - \frac{\mathcal{F}_1{}^2}{2}\right)z^2 + \cdots\right] \tag{2.32}$$

and

$$h(w) = \frac{\mathfrak{p}_2(w) - 1}{\mathfrak{p}_2(w) + 1} = \frac{1}{2}\left[j_1 w + \left(j_2 - \frac{j_1{}^2}{2}\right)w^2 + \cdots\right]. \tag{2.33}$$

It is clear that $\mathfrak{p}_1(z), \mathfrak{p}_2(z)$ are analytic and have positive real parts in $\mathfrak{A}$. In view of (2.28), (2.29), (2.32) and (2.33), clearly

$$(1-\eta)\left[\left(1+\frac{zf''(z)}{f'(z)}\right)-\gamma\left(\frac{zf'(z)}{f(z)}+\frac{f(z)}{z}-2\right)\right]+\eta\left[\frac{z^2f''(z)+(1-\gamma)zf'(z)+\gamma z}{zf'(z)}\right]$$
$$=\psi(z)\left(\pi\left(\left(\frac{\mathfrak{p}_1(z)-1}{\mathfrak{p}_2(z)+1}\right)-1\right)\right) \quad (2.34)$$

and

$$(1-\eta)\left[\left(1+\frac{wg''(w)}{g'(w)}\right)-\gamma\left(\frac{wg'(w)}{g(w)}+\frac{g(w)}{w}-2\right)\right]$$
$$+\eta\left[\frac{w^2g''(w)+(1-\gamma)zw'(z)+\gamma w}{wg'(w)}\right]\prec_q \psi(w)\left(\pi\left(\frac{\mathfrak{p}_1(w)-1}{\mathfrak{p}_2(w)+1}\right)-1\right). \quad (2.35)$$

The series expansions for $f(z)$ *and* $g(w)$ as given in (1.1) and (1.2) respectively, provide us

$$(1-\eta)\left[\left(1+\frac{zf''(z)}{f'(z)}\right)-\gamma\left(\frac{zf'(z)}{f(z)}+\frac{f(z)}{z}-2\right)\right]+\eta\left[\frac{z^2f''(z)+(1-\gamma)zf'(z)+\gamma z}{zf'(z)}\right]$$
$$=1+(2-2\gamma)a_2z+[(6-3\gamma)a_3-(4-\gamma-3\eta\gamma)a_2^2]z^2+\cdots \quad (2.36)$$

and

$$(1-\eta)\left[\left(1+\frac{wg''(w)}{g'(w)}\right)-\gamma\left(\frac{wg'(w)}{g(w)}+\frac{g(w)}{w}-2\right)\right]+\eta\left[\frac{w^2g''(w)+(1-\gamma)zw'(z)+\gamma w}{wg'(w)}\right]$$
$$=(-2+2\gamma)a_2w+\left[\big(2(6-3\gamma)-(4-\gamma-3\eta\gamma)\big)a_2^2-(6-3\gamma)a_3\right]w^2+\cdots. \quad (2.37)$$

Using (2.30) and (2.31) to gather with (1.4) and (1.5)

$$\psi(z)\left(\pi\left(\frac{\mathfrak{p}_1(z)-1}{\mathfrak{p}_2(z)+1}\right)-1\right)=\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\mathcal{F}_1z+\left[\frac{1}{2}\mathcal{D}_1\mathcal{K}_1\mathcal{F}_1+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\left(\mathcal{F}_2-\frac{\mathcal{F}_1{}^2}{2}\right)+\frac{\mathcal{D}_0\mathcal{K}_2\mathcal{F}_1{}^2}{4}\right]z^2+\cdots \quad (2.38)$$

and

$$\psi(w)\left(\pi\left(\frac{\mathfrak{p}_2(w)-1}{\mathfrak{p}_2(w)+1}\right)-1\right)=\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\jmath_1w+\left[\frac{1}{2}\mathcal{D}_1\mathcal{K}_1\jmath_1+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\left(\jmath_2-\frac{\jmath_1{}^2}{2}\right)+\frac{\mathcal{D}_0\mathcal{K}_2\jmath_1{}^2}{4}\right]w^2+\cdots. \quad (2.39)$$

Now equating (2.36) and (2.37) and comparing the coefficients to z *and* z^2, we obtain

$$(2-2\gamma)a_2=\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\mathcal{F}_1 \quad (2.40)$$

and

$$[(6-3\gamma)a_3-(4-\gamma-3\eta\gamma)a_2^2]=\frac{1}{2}\mathcal{D}_1\mathcal{K}_1\mathcal{F}_1+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\left(\mathcal{F}_2-\frac{\mathcal{F}_1{}^2}{2}\right)+\frac{\mathcal{D}_0\mathcal{K}_2\mathcal{F}_1{}^2}{4}. \quad (2.41)$$

Similarly (2.37) and (2.39), gives us

$$(-2+2\gamma)a_2=\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\jmath_1 \quad (2.42)$$

and

$$\left[\left(2(6-3\gamma)-(4-\gamma-3\eta\gamma)\right)a_2^2-(6-3\gamma)a_3\right]=\frac{1}{2}\mathcal{D}_1\mathcal{K}_1 j_1+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\left(j_2-\frac{{j_1}^2}{2}\right)+\frac{\mathcal{D}_0\mathcal{K}_2{j_1}^2}{4}. \tag{2.43}$$

From (2.40) and (2.42), we find that

$$a_2=\frac{\mathcal{D}_0\mathcal{K}_1\ell_1\mathfrak{E}}{2(2-2\gamma)}=-\left(\frac{\mathcal{D}_0\mathcal{K}_1 j_1\mathfrak{E}}{2(2-2\gamma)}\right), \tag{2.44}$$

which implies

$$|a_2|\leq\frac{|\mathcal{D}_0|\mathcal{K}_1}{2(2-2\gamma)}. \tag{2.45}$$

Adding (2.41) and (2.43), we obtain

$$(4-4\gamma+6\eta\gamma){a_2}^2=\frac{1}{2}\mathcal{D}_0\mathcal{K}_1(\mathcal{F}_2+j_2)+\frac{\mathcal{D}_0\mathcal{K}_2}{4}({\mathcal{F}_1}^2+{j_1}^2)|\mathcal{K}_2-\mathcal{K}_1|, \tag{2.46}$$

which implies

$$|a_2|^2\leq\frac{2|\mathcal{D}_0|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4-4\gamma+6\eta\gamma}. \tag{2.47}$$

Subsequently, to get the upper bound for $|a_3|$, we subtract (2.41) from (2.43), yielding

$$\left(2(6-3\gamma)\right)a_3=\left(2(6-3\gamma)\right)a_2^2+\frac{1}{2}\mathcal{D}_1\mathcal{K}_1(\mathcal{F}_1-j_1)+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1(\mathcal{F}_2-j_2), \tag{2.48}$$

by using Lemma 1.1 and (2.46) in (2.48), we obtain

$$|a_3|\leq\frac{2|\mathcal{D}_0|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4-4\gamma+6\eta\gamma}+\frac{(\mathcal{D}_1\mathcal{K}_1+2\mathcal{D}_0\mathcal{K}_1)}{6-3\gamma}. \tag{2.49}$$

Next, from (2.40) and (2.41), we have

$$(6-3\gamma)a_3=\frac{4-\gamma+3\eta\gamma}{4(2-2\gamma)^2}{\mathcal{D}_0}^2{\mathcal{K}_1}^2{\mathcal{F}_1}^2+\frac{1}{2}\mathcal{D}_1\mathcal{K}_1\mathcal{F}_1+\frac{1}{2}\mathcal{D}_0\mathcal{K}_1\mathcal{F}_2-\frac{1}{4}\mathcal{D}_0\mathcal{K}_1{\mathcal{F}_1}^2+\frac{1}{4}\mathcal{D}_0\mathcal{K}_2{\mathcal{F}_1}^2,$$

which implies

$$|a_3|\leq\frac{1}{6-3\gamma}\left[\frac{4-\gamma-3\eta\gamma}{4(2-2\gamma)^2}|\mathcal{D}_0|^2{\mathcal{K}_1}^2+\mathcal{K}_1|\mathcal{D}_1+\mathcal{D}_0|+|\mathcal{D}_0||\mathcal{K}_2-\mathcal{K}_1|\right]. \tag{2.50}$$

We have the special cases below are also previously uninhibited and they are as follows:

In virtue of Remark (2.6), (2.7) we obtain some corollaries below

Corollary(2.6) : Assume f is given by (1.1) belonging to the class $\Re\mathbb{A}^*(0,\gamma,\pi)$, then

$$|a_2|\leq\min\left\{\frac{|\mathcal{D}_0|\mathcal{K}_1}{2(2-2\gamma)},\sqrt{\frac{2|\mathcal{D}_0|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4-4\gamma}}\right\}$$

and

$$|a_3| \le \min\left\{\frac{2|\mathcal{D}_0|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4-4\gamma} + \frac{(\mathcal{D}_1\mathcal{K}_1+2\mathcal{D}_0\mathcal{K}_1)}{6-3\gamma}, \frac{1}{6-3\gamma}\left[\frac{4-\gamma}{2(2-2\gamma)^2}|\mathcal{D}_0|^2{\mathcal{K}_1}^2+\mathcal{K}_1|\mathcal{D}_1+\mathcal{D}_0|+|\mathcal{D}_0||\mathcal{K}_2-\mathcal{K}_1|\right]\right\}.$$

Corollary(2.7): Assume f is given by (1.1) belonging to the class $\Re\mathbb{A}^*(1,\gamma,\pi)$, then

$$|a_2| \le \min\left\{\frac{|\mathcal{D}_0|\mathcal{K}_1}{2(2-2\gamma)}, \sqrt{\frac{2|\mathcal{D}_0|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4+2\gamma}}\right\}$$

and

$$|a_3| \le \min\left\{\frac{2|\mathcal{D}_0|(\mathcal{K}_1+|\mathcal{K}_2-\mathcal{K}_1|)}{4+2\gamma} + \frac{(\mathcal{D}_1\mathcal{K}_1+2\mathcal{D}_0\mathcal{K}_1)}{6-3\gamma}, \frac{1}{6-3\gamma}\left[\frac{4-4\gamma}{4(2-2\gamma)^2}|\mathcal{D}_0|^2{\mathcal{K}_1}^2+\mathcal{K}_1|\mathcal{D}_1+\mathcal{D}_0|+|\mathcal{D}_0||\mathcal{K}_2-\mathcal{K}_1|\right]\right\}.$$

Conclusion

In this paper , we introduced subclass of the function of analytic and bi-univalent defined in the open unit disk $\mathfrak{U}$ by applying quasi-subordination. Some results and properties about the corresponding bound estimations of the coefficients a_2 and a_3 are given and investigated. Here, we opened some new windows to find the coefficients using quasi-subordination.

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