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Results on New Subclasses of m-fold Symmetric Bi-Univalent functions using Coefficient Inequalities

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ABSTRACT

In this paper, we introduce two new subclasses $\mathrm{SF}_{\sum m}(\rho,\delta,\mathfrak{P})$ and $\mathrm{SF}^{*}_{\sum m}(\rho,\delta,\mathfrak{P})$ of the m-fold symmetric bi-univalent functions that are defined in the open unit disc $\mathfrak{C}$. Moreover, the upper bounds for the first two Taylor-Maclaurin $|a_{m+1}|$, $|a_{2m+1}|$ $(k \geq 2)$ are obtained with some corollaries.

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1. Introduction

Letting $\mathcal{A}$ represents the set of functions f which are analytic within the open unit disc $\mathfrak{C} = \{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| < 1\}$, constrained by $f(0) = 0$ while $f'(0) = 1$, and conforming to the stated form:

$$f(\mathfrak{z}) = \mathfrak{z} + \sum_{r=2}^{\infty} a_r \mathfrak{z}^r \ . \qquad (1.1)$$

$\mathcal{S}$ denotes the subset of $\mathcal{A}$ including functions in (1.1) which are univalent in $\mathfrak{C}$. As per the Koebe one-quarter theorem (see [10]), each function $f \in \mathcal{S}$ has an inverse f^{-1} that fulfills

$$f^{-1}\big(f(\mathfrak{z})\big) = \mathfrak{z}\,, (\mathfrak{z} \in \mathfrak{C}),$$

and

$$f\big(f^{-1}(\mathrm{w})\big) = \mathrm{w}\,, \qquad \left(|\mathrm{w}| < r_0(f)\,, r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(\mathrm{w}) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \qquad (1.2)$$

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A function f is considered bi-univalent in $\mathfrak{C}$ if f and its inverse f^{-1} are univalent in $\mathfrak{C}$. We designate $\sum$ to be the collection of bi-univalent functions in $\mathfrak{C}$ as defined in (1.1). The seminal work of Srivastava et al. [19], has reinvigorated the investigation of bi-univalent functions in the past few years, where a substantial number of sequels his work are established and examined many families of the bi-univalent function family by numerous authors (see, for instance, [1,2,5,6,8,11,15,16,17,18,23,24,26]). For every function $f \in S$, the function $h(\mathfrak{z}) = \sqrt[m]{f(\mathfrak{z}^m)}$, where $\mathfrak{z} \in \mathfrak{C}$ and $m \in \mathbb{N}$, is univalent as well as maps the unit disc into an area exhibiting symmetric m-fold.

A function is classified as m-fold symmetric (see [11]) if it possesses the following normalized representation:

$$f(\mathfrak{z}) = \mathfrak{z} + \sum_{k=1}^{\infty} a_{mk+1}\mathfrak{z}^{mk+1}, \quad (\mathfrak{z} \in \mathfrak{C}, m \in \mathbb{N}). \tag{1.3}$$

Letting $\mathcal{S}_m$ be the collection of m-fold symmetric univalent functions in $\mathfrak{C}$, normalized by the series expansion (1.3). The functions of the $\mathcal{S}$ family exhibit unilateral symmetry.

In [20], Srivastava et al. delineated m-fold symmetric bi-univalent functions, paralleling the notion of m-fold symmetric univalent functions. Their findings demonstrate that each function $f \in \sum$ yields an m-fold symmetric bi-univalent function across all $m \in \mathbb{N}$. Furthermore, using the normalized form delineated by (1.3), they calculated the series expansion for f^{-1} as follows:

$$g(w) = w + a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (2m+3)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots, \tag{1.4}$$

hence f^{-1} equals g. We represent $\sum_m$ as the collection of m-fold symmetric bi-univalent functions in $\mathfrak{C}$. It is obvious that for = 1, equation (1.4) corresponds with equation (1.2) of the family.

Examples of m-fold symmetric bi-univalent functions are presented as follows:

$$\left(\frac{\mathfrak{z}^m}{1-\mathfrak{z}^m}\right)^{\frac{1}{m}}, \left[\frac{1}{2}\log\left(\frac{\mathfrak{z}^m}{1-\mathfrak{z}^m}\right)\right]^{\frac{1}{m}} \text{ and } [-\log(1-\mathfrak{z}^m)]^{\frac{1}{m}},$$

alongside the respective inverse functions in that order:

$$\left(\frac{w^m}{1-w^m}\right)^{\frac{1}{m}}, \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \text{ and } \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}}.$$

In recent years, numerous writers examined bounds for different subfamilies of m-fold bi-univalent functions ([3,4,9,7,14,20,21,22,25]). To substantiate our principal findings, we necessitate the following lemma.

Lemma 1.1 [3]. If $h \in \mathfrak{J}$, then $|c_k| \leq 2$ for every $k \in \mathbb{N}$, where $\mathfrak{J}$ denotes the set of all h such that $Re(h(\mathfrak{z})) > 0$, with $\mathfrak{z} \in \mathfrak{C}$

$$h(\mathfrak{z}) = 1 + c_1\mathfrak{z} + c_2\mathfrak{z}^2 + \cdots, \quad (\mathfrak{z} \in \mathfrak{C}).$$

2. Coefficient Bound for the Function Family $\mathrm{SF}_{\sum m}(\rho, \delta, \mathfrak{B})$

Definition 2.1. A function $f(\mathfrak{z})$ in (1.3) is classified into the class $\mathrm{SF}_{\sum m}(\rho, \delta, \mathfrak{P})$ if the subsequent requirements are met:

$$\left|\arg\left(\rho\left[\frac{\mathfrak{z}^{2-\delta}f(\mathfrak{z})''}{(\mathfrak{z}f'(\mathfrak{z}))^{1-\delta}}\right] + (1-\rho)\left[\frac{\mathfrak{z}^{1-\delta}f(\mathfrak{z})'}{(f(\mathfrak{z}))^{1-\delta}}\right]\right)\right| < \frac{\mathfrak{P}\pi}{2}, (\mathfrak{z} \in \mathfrak{C}), \tag{2.1}$$

and

$$\left|\arg\left(\rho\left[\frac{w^{2-\delta}g(w)''}{(wg'(w))^{1-\delta}}\right] + (1-\rho)\left[\frac{w^{1-\delta}g(w)'}{(g(w))^{1-\delta}}\right]\right)\right| < \frac{\mathfrak{P}\pi}{2}, \quad (w \in \mathfrak{C}), \tag{2.2}$$

where $g(w)$ given by (1.4), $g = f^{-1}, m \in \mathbb{N}, \delta \geq 0, \; 0 < \rho \leq 1, 0 < \mathfrak{P} \leq 1$.

Remark 2.1: By specializing the parameters $\mathfrak{P}$, ρ and m, one can delineate the numerous new and established subclasses of analytic bi-univalent functions previously examined in the literature.

1- Considering $m = 1$, we derive a novel class of bi-univalent functions.

$$\mathrm{SF}_{\sum m}(\rho,\delta,\mathfrak{P}) = \mathrm{SF}_{\sum}(\rho,\delta,\mathfrak{P}) .$$

2- For $\rho = 0$ and $\delta = 0$, we derive a class that comprises m-fold symmetric bi-starlike functions as defined by Altinkaya and Yalcin [3].

$$\mathrm{SF}_{\sum m}(\rho,\delta,\mathfrak{P}) = S^{\mathfrak{P}}_{\sum m} .$$

3- Assuming $\rho = 1$ and $\delta = 0$, we derive a class comprising m-fold symmetric convex bi-univalent functions as established by A. Wanas and Majeed [18].

$$\mathrm{SF}_{\sum m}(\rho,\delta,\mathfrak{P}) = E_{\sum m}(0,1,1,\mathfrak{P}).$$

4- Assuming $\rho = 0, \delta = 0$, and $m = 1$, we derive the class of bi-univalent functions presented by Brannan and Taha [9].

$$\mathrm{SF}_{\sum m}(\rho,\delta,\mathfrak{P}) = S^{*}_{\sum}(\mathfrak{P}) .$$

5- Considering $\rho = 1, m = 1$, with $\delta = 0$, we derive a family of convex bi-univalent functions as presented by Brannan and Taha [9].

$$\mathrm{SF}_{\sum m}(\rho,\delta,\mathfrak{P}) = S_{\sum 1}(\mathfrak{P}).$$

Theorem 2.1. If $f \in \mathrm{SF}_{\sum m}(\rho,\delta,\mathfrak{P})$, ($m \in \mathbb{N}$, $\delta \geq 0, 0 < \rho \leq 1, 0 < \mathfrak{P} \leq 1$), then

$$|a_{m+1}| \leq \frac{2\mathfrak{P}}{\sqrt{\left|\begin{matrix}\mathfrak{P}\left[(m+1)[2m(1+2m\rho)+\delta(1-\rho)] - 2(1-\delta)\left[m(1+2m\rho+m^2\rho)+\frac{1}{2}\delta(1-\rho)\right]\right] \\ -(\mathfrak{P}-1)[m(1+m\rho)+\delta(1-\rho)]^2\end{matrix}\right|}} \tag{2.3}$$

and

$$|a_{2m+1}| \leq \left|\frac{2\mathfrak{P}}{[2m(1+2m\rho)+\delta(1-\rho)]} + \frac{2\mathfrak{P}^2(m+1)}{[m(1+m\rho)+\delta(1-\rho)]^2}\right| . \tag{2.4}$$

Proof: Let $f \in \mathrm{SF}_{\sum m}(\rho,\delta,\mathfrak{P})$. Then

$$\rho\left[\frac{z^{2-\delta} f(z)''}{\left(z f'(z)\right)^{1-\delta}}\right] + (1-\rho)\left[\frac{z^{1-\delta} f(z)'}{\left(f(z)\right)^{1-\delta}}\right] = [p(z)]^{\mathfrak{P}} \tag{2.5}$$

and

$$\rho\left[\frac{w^{2-\delta} g(w)''}{(w g'(w))^{1-\delta}}\right] + (1-\rho)\left[\frac{w^{1-\delta} g(w)'}{(g(w))^{1-\delta}}\right] = [q(w)]^{\mathfrak{P}}, \tag{2.6}$$

where $p(z)$ as well as $q(w)$ are in $\mathfrak{J}$ and possess the following forms:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$q(w) = 1 + +q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} \ldots . \quad (2.8)$$

This yields the following relations

$$[m(1+m\rho) + \delta(1-\rho)]a_{m+1} = \mathfrak{P} p_m, \quad (2.9)$$

$$[2m(1+2m\rho) + \delta(1-\rho)]a_{2m+1} - (1-\delta)\left[m(1+2m\rho+m^2\rho) + \frac{1}{2}\delta(1-\rho)\right]a_{m+1}^2$$

$$= \mathfrak{P} p_{2m} + \frac{\mathfrak{P}(\mathfrak{P}-1)}{2} {p_m}^2 \quad (2.10)$$

and

$$-[m(1+m\rho) + \delta(1-\rho)]a_{m+1} = \mathfrak{P} q_m, \quad (2.11)$$

$$\left[(m+1)[2m(1+2m\rho) + \delta(1-\rho)] - (1-\delta)\left[m(1+2m\rho+m^2\rho) + \frac{1}{2}\delta(1-\rho)\right]\right]a_{m+1}^2$$

$$-[2m(1+2m\rho) + \delta(1-\rho)]a_{2m+1} = \mathfrak{P} q_{2m} + \frac{\mathfrak{P}(\mathfrak{P}-1)}{2} {q_m}^2 \quad (2.12)$$

By (2.9) and (2.11), we get that

$$p_m = -q_m \quad (2.13)$$

and

$$2[m(1+m\rho) + \delta(1-\rho)]^2 a_{m+1}^2 = \mathfrak{P}^2({p_m}^2 + {q_m}^2) \; . \quad (2.14)$$

Now, from (2.10), (2.12) and (2.14), we obtain that

$${a_{m+1}}^2 = \frac{\mathfrak{P}^2(p_2 + q_2)}{\mathfrak{P}\left[(m+1)[2m(1+2m\rho) + \delta(1-\rho)] - 2(1-\delta)\left[m(1+2m\rho+m^2\rho) + \frac{1}{2}\delta(1-\rho)\right]\right] - (\mathfrak{P}-1)[m(1+m\rho) + \delta(1-\rho)]^2}. \quad (2.15)$$

Utilizing Lemma 1.1 for the coefficients p_{2m} with q_{2m}, we promptly obtain

$$|a_{m+1}| \le \frac{2\mathfrak{P}}{\sqrt{\left|\mathfrak{P}\left[(m+1)[2m(1+2m\rho) + \delta(1-\rho)] - 2(1-\delta)\left[m(1+2m\rho+m^2\rho) + \frac{1}{2}\delta(1-\rho)\right]\right] - (\mathfrak{P}-1)[m(1+m\rho) + \delta(1-\rho)]^2\right|}},$$

which gives us the desired estimate on $|a_{m+1}|$ as asserted within (2.3).

To determine the bound on $|a_{2m+1}|$, we subtract (2.12) and (2.10), resulting in

$$2[2m(1+2m\rho) + \delta(1-\rho)]a_{2m+1} - \left[(m+1)[2m(1+2m\rho) + \delta(1-\rho)]\right]a_{m+1}^2$$

$$= \mathfrak{P}(p_{2m} - q_{2m}) + \frac{\mathfrak{P}(\mathfrak{P}-1)}{2}({p_m}^2 - {q_m}^2). \quad (2.16)$$

From (2.13), (2.14) and (2.16), we obtain

$$a_{2m+1} = \frac{\mathfrak{P}(p_{2m} - q_{2m})}{2[2m(1+2m\rho) + \delta(1-\rho)]} + \frac{m+1}{2}\frac{\mathfrak{P}^2({p_m}^2 - {q_m}^2)}{2[m(1+m\rho) + \delta(1-\rho)]^2} \; . \quad (2.17)$$

Using Lemma 1.1 to the coefficients p_{2m} and q_{2m}, yields that

$$|a_{2m+1}| \leq \left|\frac{2\mathfrak{P}}{[2m(1+2m\rho)+\delta(1-\rho)]}+\frac{2\mathfrak{P}^2(m+1)}{[m(1+m\rho)+\delta(1-\rho)]^2}\right|,$$

which complete the theorem.

When $m = 1, \delta = 0$ in Theorem 2.1, we obtain the subsequent Corollary.

Corollary 2.1. Letting f, as defined in (1.3), belongs to the class $\mathrm{SF}_{\sum m}(\rho,\delta,\mathfrak{P})$. Subsequently

$$|a_2| \leq \frac{2\mathfrak{P}}{\sqrt{2\mathfrak{P}(1+\rho)+(1-\mathfrak{P})(1+\rho)^2}},$$

and

$$|a_3| \leq \frac{\mathfrak{P}}{(1+2\rho)}+\frac{4\mathfrak{P}^2}{(1+\rho)^2}.$$

Assuming $\rho = 0$ as well as $\delta = 0$ in Theorem 2.1, we deduce the subsequent Corollary.

Corollary 2.2. Letting f, as specified in (1.3), belongs to the class $S_{\sum m}^{\mathfrak{P}}$. Subsequently

$$|a_{m+1}| \leq \frac{2\mathfrak{P}}{m\sqrt{\mathfrak{P}+1}},$$

and

$$|a_{2m+1}| \leq \frac{\mathfrak{P}}{m}+\frac{2\mathfrak{P}^2(m+1)}{m^2}.$$

Assuming $\rho = 0, m = 1$, as well as $\delta = 0$ in Theorem 2.1, the following corollary is established.

Corollary 2.3. Suppose f, as defined in (1.3), belongs to the class $S_{\sum}^{*}(\mathfrak{P})$. Subsequently

$$|a_2| \leq \frac{2\mathfrak{P}}{\sqrt{\mathfrak{P}+1}},$$

and

$$|a_3| \leq \mathfrak{P}(4\mathfrak{P}+1).$$

By substituting $\rho = 1$ as well as $\delta = 0$ in Theorem 2.1, we derive the subsequent conclusion.

Corollary 2.4. Assume f, as defined in (1.3), belongs to the class $E_{\sum m}(0,1,1,\mathfrak{P})$. Subsequently

$$|a_{m+1}| \leq \frac{2\mathfrak{P}}{m\sqrt{2\mathfrak{P}(1+m)+(1-\mathfrak{P})(1+m\rho)^2}},$$

and

$$|a_{2m+1}| \leq \frac{\mathfrak{P}}{m(1+2m)}+\frac{2\mathfrak{P}^2}{m^2(1+m)^2}.$$

Assuming $\rho = 1, m = 1$, with $\delta = 0$ in Theorem 2.1, we deduce the subsequent Corollary.

Corollary 2.5 Assume f, as defined in (1.3), belongs to the class $S_{\sum 1}(\mathfrak{P})$. Subsequently

$$|a_2| \leq \mathfrak{P},$$

and

$$|a_3| \leq \frac{\mathfrak{P}}{3} + \mathfrak{P}^2$$

3. 2. Coefficient Bound for the Function Family $\mathrm{SF}^*_{\Sigma m}(\rho, \delta, \mathfrak{P})$

Definition 3.1. A function, such as $f(z)$ stated in (1.3) is classified as belonging to the class $\mathrm{SF}^*_{\Sigma m}(\rho, \delta, \mathfrak{B})$ if it meets the subsequent requirements.

$$\mathrm{Re}\left(\rho\left[\frac{z^{2-\delta} f(z)''}{\left(z f'(z)\right)^{1-\delta}}\right] + (1-\rho)\left[\frac{z^{1-\delta} f(z)'}{\left(f(z)\right)^{1-\delta}}\right]\right) > \mathfrak{P}, \qquad (z \in \mathfrak{C}) \tag{3.1}$$

and

$$\mathrm{Re}\left(\rho\left[\frac{w^{2-\delta} g(w)''}{(w g'(w))^{1-\delta}}\right] + (1-\rho)\left[\frac{w^{1-\delta} g(w)'}{(g(w))^{1-\delta}}\right]\right) > \mathfrak{P}, \ (w \in \mathfrak{C}) \tag{3.2}$$

where $g(w)$ given by (1.4),

Remark 3.1 By specializing the parameters $\mathfrak{P}, \rho$, as well as m, one can delineate the numerous new and established subclasses of analytic bi-univalent functions previously examined in the academic literature.

1- Considering $m = 1$, we derive a novel class of bi-univalent functions.

$$\mathrm{SF}^*_{\Sigma m}(\rho, \delta, \mathfrak{P}) = \mathrm{SF}^*_{\Sigma}(\rho, \delta, \mathfrak{P}).$$

2- Assuming $\rho = 0$ and $\delta = 0$, we derive a class that comprises m-fold symmetric bi-starlike functions as established by Altinkaya and Yalcin [3].

$$\mathrm{SF}^*_{\Sigma m}(\rho, \delta, \mathfrak{P}) = \mathbb{N}^0_{\Sigma, m}(\mathfrak{P}, 1) .$$

3- Considering $\rho = 1$ and $\delta = 0$, we derive a class comprising m-fold symmetric convex bi-univalent functions as established by Wanas with Majeed [18].

$$\mathrm{SF}^*_{\Sigma m}(\rho, \delta, \mathfrak{P}) = E^*_{\Sigma m}(0,1,1,\mathfrak{P}).$$

4- Assuming $\rho = 0, \delta = 0$, and $m = 1$, we derive the class of bi-univalent functions presented by Brannan with Taha [9].

$$\mathrm{SF}^*_{\Sigma m}(\rho, \delta, \mathfrak{P}) = S^*_{\Sigma}(\mathfrak{P}).$$

5- Assuming $\rho = 1, m = 1$, with $\delta = 0$, we derive a class of convex bi-univalent functions as presented by Brannan with Taha [9].

$$\mathrm{SF}^*_{\Sigma m}(\rho, \delta, \mathfrak{P}) = S_{\Sigma 1}(\mathfrak{P}).$$

Theorem 3.1 If $f \in \mathrm{SF}^*_{\Sigma m}(\rho, \delta, \mathfrak{P})$, $(m \in \mathbb{N},\ \delta \geq 0, 0 < \rho \leq 1, 0 < \mathfrak{P} \leq 1)$, then

$$|a_{m+1}| \leq \sqrt{\left|\frac{4(1-\mathfrak{P})}{\left[[2m(1+2m\rho)+\delta(1-\rho)](1+m) - 2(1-\delta)\left[m(1+2m\rho+m^2\rho)+\frac{1}{2}\delta(1-\rho)\right]\right]}\right|}, \tag{3.3}$$

and

$$a_{2m+1} \leq \left| \frac{2(1-\mathfrak{P})}{m(1+2m\rho)+\delta(1-\rho)} + \frac{2(1-\mathfrak{P})^2(1+m)}{[m(1+m\rho)+\delta(1-\rho)]^2} \right| . \tag{3.4}$$

Proof. Consequently, using (4.23) along with (4.24), there exist p and $q \in \mathfrak{J}$ which means that

$$\rho\left[\frac{z^{2-\delta}f(z)''}{(zf'(z))^{1-\delta}}\right] + (1-\rho)\left[\frac{z^{1-\delta}f(z)'}{(f(z))^{1-\delta}}\right] = \mathfrak{P} + (1-\mathfrak{P})p(z) \tag{3.5}$$

and

$$\left(\rho\left[\frac{w^{2-\delta}g(w)''}{(wg'(w))^{1-\delta}}\right] + (1-\rho)\left[\frac{w^{1-\delta}g(w)'}{(g(w))^{1-\delta}}\right]\right) = \mathfrak{P} + (1-\mathfrak{P})q(w), \tag{3.6}$$

where $p(z)$ and $q(w)$ in $\mathfrak{J}$ given by (2.7) and (2.8).

This, we get the following relations

$$[m(1+m\rho)+\delta(1-\rho)]a_{m+1} = (1-\mathfrak{P})p_m, \tag{3.7}$$

$$[2m(1+2m\rho)+\delta(1-\rho)]a_{2m+1} - (1-\delta)\left[m(1+2m\rho+m^2\rho)+\tfrac{1}{2}\delta(1-\rho)\right]a_{m+1}^2 = (1-\mathfrak{P})p_{2m} \tag{3.8}$$

and

$$-[m(1+m\rho)+\delta(1-\rho)]a_{m+1} = (1-\mathfrak{P})q_m, \tag{3.9}$$

$$\left[[2m(1+2m\rho)+\delta(1-\rho)](1+m) - (1-\delta)\left[m(1+2m\rho+m^2\rho)+\frac{1}{2}\delta(1-\rho)\right]\right]a_{m+1}^2$$
$$-[2m(1+2m\rho)+\delta(1-\rho)]a_{2m+1} = (1-\mathfrak{P})q_{2m}. \tag{3.10}$$

According to (3.7) as well as (3.9):

$$p_m = -q_m \tag{3.11}$$

and

$$2[m(1+m\rho)+\delta(1-\rho)]^2 a_{m+1}^2 = (1-\mathcal{T})^2({p_m}^2+{q_m}^2). \tag{3.12}$$

Now, from (3.8) and (3.10), we obtain that

$$a_{m+1}^2 = \frac{(1-\mathfrak{P})(p_{2m}+q_{2m})}{\left[[2m(1+2m\rho)+\delta(1-\rho)](1+m) - 2(1-\delta)\left[m(1+2m\rho+m^2\rho)+\frac{1}{2}\delta(1-\rho)\right]\right]}. \tag{3.13}$$

By applying Lemma 1.1 to the coefficients p_{2m} as well as q_{2m}, yields

$$|a_{m+1}| \leq \sqrt{\left|\frac{4(1-\mathfrak{P})}{\left[[2m(1+2m\rho)+\delta(1-\rho)](1+m) - 2(1-\delta)\left[m(1+2m\rho+m^2\rho)+\frac{1}{2}\delta(1-\rho)\right]\right]}\right|}\,.$$

This provides the requisite estimate for $|a_{m+1}|$ as stated within (3.3). To determine the bound on $|a_{2m+1}|$, we remove (3.10) and (3.8), resulting in

$$a_{2m+1} = \frac{(1-\mathfrak{P})(p_{2m}+q_{2m})}{2m(1+2m\rho)+\delta(1-\rho)} + \frac{m+1}{2}a_{m+1}^2,$$

and a_{m+1}^2 from (3.12), we obtain

$$a_{2m+1}=\frac{(1-\mathfrak{P})(p_{2m}+q_{2m})}{2m(1+2m\rho)+\delta(1-\rho)}+\frac{m+1}{2}\frac{(1-\mathfrak{P})^2({p_m}^2+{q_m}^2)}{2[m(1+m\rho)+\delta(1-\rho)]^2}.$$

By computing the absolute value of (39) and reapplying Lemma 1.1 to the coefficients p_{2m} as well as q_{2m}, we derive

$$a_{2m+1}\le\left|\frac{2(1-\mathfrak{P})}{m(1+2m\rho)+\delta(1-\rho)}+\frac{2(1-\mathfrak{P})^2(m+1)}{[m(1+m\rho)+\delta(1-\rho)]^2}\right|,$$

This concludes the evidence of Theorem 3.1.

Assuming $m = 1$ with $\delta = 0$ in Theorem 3.1, we obtain the subsequent corollary.

Corollary 3.1. Suppose f, as defined by (1.3), belong to the class $\mathrm{SF}^*_{\Sigma m}(\rho,\delta,\mathfrak{B})$. Subsequently

$$|a_2|\le\sqrt{\frac{2(1-\mathfrak{P})}{(1+\rho)}},$$

and

$$|a_3|\le\frac{2(1-\mathfrak{P})}{(1+2\rho)}+\frac{4(1-\mathfrak{P})^2}{(1+\rho)^2}.$$

Assuming $\rho = 0$ and $\delta = 0$ in Theorem 3.1, we obtain the subsequent corollary.

Corollary 3.2. Suppose f, as specified by (1.3), belong to the class $\mathbb{N}^0_{\Sigma,m}(\mathfrak{P},1)$. Subsequently

$$|a_{m+1}|\le\frac{1}{m}\sqrt{2(1-\mathfrak{P})},$$

and

$$|a_{2m+1}|\le\frac{(1-\mathfrak{P})}{m}+\frac{2(1-\mathfrak{P})^2(m+1)}{m^2}.$$

Assuming $\rho = 0, m = 1$, and $\delta = 0$, Theorem 3.1 yields an additional corollary.

Corollary 3.3. Suppose f, as defined in (1.3), belongs to the class $S^*_\Sigma(\mathfrak{P})$. Subsequently

$$|a_2|\le\sqrt{2(1-\mathfrak{P})},$$

and

$$|a_3|\le(1-\mathfrak{P})+4(1-\mathfrak{P})^2.$$

Assuming $\rho = 1$ as well as $\delta = 0$ in Theorem 3.1, the subsequent corollary is derived.

Corollary 3.4. Suppose f, as defined in (1.3), belongs to the class $E^*_{\Sigma m}(0,1,1,\mathfrak{P})$. Subsequently

$$|a_{m+1}|\le\frac{1}{m}\sqrt{\frac{2(1-\mathfrak{P})}{m+1}},$$

and

$$|a_{2m+1}|\le\frac{(1-\mathfrak{P})}{m(1+2m)}+\frac{2(1-\mathfrak{P})^2}{m^2(1+m)}.$$

Assuming $\rho = 1, m = 1$, with $\delta = 0$ in Theorem 3.1 yields the subsequent corollary.

Corollary 3.5. Suppose that f, defined in (1.3), belongs to the class $S_{\sum 1}(\mathfrak{P})$. Subsequently

$$|a_2| \leq \sqrt{(1-\mathfrak{P})}\,,$$

and

$$|a_3| \leq \frac{(1-\mathfrak{P})}{3} + (1-\mathfrak{P})^2.$$

Conclusion

This study has introduced new subfamilies $\mathrm{SF}_{\sum m}(\rho,\delta,\mathfrak{P})$ and $\mathrm{SF}^{*}_{\sum m}(\rho,\delta,\mathfrak{P})$ of $\sum_m$ with have derived estimates for the coefficients $|a_{m+1}|$, and $|a_{2m+1}|$ for functions within each of these subfamilies.

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