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New Families of Bi-Univalent Functions Associated with the Quotient of Analytic Functions

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ABSTRACT

The purpose of this paper is to obtain the upper bounds for the first two Taylor-Maclaurin $|a_2|$ and $|a_3|$ for a new families $\Psi_\Sigma(\gamma;\alpha), \Psi_\Sigma^*(\gamma;\beta), \Phi_\Sigma(\lambda;\alpha)$ and $\Phi_\Sigma^*(\lambda;\beta)$ of holomorphic and bi-univalent functions defined by the ratio of analytic representations of convex and starlike functions in the open unit disk U.

MSC..

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1. Introduction

Indicate by $\mathcal{A}$ the collection of all holomorphic functions in the unit disk $U=\{z\in\mathbb{C}: |z|<1\}$ that have the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k\ . \qquad (1.1)$$

Further, assume that S stands for the sub-collection of $\mathcal{A}$ consisting of functions U satisfying (1.1) which are also univalent in U.

A function $f\in\mathcal{A}$ is called starlike of order δ $(0\le\delta<1)$, if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\delta,\ \ (z\in U).$$

Singh [13] introduced and studied Bazilevič function that is the function f such that

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$$Re\left\{\frac{z^{1-\gamma}f'(z)}{(f'(z))^{1-\gamma}}\right\} > 0, \quad (z \in U, \gamma \geq 0).$$

On the order hand, a function $f \in \mathcal{A}$ is called a λ −pseudo-starlike function in U if (see [3])

$$Re\left\{\frac{z(f'(z))^{\lambda}}{f(z)}\right\} > 0, (z \in U, \lambda \geq 1).$$

Recently, several authors introduced and studied different subfamilies of associated with Bazilevič and λ −pseudo functions (see, for example, [4, 5, 8, 10, 21, 22, 23, 26, 27]).

According to the Koebe one-quarter theorem (see [5]) every function $f \in S$ has an inverse f^{-1} which satisfies

$f^{-1}(f(z)) = z, (z \in U)$ and $f(f^{-1}(w)) = w, (|w| < r_0(f) \geq \frac{1}{4})$, where

$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots$ (1.2)

For $f \in \mathcal{A}$, if both f and f^{-1} are univalent in U, we say that f bi-univalent function in U.

We indicate by Σ the family of bi-univalent functions in U given by (1.1).

For a brief historical account and for several interesting examples of functions in the family Σ, see the pioneering work on this subject by Srivastava et al. [20], which actually revived the study of bi-univalent functions in recent tears. From the work of Srivastava rt al. [20], we choose to recall here the following examples of functions in the family Σ :

$$\frac{z}{1-z}, -log\,(1-z) \text{ and } \frac{1}{2}log\left(\frac{1+z}{1-z}\right).$$

We notice that the class Σ is not empty. However, the koebe function is not a member of Σ.

Recently, many authors introduced various subfamilies of the bi-univalent functions family Σ and investigated upper bounds for the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) (see, for example [1, 2, 7, 9, 11, 12, 14, 15, 16, 17, 18, 19, 21, 24, 25, 28]). The problem to find the general coefficient bounds on the Taylor- Maclaurin coefficients

$$|a_n| \quad (n \in \mathbb{N} \setminus \{1,2\}; \mathbb{N} := 1,2,3, \ldots)$$

for functions $f \in \Sigma$ is still not completely addressed for many of the subclasses of the bi-univalent function class Σ .

We require the following lemma that will be used to prove our main results.

Lemma 1.1. [6] If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h holomorphic in U for which

$$Re(h(z)) > 0, \quad (z \in U),$$

where

$$h(z) = 1 + c_1z + c_2z^2 + \ldots, \quad (z \in U).$$

2. Coefficient Estimates for the Family $\Psi_\Sigma(\gamma;\alpha)$

Definition 2.1. A function $f \in \Sigma$ given by (1.1) is called in the family $\Psi_\Sigma(\gamma;\alpha)$ if it fulfills the conditions:

$$\left|arg\left(\frac{1+\frac{z^{2-\gamma}f''(z)}{(zf'(z))^{1-\gamma}}}{\frac{z^{1-\gamma}f'(z)}{(f'(z))^{1-\gamma}}}\right)\right| < \frac{\alpha\pi}{2}, \qquad (z \in U) \tag{2.1}$$

and

$$\left|arg\left(\frac{1+\frac{w^{2-\gamma}g''(w)}{(wg'(w))^{1-\gamma}}}{\frac{w^{1-\gamma}g'(w)}{(g'(w))^{1-\gamma}}}\right)\right| < \frac{\alpha\pi}{2}, \qquad (w \in U), \tag{2.2}$$

where $0 < \alpha \leq 1, \gamma \geq 0$ and the function $g = f^{-1}(z)$ is given by (1.2).

Theorem 2.1. Let $f \in \Psi_\Sigma(\gamma;\alpha)$ $(0 < \alpha \leq 1, \gamma \geq 0)$ by given by (1.1). Then

$$|a_2| \leq 2\alpha\sqrt{\frac{2}{\alpha\gamma(\gamma(\gamma+3)-5)+\gamma(\gamma-2)+\alpha+11}}$$

and

$$|a_3| \leq \frac{2\alpha}{4-\gamma} + \frac{4\alpha^2\left(-\frac{1}{2}\gamma(\gamma^2+2\gamma+7)-7\right)}{(4-\gamma)(1-\gamma)^2}$$

Proof. It follows from conditions (2.1) and (2.2) that

$$\frac{1+\frac{z^{2-\gamma}f''(z)}{(zf'(z))^{1-\gamma}}}{\frac{z^{1-\gamma}f'(z)}{(f'(z))^{1-\gamma}}} = [p(z)]^{\alpha} \tag{2.3}$$

and

$$\frac{1+\frac{w^{2-\gamma}g''(w)}{(wg'(w))^{1-\gamma}}}{\frac{w^{1-\gamma}g'(w)}{(g'(w))^{1-\gamma}}}=[q(w)]^{\alpha}, \tag{2.4}$$

where $g=f^{-1}$ and p,q in $\mathcal{P}$ have the following series representations:

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\ldots \tag{2.5}$$

and

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\ldots\ . \tag{2.6}$$

Comparing the corresponding coefficients of (2.3) and (2.4) yields

$$(1-\gamma)a_2=\alpha p_1\,, \tag{2.7}$$

$$(4-\gamma)a_3+\left(\frac{1}{2}\gamma(\gamma-7)-3\right)a_2^2=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2\ , \tag{2.8}$$

$$-(1-\gamma)a_2=\alpha q_1 \tag{2.9}$$

and

$$\left(\frac{1}{2}\gamma^2(\gamma+3)+4\right)a_2^2-(4-\gamma)a_3=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.10}$$

In view of (2.7) and (2.9), we conclude that

$$p_1=-q_1 \tag{2.11}$$

and

$$(1-\gamma)^2a_2^2=\alpha^2(p_1^2+q_1^2)\,. \tag{2.12}$$

Also, by using (2.8) and (2.10), together with (2.12), we find that

$$\left(\frac{1}{2}\gamma(\gamma(\gamma+4)-7)+1\right)a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)$$
$$=\alpha(p_2+q_2)+\frac{(\alpha-1)\,(1-\gamma)^2}{2\alpha}a_2^2\,.$$

Further computations show that

$$a_2^2=\frac{2\alpha^2(p_2+q_2)}{\alpha\gamma(\gamma(\gamma+3)-5)+\gamma(\gamma-2)+\alpha+1}. \tag{2.13}$$

By taking the absolute value of (2.13) and applying Lemma 1.1 for the coefficient p_2 and q_2, we have

$$|a_2|\le 2\alpha\sqrt{\frac{2}{\alpha\gamma(\gamma(\gamma+3)-5)+\gamma(\gamma-2)+\alpha+11}}.$$

To determinate the bound on $|a_3|$, by subtracting (2.10) from (2.8), we get

$$2(4-\gamma)a_3+\left(\frac{1}{2}\gamma^2-\frac{7}{2}\gamma-3-\frac{1}{2}\gamma^3-\frac{3}{2}\gamma^2-4\right)a_2^2$$
$$=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2). \tag{2.14}$$

Now, substituting the value of a_2^2 from (2.12) into (2.14) and using (2.11), we deduce that

$$a_3=\frac{\alpha(p_2-q_2)}{2(4-\gamma)}-\frac{\alpha^2(p_1^2+q_1^2)\left(-\frac{1}{2}\gamma(\gamma^2+2\gamma+7)-7\right)}{2(4-\gamma)(1-\gamma)^2}\,. \tag{2.15}$$

Taking the absolute value of (2.15) and applying Lemma 1.1 once again for the coefficient p_1, p_2, q_1 and q_2, it following that

$$|a_3|\le\frac{2\alpha}{4-\gamma}+\frac{4\alpha^2\left(-\frac{1}{2}\gamma(\gamma^2+2\gamma+7)-7\right)}{(4-\gamma)(1-\gamma)^2}.$$

Putting $\gamma=0$ in Theorem 2.1, we conclude the following corollary:

Corollary 2.1. If $f\in\Psi_\Sigma(0;\alpha)$, then

$$|a_2|\le 2\alpha\sqrt{\frac{2}{\alpha+11}}$$

and

$$|a_3|\le\frac{\alpha}{2}-7\alpha^2\,.$$

3. Coefficient Estimates for the Family $\boldsymbol{\Psi_{\Sigma}^{*}(\gamma;\beta)}$

Definition 3.1. A function $f\in\Sigma$ given by (1.1), is said to be in the family $\Psi_{\Sigma}^{*}(\gamma;\beta)$

$$\operatorname{Re}\left(\frac{1+\frac{z^{2-\gamma}f''(z)}{(zf'(z))^{1-\gamma}}}{\frac{z^{1-\gamma}f'(z)}{(f'(z))^{1-\gamma}}}\right)>\beta\,,\qquad (z\in U),\qquad (3.1)$$

and

$$\operatorname{Re}\left(\frac{1+\frac{w^{2-\gamma}g''(w)}{(wg'(w))^{1-\gamma}}}{\frac{w^{1-\gamma}g'(w)}{(g'(w))^{1-\gamma}}}\right)>\beta\,,\qquad (w\in U),\qquad (3.2)$$

where $0\le\beta<1,\gamma\ge 0$ and the function $g=f^{-1}$ is given by (1.2).

Theorem 3.1. let $f\in\Psi_{\Sigma}^{*}(\gamma;\beta)$ $(0\le\beta<1,\gamma\ge 0)$ by given by (1.1). Then

$$|a_2|\le 2\sqrt{\frac{1-\beta}{\left(\frac{1}{2}\gamma(\gamma(\gamma+4)-7)+1\right)}}$$

and

$$|a_3|\le\frac{4(1-\beta)^2\left(\frac{1}{2}\gamma(\gamma(\gamma+2)+7)+7\right)}{(1-\gamma)^2(4-\gamma)}+\frac{2(1-\beta)}{4-\gamma}.$$

Proof. In the light of the conditions (3.1) and (3.2), there are $p,q\in\mathcal{P}$ such that

$$\frac{1+\frac{z^{2-\gamma}f''(z)}{(zf'(z))^{1-\gamma}}}{\frac{z^{1-\gamma}f'(z)}{(f'(z))^{1-\gamma}}}=\beta+(1-\beta)p(z)\qquad (3.3)$$

and

$$\frac{1+\frac{w^{2-\gamma}g''(w)}{(wg'(w))^{1-\gamma}}}{\frac{w^{1-\gamma}g'(w)}{(g'(w))^{1-\gamma}}}=\beta+(1-\beta)q(w),\qquad (3.4)$$

where $p(z)$ and $q(w)$ given by (2.5) and (2.6), respectively. Thus, by comparing the corresponding coefficient in (3.3) and (3.4), we find that

$$(1-\gamma)a_2=(1-\beta)p_1,\qquad (3.5)$$

$$(4-\gamma)a_3+\left(\frac{1}{2}\gamma(\gamma-7)-3\right)a_2^2=(1-\beta)p_2,\qquad (3.6)$$

$$-(1-\gamma)\,a_2=(1-\beta)q_1,\qquad (3.7)$$

and

$$\left(\frac{1}{2}\gamma^2(\gamma+3)+4\right)a_2^2-(4-\gamma)a_3=(1-\beta)q_2.\qquad (3.8)$$

From (3.5) and (3.7), we get

$$p=-q\qquad (3.9)$$

and

$$(1-\gamma)^2a_2^2=(1-\beta)^2(p_1^2+q_1^2).\qquad (3.10)$$

If we add (3.6) to (3.8), we obtain

$$\left(\frac{1}{2}\gamma(\gamma(\gamma+4)-7)+1\right)a_2^2=(1-\beta)(p_2+q_2).\qquad (3.11)$$

Hance, we find that

$$a_2^2=\frac{(1-\beta)(p_2+q_2)}{\left(\frac{1}{2}\gamma(\gamma(\gamma+4)-7)+1\right)}.\qquad (3.12)$$

Next, by applying the Lemma 1. 1 for the coefficients p_2 and q_2, we have

$$|a_2|\le 2\sqrt{\frac{1-\beta}{\left(\frac{1}{2}\gamma(\gamma(\gamma+4)-7)+1\right)}}.$$

In order to determinate the bound on $|a_3|$, by subtracting (3.8) from (3.6), we get

$$2(4-\gamma)a_3-\left(\frac{1}{2}\gamma(\gamma(\gamma+2)+7)+7\right)a_2^2=(1-\beta)(p_2-q_2),$$

or equivalently,

$$a_3 = \frac{\left(\frac{1}{2}\gamma(\gamma(\gamma+2)+7)+7\right)}{2(4-\gamma)} a_2^2 + \frac{(1-\beta)(p_2-q_2)}{2(4-\gamma)}. \tag{3.13}$$

Now, upon substituting the value of a_2^2 from (3.10) in to (3.13), it follows that

$$a_3 = \frac{\left(\frac{1}{2}\gamma(\gamma(\gamma+2)+7)+7\right)(1-\beta)^2(p_1^2+q_1^2)}{2(4-\gamma)(1-\gamma)^2} + \frac{(1-\beta)(p_2-q_2)}{2(4-\gamma)}.$$

Finally, by applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \le \frac{4(1-\beta)^2\left(\frac{1}{2}\gamma(\gamma(\gamma+2)+7)+7\right)}{(1-\gamma)^2(4-\gamma)} + \frac{2(1-\beta)}{4-\gamma}.$$

This completes the proof of Theorem 3.1.

Putting $\gamma = 0$ in Theorem 3.1, we conclude the following corollary:

Corollary 3.1. If $f \in \Psi_\Sigma^*(0;\beta)$ if it fulfills the conditions:

$$|a_2| \le 2\sqrt{1-\beta}$$

and

$$|a_3| \le 7(1-\beta)^2 + \frac{1-\beta}{2}.$$

4. Coefficient Estimates for the Family $\Phi_\Sigma(\lambda;\alpha)$

Definition 4.1. A function $f \in \Sigma$ given by (1.1) is called in the family $\Phi_\Sigma(\lambda;\alpha)$ if it fulfills the conditions:

$$\left| arg\left(\frac{\left(\frac{(zf'(z))'}{f(z)}\right)^\lambda}{\frac{z(f'(z))^\lambda}{f(z)}} \right) \right| < \frac{\alpha\pi}{2}, \tag{4.1}$$

and

$$\left| arg\left(\frac{\left(\frac{(wg'(w))'}{g(w)}\right)^\lambda}{\frac{w(g'(w))^\lambda}{g(w)}} \right) \right| < \frac{\alpha\pi}{2}, \tag{4.2}$$

where $0 < \alpha \le 1, \lambda \ge 1$ and the function $g = f^{-1}$.

Theorem 4.1. Let $f \in \Phi_\Sigma(\lambda;\alpha)$ $(0 < \alpha \le 1, \lambda \ge 1)$ by given by (1.1). Then

$$|a_2| \le \alpha \frac{2}{\sqrt{(\lambda-1)(2\lambda+4)+1}}$$

and

$$|a_3| \le \frac{(18\lambda-5)\alpha^2}{(2\lambda-1)^2(3\lambda-1)} + \frac{\alpha}{3\lambda-1}$$

Proof. It follows from conditions (4.1) and (4.2) that

$$\frac{\left(\frac{(zf'(z))'}{f(z)}\right)^\lambda}{\frac{z(f'(z))^\lambda}{f(z)}} = [p(z)]^\alpha \tag{4.3}$$

and

$$\frac{\left(\frac{(wg'(w))'}{g(w)}\right)^\lambda}{\frac{w(g'(w))^\lambda}{g(w)}} = [q(w)]^\alpha, \tag{4.4}$$

where $g = f^{-1}$ and p, q in $\mathcal{P}$ have the following series representations:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \tag{4.5}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \dots \, . \tag{4.6}$$

Comparing the corresponding coefficients of (4.3) and (4.4) yields

$$(2\lambda - 1)a_2 = \alpha p_1 \, , \tag{4.7}$$

$$(2\lambda(\lambda - 5) + 2)a_2^2 + 2(3\lambda - 1)a_3 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 \, , \tag{4.8}$$

$$-(2\lambda - 1)a_2 = \alpha q_1 \tag{4.9}$$

and

$$(2\lambda(\lambda + 4) - 3)a_2^2 - 2(3\lambda - 1)a_3 = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2 \, . \tag{4.10}$$

In view of (4.7) and (4.9), we conclude that

$$p_1 = -q_1 \tag{4.11}$$

and

$$2(2\lambda - 1)^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2). \tag{4.12}$$

Also, by using (4.8) and (4.10), together with (4.12), we find that

$$(2\lambda(2\lambda - 1) - 1)a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2)$$

$$= \alpha(p_2 + q_2) + \frac{(\alpha - 1)(2\lambda - 1)^2}{\alpha} a_2^2.$$

Further computations show that

$$a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{(\lambda - 1)(2\alpha + 4\lambda) + 1}. \tag{4.13}$$

By taking the absolute value of (4.13) and applying Lemma 1.1 for the coefficient p_2 and q_2, we have

$$|a_2| \le \alpha \frac{2}{\sqrt{(\lambda - 1)(2\alpha + 4\lambda) + 1}}.$$

To determinate the bound on $|a_3|$, by subtracting (4.10) from (4.8), we get

$$(18\lambda - 5)a_2^2 + 4(3\lambda - 1)a_3 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 - q_1^2). \tag{4.14}$$

Now, substituting the value of a_2^2 from (4.12) into (4.14) and using (4.11), we deduce that

$$a_3 = \frac{\alpha(p_2 - q_2)}{4(3\lambda - 1)} + \frac{\alpha^2 (p_1^2 + q_1^2)(18\lambda - 5)}{8(2\lambda - 1)^2 (3\lambda - 1)}. \tag{4.15}$$

Taking the absolute value of (4.15) and applying Lemma 1.1 once again for the coefficient p_1, p_2, q_1 and q_2, it following that

$$|a_3| \le \frac{(18\lambda - 5)\alpha^2}{(2\lambda - 1)^2 (3\lambda - 1)} + \frac{\alpha}{3\lambda - 1}.$$

Putting $\lambda = 1$ in Theorem 4.1, we conclude the following corollary:

Corollary 4.1. If $f \in \Phi_\Sigma(1; \alpha)$ if it fulfills the conditions:

$$|a_2| \le 2\alpha$$

and

$$|a_3| \le \frac{1}{2}\alpha(13\alpha + 1).$$

5. Coefficient Estimates for the Family $\boldsymbol{\Phi_\Sigma^*(\lambda; \beta)}$

Definition 5.1. A function $f \in \Sigma$ given by (1.1) is called in the family $\Phi_\Sigma^*(\lambda; \beta)$ if it fulfills the conditions:

$$Re\left(\frac{\left(\frac{(zf'(z))'}{f(z)}\right)^\lambda}{\frac{z(f'(z))^\lambda}{f(z)}}\right) > \beta \tag{5.1}$$

and

$$Re\left(\frac{\left(\frac{(wg'(w))'}{g(w)}\right)^\lambda}{\frac{w(g'(w))^\lambda}{g(w)}}\right) > \beta \tag{5.2}$$

where $0 \le \beta < 1, \lambda \ge 1$ and the function $g = f^{-1}$.

Theorem 5.1. Let $f \in \Phi_\Sigma^*(\lambda; \beta)$ $(0 \le \beta < 1, \lambda \ge 1 \,)$ by given by (1.1). Then

$$|a_2| \leq 2\sqrt{\frac{(1-\beta)}{(2\lambda(2\lambda-1)-1}}. \tag{5.1}$$

and

$$|a_3| \leq \frac{1-\beta}{3\lambda-1} + \frac{2(1-\beta)^2(18\lambda-5)}{(3\lambda-1)(2\lambda-1)^2}. \tag{5.2}$$

Proof. In the light of the conditions (5.1) and (5.2), there are $p, q \in \mathcal{P}$ such that

$$\frac{\left(\frac{(zf'(z))'}{f(z)}\right)^{\lambda}}{\frac{z(f'(z))^{\lambda}}{f(z)}} = \beta + (1-\beta)p(z) \tag{5.3}$$

and

$$\frac{\left(\frac{(wg'(w))'}{g(w)}\right)^{\lambda}}{\frac{w(g'(w))^{\lambda}}{g(w)}} = \beta + (1-\beta)q(w), \tag{5.4}$$

where $p(z)$ and $q(w)$ given by (2.5) and (2.6), respectively. Thus, by comparing the corresponding coefficient in (5.3) and (5.4), we find that

$$(2\lambda-1)a_2 = (1-\beta)p_1, \tag{5.5}$$

$$(2\lambda(\lambda-5)+2)a_2^2 + 2(3\lambda-1)a_3 = (1-\beta)p_2, \tag{5.6}$$

$$-(2\lambda-1)\, a_2 = (1-\beta)q_1, \tag{5.7}$$

and

$$(2\lambda(\lambda+4)-3)a_2^2 - 2(3\lambda-1)\, a_3 = (1-\beta)q_2. \tag{5.8}$$

From (5.5) and (5.7), we get

$$p = -q \tag{5.9}$$

and

$$(2\lambda-1)^2 a_2^2 = (1-\beta)^2(p_1^2+q_1^2). \tag{5.10}$$

If we add (5.6) to (5.8), we obtain

$$(2\lambda(2\lambda-1)-1)a_2^2 = (1-\beta)(p_2+q_2). \tag{5.11}$$

Hance, we find that

$$a_2^2 = \frac{(1-\beta)(p_2+q_2)}{(2\lambda(2\lambda-1)-1)}. \tag{5.12}$$

Next, by applying the Lemma 1.1 for the coefficients p_2 and q_2, we have

$$|a_2| \leq 2\sqrt{\frac{(1-\beta)}{2\lambda(2\lambda-1)-1}}.$$

In order to determinate the bound on $|a_3|$, by subtracting (5.8) from (5.6), we get

$$4(3\lambda-1)a_3 - (18\lambda-5)a_2^2 = (1-\beta)(p_2-q_2),$$

or, equivalently,

$$a_3 = \frac{(18\lambda-5)}{4(3\lambda-1)}a_2^2 - \frac{(1-\beta)(p_2-q_2)}{4(3\lambda-1)}. \tag{5.13}$$

Now, upon substituting the value of a_2^2 from (5.10) in to (5.13), it follows that

$$a_3 = \frac{(18\lambda-5)(1-\beta)^2(p_1^2+q_1^2)}{4(3\lambda-1)(2\lambda-1)^2} - \frac{(1-\beta)(p_2-q_2)}{4(3\lambda-1)}.$$

Finally, by applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \leq \frac{1-\beta}{3\lambda-1} + \frac{2(1-\beta)^2(18\lambda-5)}{(3\lambda-1)(2\lambda-1)^2}.$$

This completes the proof of Theorem 5.1.

Putting $\lambda = 1$ in Theorem 5.1, we conclude the following corollary:

Corollary 5.1. If $f \in \Phi_\Sigma^*(1;\beta)$ if it fulfills the conditions:

$$|a_2| \leq 2\sqrt{1-\beta}$$

and

$$|a_3| \leq \frac{1-\beta}{2} + 13(1-\beta)^2.$$

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