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Approximate of fuzzy homomorphism and fuzzy derivation on Banach Algebra

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In this paper, we proved the approximation of homomorphism and derivation related to the following functional equation:

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)+2f(x)$$

On Fuzzy Banach Algebras space by means of direct and fixed point methods.

MSC..

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1. Introduction

Starting with Ulam's question on stability posed in 1940 [16], and following the first response to Ulam's question by Hyers [8] in Banach spaces, along with the extension of $Hyers'$ result by $Aoki$ [3] and Rassias [15], the study of stability became a focal point for mathematicians due to its significance and applications.

Concurrently, with the emergence of the *concept of fuzzy sets introduced* by Lotfi Zadeh in 1965 [17] as a generalization of classical sets, the theory of fuzzy spaces witnessed significant development when Katrasas [9] defined a fuzzy norm on a linear space to create a fuzzy vector topological structure on this space.

Subsequently, intensive efforts were made by mathematicians to define fuzzy norms on linear spaces. With the substantial progress made in fuzzy theory, efforts to study the stability of functional equations in fuzzy spaces

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increased, leading to important results. In this context, we have reference some significant research papers on the topic [1, 2, 4, 5, 6, 10, 11, 12, 13 and 14].

In this paper, we studied the stability of the *additive functional equation*:

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)+2f(x). \tag{1.1}$$

Furthermore, we discussed the homomorphism and derivation of the previous functional equation in the fuzzy Banach algebra space in the case of the function being odd, and we reached valuable results and good conclusions, whether by the direct method or the fixed-point method.

2. Preliminaries.

Definition2.1. [12] Let X be a *real vector space*. A function $N: X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X *if for all* $x, y \in X$ *and all* $s, t \in \mathbb{R}$,

(N_1) $N(x,t)=0$ for $t \leq 0$,
(N_2) $x=0$ *if and only if* $N(x,t)=1$ *for all* $t>0$,
(N_3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c \neq 0$,
(N_4) $N(x+y,s+t) \geq \min\{N(x+s),(y+t)\}$,
(N_5) $N(x,\ \cdot)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x,t)=1$,
(N_6) for $x \neq 0$, $N(x,\cdot)$ *is continuous on* $\mathbb{R}$.

The pair (X,N) is a called a *fuzzy normed vector space*.

Definition2.2. [1] Let (X,N) be a fuzzy normed vector space. A sequence x_n *in* X is said to be convergent or converge if there exists an x *in* X such that $\lim_{n\to\infty} N(x_n-x,t)=1$, for all $t>0$. In this case x is called the limit of the sequence $\{x_n\}$ and we denote it by $N-\lim_{n\to\infty} x_n = x$.

Definition2.3. [1] Let (X,N) be a fuzzy normed vector space. A sequence x_n in X is called Cauchy if for each $\varepsilon>0$ and each $t>0$ there exists an $n_0 \in N$ such that for all $n \geq n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,t)>1-\varepsilon$. It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space

Definition2.4. [12] Let X be an *algebra* and (X,N) a *fuzzy normed space*.

1. The *fuzzy normed* space (X,N) is called a *fuzzy normed algebra* if

$$N(xy,st) \geq N(x,s)\,.\,N(y,t)$$

for all $x,y \in X$, for all $s,t \in \mathbb{R}$.

2. A complete *fuzzy normed algebra* is called *fuzzy Banach algebra*.

Definition2.5. [11] Let (X,N_x) and (Y,N) be *fuzzy normed algebras*. Then a multiplicative $\mathbb{R}$-*linear mapping* $H:(X,N_x)\to(Y,N)$ is called a *fuzzy algebra homomorphism*.

Definition2.6. [11] Let (Y,N) be a *fuzzy normed algebra*. Then an $\mathbb{R}$-linear mapping $\delta:(Y,N)\to(Y,N)$ is called a *fuzzy algebra derivation* if $\delta(x,y)=y\delta(x)+x\delta(y)$ *for all* $x,y \in X$.

Definition2.7. [7] Let X be a set. A function $d: X \times X \to [0,\infty]$ is called a *generalized metric* on X if,

(i) $d(x,y)=0$ *if and only if* $x=y$,

(ii) $d(x, y) = d(y, x) \quad for\ all\ \ x, y \in X,$

(iii) $d(x, z) \leq d(x, y) + \ d(y, z)\ for\ all\ x, y, z \in X.$

Theorem2.6. [7] Let (X, d) be a complete *generalized metric* space, and let $J: X \to X$ be a strictly contractive mapping with the Lipschitz constant$L < 1$. Then, for each $x \in X$, either

$d(J^n x, J^{n+1}x) = +\infty,\ \ for\ all\ n \geq 0$ or there exists a natural number n_0 such that

(1) $d(J^n x, J^{n+1}x) < +\infty, \forall n \geq n0,$

(2) The sequence $J^n x$ is convergent to a fixed point $y^*\ of\ \ J,$

(3) y^* is the unique fixed point of J in the set$Y = \{y \in X, d(J^{n_0}x, y) < \infty\},$

(4)$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)\ for\ all\ y\ \in Y.$

3. Stability of function equation (1.1) in F.B.A. space using the direct method.

Theorem3.1 Let X, Y be two F. B. A. spaces, $f: X \to Y$ odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x)\ for\ all\ x, y \in X, \lambda \in \mathbb{R}.$ Suppose function $\theta: X^2 \to [0,\ \infty)$ such that $\rho\theta(x, y) = \theta(\rho x, \rho y), for\ all\ \rho \in \mathbb{R}$, and

$$\sum_{i=0}^{\infty} 3^i\, \theta\left(\frac{x}{3^{i+1}}, \frac{y}{3^{i+1}}\right) < \infty, for\ all\ x, y \in X, \tag{3.1}$$

$$\lim_{t\to\infty} N\left(D_f(x, y), t\theta(x, y)\right) = 1. \tag{3.2}$$

where

$$D_f(x, y) = f(2x + y) - f(2x - y) - f(x + y) - f(x - y) + 2f(2x) + 2f(x),$$

$$\lim_{t\to\infty} N\big(f(xy) - f(x)f(y), t\tilde{\theta}(x, y)\big) = 1. \tag{3.3}$$

$$\tilde{\theta}(x, y) = \sum_{i=0}^{\infty} 3^i \theta\left(\frac{x}{3^{i+1}}, \frac{y}{3^{i+1}}\right).$$

$for\ all\ x, y \in X$. Then $H(x) = \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$ exists for each $x \in X$ and defines *fuzzy algebra homomorphism* $H: X \to Y$ such that if for some $\beta > 0, \alpha > 0$

$$N\big(D_f(x, y), \beta\theta_n(x, x)\big) \geq \alpha, for\ all\ x \in X, \tag{3.4}$$

Then $N\left(f(x) - H(x), \beta\tilde{\theta}(x, x)\right) \geq \alpha, for\ all\ x \in X.$ $\quad$ (3.5)

Hence the *fuzzy algebra homomorphism* $H: X \to Y$ is a *unique mapping such that*

$$\lim_{t\to\infty} N\big(f(x) - H(x), t\tilde{\theta}(x, x)\big) = 1,\ \ for\ all\ x \in X, \tag{3.6}$$

Proof. Let $x = y$ in (3.2). *for a given* $\varepsilon > 0$ *such that*

$$N\left(f(x) - 3f\left(\frac{x}{3}\right), t\theta(\frac{x}{3}, \frac{x}{3})\right) \geq 1 - \varepsilon, \text{for all } x, y\ \in X. \tag{3.7}$$

By induction on n, we will show that

$N\left(f(x) - 3^n f\left(\frac{x}{3^n}\right), t\sum_{k=0}^{n-1} 3^k \theta(\frac{x}{3^{k+1}}, \frac{x}{3^{k+1}})\right) \geq 1 - \varepsilon$ (3.8)

$for\ all\ x \in X, n \in \mathbb{N}$, and all $t \geq t_0$.

It follows form (3.7) that (3.8) holds for $n = 1$.

Assume that (3.8) holds $all\ n \in \mathbb{N}$. Then

$$N\left(f(x) - 3^{n+1} f\left(\frac{x}{3^{n+1}}\right), t\sum_{k=0}^{n} 3^k \theta(\frac{x}{3^{k+1}}, \frac{x}{3^{k+1}})\right) \geq \min\left\{\begin{matrix} N\left(f(x) - 3^n f\left(\frac{x}{3^n}\right), t\sum_{k=0}^{n-1} 3^k \theta\left(\frac{x}{3^{k+1}}, \frac{x}{3^{k+1}}\right)\right), \\ N\left(3^n f\left(\frac{x}{3^n}\right) - 3^{n+1} f\left(\frac{x}{3^{n+1}}\right), 3^n t\theta\left(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}\right)\right) \end{matrix}\right\}$$

$$\geq \min\{1 - \varepsilon, 1 - \varepsilon\} = 1 - \varepsilon$$

Let $t = t_0$ and repealing $n\ and\ x\ by\ p\ and\ \frac{x}{3^n}$ in (3.8), respectively, we get

$N\left(3^n f\left(\frac{x}{3^n}\right) - 3^{n+p} f\left(\frac{x}{3^{n+p}}\right), 3^n t_0 \sum_{k=0}^{p-1} 3^k\ \theta(\frac{x}{3^{n+k+1}}, \frac{x}{3^{n+k+1}})\right) \geq 1 - \varepsilon,$

$for\ all\ n \geq 0, p > 0.$ (3.9)

It follows from (3.1) and the equality

$$\sum_{k=0}^{p-1} 3^{n+k} \theta(\frac{x}{3^{n+k+1}}, \frac{x}{3^{n+k+1}}) = \sum_{k=n}^{n+p-1} 3^k \theta(\frac{x}{3^{k+1}}, \frac{x}{3^{k+1}})$$

$$= t_0 \sum_{k=n}^{n+p-1} 3^k\ \theta(\frac{x}{3^{k+1}}, \frac{x}{3^{k+1}}) < \beta$$

$for\ all\ n \geq n_0$ and $p > 0$. From (3.9) we get

$N\left(3^n f\left(\frac{x}{3^n}\right) - 3^{n+p} f\left(\frac{x}{3^{n+p}}\right), \beta\right) \geq N\left(3^n f\left(\frac{x}{3^n}\right) - 3^{n+p} f\left(\frac{x}{3^{n+p}}\right), 3^n t_0 \sum_{k=n}^{p-1} 3^k\ \theta\left(\frac{x}{3^{k+n+1}}, \frac{x}{3^{k+n+1}}\right)\right) \geq 1 - \varepsilon$

Thus the sequence $\left\{3^k f(\frac{x}{3^k})\right\}$ is Cauchy in$Y$. Since Y F. B. A, the sequence$\left\{3^n f(\frac{x}{3^n})\right\}$ converges to some$H(x) \in Y$. $H(x) = N - \lim_{n\to\infty} 3^n\ f(\frac{x}{3^n})$, for each $t > 0\ and\ x \in X$.

$$\lim_{n\to\infty} N\left(3^n f\left(\frac{x}{3^n}\right) - H(x), t\right) = 1.$$

$for\ all\ t > 0\ and\ \ 0 < \varepsilon < 1$. Since $\lim_{n\to\infty} 3^n\ \theta\left(\frac{x}{3^{n+1}}, \frac{y}{3^{n+1}}\right) = 0$, there is an$n_1 > n_0$ such that $t_0 3^n \theta(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}) < \frac{t}{7}$ for all $n \geq n_1$.

Hence for each $k \geq n_1$ we have

$N(H(2x + y) + H(2x - y) + H(x + y) + H(x - y) - 2H(2x) - 2H(x), t)$

$$\geq \min\left\{\begin{array}{c} N\left(H(2x+y)-3^k f\left(\frac{(2x+y)}{3^k}\right),\frac{t}{7}\right), \\ N\left(H(2x-y)-3^k f\left(\frac{(2x-y)}{3^k}\right),\frac{t}{7}\right), \\ N\left(H(x+y)-3^k f\left(\frac{(x+y)}{3^k}\right),\frac{t}{7}\right), \\ N\left(H(x-y)-3^k f\left(\frac{(x-y)}{3^k}\right),\frac{t}{7}\right), \\ N\left(2H(2x)-3^k 2f\left(\frac{2x}{3^k}\right),\frac{t}{7}\right), \\ N\left(2H(x)-3^k 2f\left(\frac{x}{3^k}\right),\frac{t}{7}\right), \\ N\left(\begin{array}{c} 3^k f\left(\frac{(2x+y)}{3^k}\right)+3^k f\left(\frac{(2x-y)}{3^k}\right)-3^k f\left(\frac{(x+y)}{3^k}\right)- \\ 3^k f\left(\frac{(x-y)}{3^k}\right)-3^k 2f\left(\frac{2x}{3^k}\right)-3^k 2f\left(\frac{x}{3^k}\right),\frac{t}{7} \end{array}\right) \end{array}\right\}$$

The $first\ six$ terms on the right-hand side of the a above inequality end to 1 as $k \to \infty$ and last term is greater then

$$N\left(\begin{array}{c} 3^k f\left(\frac{(2x+y)}{3^k}\right)+3^k f\left(\frac{(2x-y)}{3^k}\right)-3^k f\left(\frac{(x+y)}{3^k}\right)- \\ 3^k f\left(\frac{(x-y)}{3^k}\right)-3^k 2f\left(\frac{2x}{3^k}\right)-3^k 2f\left(\frac{x}{3^k}\right),\frac{t}{7} \end{array}\right)$$

$$> N\left(\begin{array}{c} 3^k f\left(\frac{(2x+y)}{3^k}\right)+3^k f\left(\frac{(2x-y)}{3^k}\right)-3^k f\left(\frac{(x+y)}{3^k}\right)- \\ 3^k f\left(\frac{(x-y)}{3^k}\right)-3^k 2f\left(\frac{2x}{3^k}\right)-3^k 2f\left(\frac{x}{3^k}\right), t_0 3^k\theta\left(\frac{x}{3^{k+1}},\frac{x}{3^{k+1}}\right) \end{array}\right) \geq 1-\varepsilon$$

$for\ all\ t > 0$. Since $N(H(2x+y)+H(2x-y)+H(x+y)+H(x-y)-2H(2x)-2H(x),t) = 1$. For all $t > 0$, by (N_2).

$$H(2x+y)+H(2x-y)+H(x+y)+H(x-y)-2H(2x)-2H(x) = 0.$$

Hence H is satisfying the functional equation.

Hence the $mapping\ H: X \to Y$ is$additive$.

$for\ all\ x \in X, \lambda \in \mathbb{R}$. $H(\lambda x) = \lim_{n\to\infty} 3^n f\left(\frac{\lambda x}{3^n}\right) = \lambda \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right) = \lambda H(x)$

It results from (3.3) that $H(xy) = H(x)H(y)$, for all $x, y \in X$.

Let β and α be positive.

$$\theta_n(x,y) = \sum_{k=0}^{n-1} 3^k \theta\left(\frac{x}{3^{k+1}},\frac{y}{3^{k+1}}\right), for\ all\ x,y \in X.$$

It can be inferred from (3.4) that $N\left(f(x)-3^n f\left(\frac{x}{3^n}\right), \beta\sum_{k=0}^{n-1} 3^k\theta\left(\frac{x}{3^{k+1}},\frac{x}{3^{k+1}}\right)\right) \geq \alpha$ (3.10)

$for\ all\ n \in \mathbb{N}$. Let $t > 0$. We have

$$N(f(x)-H(x),\beta\theta_n(x,x)+t)$$
$$\geq \min\left\{N\left(f(x)-3^n f\left(\frac{x}{3^n}\right),\beta\theta_n(x,x)\right),N\left(f(x)-3^n f\left(\frac{x}{3^n}\right),t\right)\right\} \quad (3.11)$$

From (3.10) and (3.11) and $\lim_{n\to\infty} N\left(3^n f\left(\frac{x}{3^n}\right)-H(x),t\right)=1$, we get

$$N(f(x)-H(x),\beta\theta_n(x,x)+t)\geq\alpha$$

Letting $t\to 0$,

$$N\left(f(x)-H(x),\beta\tilde{\theta}(x,x)\right)\geq\alpha$$

Let $\grave{H}$ be another *additive mapping* satisfying (3.5) and (3.6). Suppose that $c>0,\varepsilon>0$ *such that*

$$N\left(f(x)-H(x),t\tilde{\theta}(x,x)\right)\geq 1-\varepsilon$$

$$N\left(f(x)-\grave{H}(x),t\tilde{\theta}(x,x)\right)\geq 1-\varepsilon$$

for all $x\in X$ and $t>0$ and $t\geq t_0$. Find some integer n_0 such that

$$t_0\sum_{k=n}^{\infty}3^k\theta\left(\frac{x}{3^{k+1}},\frac{x}{3^{k+1}}\right)<\frac{c}{3}$$

Since

$$\sum_{k=n}^{\infty}3^k\theta\left(\frac{x}{3^{k+1}},\frac{x}{3^{k+1}}\right)=3^n\sum_{k=n}^{\infty}3^{(k-n)}\theta\left(3^{(n-k)}\frac{x}{3^{n+1}},3^{(n-k)}\frac{x}{3^{n+1}}\right)$$
$$=3^n\sum_{i=0}^{\infty}3^i\theta\left(\frac{1}{3^i}\frac{x}{3^{n+1}},\frac{1}{3^i}\frac{x}{3^{n+1}}\right)=3^n\tilde{\theta}\left(\frac{x}{3^{n+1}},\frac{x}{3^{n+1}}\right)$$

we have

$$N\left(H(x)-\grave{H}(x),c\right)$$
$$\geq\min\left\{N\left(3^n f\left(\frac{x}{3^n}\right)-H(x),\frac{c}{3}\right),N\left(\grave{H}(x)-3^n f\left(\frac{x}{3^n}\right),\frac{c}{3}\right),N\left(H(x)-H(x),\frac{c}{3}\right)\right\}$$
$$=\min\left\{N\left(f\left(\frac{x}{3^n}\right)-H\left(\frac{x}{3^n}\right),3^{-n}\frac{c}{3}\right),N\left(\grave{H}\left(\frac{x}{3^n}\right)-f\left(\frac{x}{3^n}\right),3^{-n}\frac{c}{3}\right),1\right\}$$
$$\geq\min\left\{\begin{array}{l}N\left(f\left(\frac{x}{3^n}\right)-H\left(\frac{x}{3^n}\right),3^{-n}t_0\sum_{k=n}^{\infty}3^k\theta\left(\frac{x}{3^{k+1}},\frac{x}{3^{k+1}}\right)\right),\\ N\left(\grave{H}\left(\frac{x}{3^n}\right)-f\left(\frac{x}{3^n}\right),3^{-n}t_0\sum_{k=n}^{\infty}3^k\theta\left(\frac{x}{3^{k+1}},\frac{x}{3^{k+1}}\right)\right),1\end{array}\right\}$$
$$=\min\left\{\begin{array}{l}N(f\left(\frac{x}{3^n}\right)-H\left(\frac{x}{3^n}\right),t_0\tilde{\theta}\left(\frac{x}{3^{n+1}},\frac{x}{3^{n+1}}\right),\\ N(\grave{H}\left(\frac{x}{3^n}\right)-f\left(\frac{x}{3^n}\right),t_0\tilde{\theta}\left(\frac{x}{3^{n+1}},\frac{x}{3^{n+1}}\right),1\end{array}\right\}$$
$$=\min\left\{\begin{array}{l}N(f\left(\frac{x}{3^n}\right)-H\left(\frac{x}{3^n}\right),\grave{t}_0\tilde{\theta}\left(\frac{x}{3^n},\frac{x}{3^n}\right),\\ N(\grave{H}\left(\frac{x}{3^n}\right)-f\left(\frac{x}{3^n}\right),\grave{t}_0\tilde{\theta}\left(\frac{x}{3^n},\frac{x}{3^n}\right),1\end{array}\right\}$$
$$\geq\min\{1-\varepsilon,1-\varepsilon,1\}=1-\varepsilon \qquad \square$$

Theorem3.2 Let X, Y be two F. B. A. spaces, $f: X \to Y$ odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x)$ $for\ all\ x, y \in X, \lambda \in \mathbb{R}$. Suppose function $\theta: X^2 \to [0,\ \infty)$ such that $\rho\theta(x, y) = \theta(\rho x, \rho y), for\ all\ \rho \in \mathbb{R}$, and

$$\sum_{i=0}^{\infty} 3^i\, \theta\left(\frac{x}{3^{i+1}}, \frac{y}{3^{i+1}}\right) < \infty, for\ all\ x, y \in X, \tag{3.11}$$

$$\lim_{t\to\infty} N\left(D_f(x, y), t\theta(x, y)\right) = 1. \tag{3.12}$$

$$\lim_{t\to\infty} N\big(f(xy) - f(x)y - f(y)x, t\tilde{\theta}(x, y)\big) = 1. \tag{3.13}$$

$$\tilde{\theta}(x, y) = \sum_{i=0}^{\infty} 3^i\theta\left(\frac{x}{3^{i+1}}, \frac{y}{3^{i+1}}\right).$$

$for\ all\ x, y \in X$. Then $\delta(x) = \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$ exists $for\ each\ x \in X$ and defines $fuzzy\ algebra\ derivation$ $\delta: X \to Y$ such that if for some $\beta > 0, \alpha > 0$

$$N\big(D_f(x, y), \beta\theta_n(x, x)\big) \geq \alpha, for\ all\ x, y \in X, \tag{3.14}$$

Then $N\left(f(x) - H(x), \beta\tilde{\theta}(x, x)\right) \geq \alpha, for\ all\ x \in X,$ (3.15)

The $fuzzy\ algebra\ derivation$ $\delta: X \to Y$ is a $unique\ mapping$ such that

$$\lim_{t\to\infty} N\big(f(x) - \delta(x), t\tilde{\theta}(x, x)\big) = 1. \tag{3.16}$$

Proof. This Theorem can be readily demonstrated utilizing the same way to the Theorem 3.1.

Theorem3.3. Let X, Y be two F.B.A. spaces, $f: X \to Y$ odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x)$ $for\ all\ x, y \in X, \lambda \in \mathbb{R}$. Suppose function $\theta: X^2 \to [0,\ \infty)$ such that $\rho\theta(x, y) = \theta(\rho x, \rho y), for\ all\ \rho \in \mathbb{R}, and$

$$\sum_{i=0}^{\infty} \frac{1}{3^{i+1}} \theta(3^i x, 3^i y) < \infty, for\ all\ x, y \in X, \tag{3.17}$$

$$\lim_{t\to\infty} N\left(D_f(x, y), t\theta(x, y)\right) = 1. \tag{3.18}$$

$$\lim_{t\to\infty} N\big(f(xy) - f(x)f(y), t\tilde{\theta}(x, y)\big) = 1. \tag{3.19}$$

$$\tilde{\theta}(x, y) = \sum_{i=0}^{\infty} \frac{1}{3^{i+1}} \theta(3^i x, 3^i\ y).$$

$for\ all\ x, y \in X$. Then $H(x) = \lim_{n\to\infty} \frac{f(3^{nx})}{3^n}$ exists for each $x \in X$ and defines $fuzzy\ algebra\ homomorphism$ $H: X \to Y$ such that if for some $\beta > 0, \alpha > 0$

$$N\big(D_f(x, y), \beta\tilde{\theta}(x, x)\big) \geq \alpha, for\ all\ x, y \in X, \tag{3.20}$$

$$N\left(f(x) - H(x), \beta\tilde{\theta}(x, x)\right) \geq \alpha, for\ all\ x \in X, \tag{3.21}$$

The $fuzzy\ algebra\ homomorphism$ $H: X \to Y$ is a $unique\ mapping$ such that

$$\lim_{t\to\infty} N\big(f(x) - H(x), t\tilde{\theta}(x, x)\big) = 1. \tag{3.22}$$

Proof. . This Theorem can be readily demonstrated utilizing the same way to the Theorem 3.1.

Theorem3.4. Let X, Y be two F.B.A. spaces. $f: X \to Y$ Odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x)$ $for\ all\ x, y \in X, \lambda \in \mathbb{R}$. Suppose function $\theta: X^2 \to [0, \infty)$ such that $\rho\theta(x, y) = \theta(\rho x, \rho y)$, $for\ all\ \rho \in \mathbb{R}$,and

$$\sum_{i=0}^{\infty} \frac{1}{3^{i+1}} \theta(3^i x, 3^i y) < \infty, for\ all\ x, y \in X, \tag{3.27}$$

$$\lim_{t\to\infty} N\left(D_f(x, y), t\theta(x, y)\right) = 1. \tag{3.28}$$

$$\lim_{t\to\infty} N\big(f(xy) - f(x)y - f(y)x, t\tilde{\theta}(x, y)\big) = 1. \tag{3.29}$$

$$\tilde{\theta}(x, y) = \sum_{i=0}^{\infty} \frac{1}{3^{i+1}} \theta(3^i x, 3^i y).$$

$for\ all\ x, y \in X$. Then $\delta(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n}$ exists for each $x \in X$ and defines $fuzzy\ algebra\ derivation\ \delta: X \to Y$ such that if for some $\beta > 0, \alpha > 0$

$$N\big(D_f(x, y), \beta\theta_n(x, x)\big) \geq \alpha, \text{ for all } x, y \in X, \tag{3.30}$$

Then $N(f(x) - \delta(x), \beta\tilde{\theta}(x, x)) \geq \alpha$, for all $x, y \in X$, (3.31)

Hence the $fuzzy\ algebra\ derivation\ \delta: X \to Y$ is a $unique\ mapping$ such that

$$\lim_{t\to\infty} N\big(f(x) - \delta(x), t\tilde{\theta}(x, x)\big) = 1. \tag{3.32}$$

Proof. This Theorem can be readily demonstrated utilizing the same way to the Theorem 3.1.

4. Stability of function equation (1.1) in F.B.A. space using the fixed point method.

Theorem 4.1 Let X, Y be two F.B.A. spaces, $f: X \to Y$ odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x), x \in X, \lambda \in \mathbb{R}$. Suppose function $\theta: X^2 \to [0,\infty)$ there exist $L < 1$ such that

$$\theta\left(\frac{x}{3}, \frac{y}{3}\right) \leq \frac{L}{3}(x, y)$$

$$N\big(D_f(x, y), t\big) \geq \frac{t}{t + \theta(x,y)}. \tag{4.1}$$

$$N(f(xy) - f(x)f(y), t) \geq \frac{t}{t + \theta(x,y)}. \tag{4.2}$$

Then $H(x) = N - \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$ exists $for\ each\ x \in X$ and defines a $fuzzy\ algebra\ homomorphism$ $H: X \to Y$ such that

$$N(f(x) - H(x), t) \geq \frac{3(1-L)t}{3(1-L)t + L\theta(x,y)}. \tag{4.3}$$

Proof. if $x = y$ in (4.1), then

$$D_f(x, x) = f(3x) - 3f(x)$$

$$N(f(3x) - 3f(x), t) \geq \frac{t}{t + \theta(x,x)}$$

$$N\left(f(x) - 3f\left(\frac{x}{3}\right), t\right) \geq \frac{3t}{3t + L\theta(x,x)} \tag{4.4}$$

Let the set $S = \{g: X \to Y\}$and introduce the generalize matric on$S$:

$d(g,h) = \inf\left\{c \in \mathbb{R}: N(g(x) - h(x), ct) \geq \frac{3t}{3t+L\theta(x,x)}, \forall x \in X, \forall t > 0\right\}.$

(S,d) is complete. By [7]

Let $J: S \to S$ the linear $mapping$ such that $J\big(g(x)\big) = 3g(\frac{x}{3}),\ all\ x \in X.$

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$N(g(x) - h(x), \varepsilon t) \geq \frac{3t}{3t+\theta(x,x)}$, for all $x \in X, t > 0$.

Hence

$$N(J(g(x)) - J(h(x)), L\varepsilon t) = N\left(3g\left(\frac{x}{3}\right) - 3h\left(\frac{x}{3}\right), L\varepsilon t\right)$$

$$= N\left(g\left(\frac{x}{3}\right) - h\left(\frac{x}{3}\right), \frac{L}{3}\varepsilon t\right)$$

$$\geq \frac{3\frac{Lt}{3}}{3\frac{Lt}{3}+L\theta\left(\frac{x}{3},\frac{x}{3}\right)} \geq \frac{t}{t+\theta\left(\frac{x}{3},\frac{x}{3}\right)}$$

$$\geq \frac{t}{t+\frac{L}{3}\theta(x,x)}$$

$$= \frac{3t}{3t+\theta(x,x)}, for\ all\ x \in X, t > 0.$$

we have , $d(g,h) = \varepsilon$,

$d(Jg, Jh) \leq L\varepsilon$

$d(Jg, Jh) \leq Ld(g,h), for\ all\ g, h \in S.$

form (4.4) $d(f, Jf) \leq 1$.

By [7], there exists a $mapping$ $H: X \to Y$ satisfying the following:

1. H is a fixed point of J, i.e.,

$$H\left(\frac{x}{3}\right) = \frac{1}{3}H(x),$$

$for\ all\ x \in X$, the $mapping\ H\ is\ a\ unique\ fixed\ point$ of J in the set

$A = \{g \in S: d(f,g) < \infty\}$. There exists a $c \in (0,\infty)$ such that $d(f,h) < c$

$$N(f(x) - H(x), ct) \geq \frac{3t}{3t + L\theta(x,x)},$$

2. $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$N - \lim_{n\to\infty} 3^n f(\frac{x}{3^n}) = H(x),$$

3. $$d(f,H) \leq \frac{1}{1-L}d(f,Jf)$$

$$d(f,H) \leq \frac{1}{1-L}.$$

By (4.2) we get,

$$N\left(3^i f\left(\frac{(2x+y)}{3^i}\right) + 3^i f\left(\frac{(2x-y)}{3^i}\right) - 3^i f\left(\frac{(x+y)}{3^i}\right) - 3^i f\left(\frac{(x-y)}{3^i}\right) - 3^i 2f\left(\frac{(2x)}{3^i}\right) - 3^i 2f\left(\frac{(x)}{3^i}\right), 3^i t\right) \geq \frac{t}{t+\theta\left(\frac{x}{3^i},\frac{y}{3^i}\right)}$$

$$N\left(3^i f\left(\frac{(2x+y)}{3^i}\right) + 3^i f\left(\frac{(2x-y)}{3^i}\right) - 3^i f\left(\frac{(x+y)}{3^i}\right) - 3^i f\left(\frac{(x-y)}{3^i}\right) - 3^i 2f\left(\frac{(2x)}{3^i}\right) - 3^i 2f\left(\frac{(x)}{3^i}\right), t\right) \geq \frac{\frac{t}{3^i}}{\frac{t}{3^i}+\frac{L^i}{3^i}\theta(x,y)},$$ for all $x, y \in X, t > 0$, (4.5)

Since $\lim_{i\to\infty} \frac{\frac{t}{3^i}}{\frac{t}{3^i}+\frac{L^i}{3^i}\theta(x,y)} = 1, \textit{for all } x, y \in X, t > 0.$

$$N\ (H(2x+y) + H(2x-y) + H(x+y) + H(x-y) - 2H(2x) - 2H(x), t) = 1.$$

Thus $H(2x+y) + H(2x-y) + H(x+y) + H(x-y) - 2H(2x) - 2H(x) = 0$. The $H: X \to Y$ is additive and $\mathbb{R}$ −linear. By (4.3),

$$N\left(3^{2i} f\left(\frac{xy}{3^{2i}}\right) - 3^i f\left(\frac{x}{3^i}\right) 3^i f\left(\frac{y}{3^i}\right), 3^{2i} t\right) \geq \frac{t}{t+\theta\left(\frac{x}{3},\frac{y}{3}\right)}$$

$$N\left(3^{2i} f\left(\frac{xy}{3^{2i}}\right) - 3^i f\left(\frac{x}{3^i}\right) 3^i f\left(\frac{y}{3^i}\right), t\right) \geq \frac{\frac{t}{3^{2i}}}{\frac{t}{3^{2i}}+\frac{L^i}{3^i}\theta(x,y)},$$ for all $x, y \in X, t > 0$. (4.6)

Since $\lim_{i\to\infty} \frac{\frac{t}{3^{2i}}}{\frac{t}{3^{2i}}+\frac{L^i}{3^i}\theta(x,y)} = 1$, for all $x, y \in X, t > 0$,

$$N\big(H(xy) - H(x)H(y)\big) = 1$$

Thus $H(xy) - H(x)H(y) = 0$. $H(x)$ is a *fuzzy algebra homomorphism*.

Theorem 4.2 Let X, Y be two F.B.A. spaces, $f: X \to Y$ odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x), x \in X, \lambda \in \mathbb{R}$. Suppose function $\theta: X^2 \to [0,\infty)$ there exist $L < 1$ *such that*

$$\theta\left(\frac{x}{3},\frac{y}{3}\right) \leq \frac{L}{3}(x,y)$$

$$N\big(D_f(x,y), t\big) \geq \frac{t}{t+\theta(x,y)}. \tag{4.7}$$

$$N(f(xy) - yf(x) - xf(y), t) \geq \frac{t}{t+\theta(x,y)}. \tag{4.8}$$

Then $\delta(x) = N - \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$ exists for each $x \in X$ and defines a *fuzzy algebra derivation* $\delta: X \to Y$ such that

$$N(f(x) - \delta(x), t) \geq \frac{3(1-L)t}{3(1-L)t+L\theta(x,y)}. \tag{4.9}$$

Proof. This theorem can be readily demonstrated utilizing the same way to the Theorem 4.1.

Theorem 4.3 Let X, Y be two F.B.A. spaces, $f: X \to Y$ odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x), x \in X, \lambda \in \mathbb{R}$. Suppose function $\theta: X^2 \to [0,\infty)$ there exist $L < 1$ such that

$$\theta(3x, 3y) \leq 3L\theta(x,y)$$

$$N\big(D_f(x,y), t\big) \geq \frac{t}{t+\theta(x,y)}. \tag{4.10}$$

$$N(f(xy) - f(x)f(y), t) \geq \frac{t}{t+\theta(x,y)}. \quad (4.11)$$

Then $H(x) = N - \lim_{n\to\infty} \frac{f(3^n x)}{3^n}$ exists for each $x \in X$ and defines a $fuzzy\ algebra\ homomorphism\ H: X \to Y$ such that

$$N(f(x) - H(x), t) \geq \frac{3(1-L)t}{3(1-L)t+\theta(x,y)} \quad (4.12)$$

Proof. This theorem can be readily demonstrated utilizing the same way to the Theorem 4.1.

Theorem 4.4 Let X, Y be two F.B.A. spaces, $f: X \to Y$ odd mapping, $f(0) = 0, f(\lambda x) = \lambda f(x), x \in X, \lambda \in \mathbb{R}$. Suppose function $\theta: X^2 \to [0,\infty)$ there exist $L < 1$ such that

$$\theta(3x, 3y) \leq 3L\theta(x, y)$$

$$N\left(D_f(x, y), t\right) \geq \frac{t}{t+\theta(x,y)}. \quad (4.16)$$

$$N(f(xy) - f(x)y - xf(y), t) \geq \frac{t}{t+\theta(x,y)}. \quad (4.17)$$

Then $\delta(x) = N - \lim_{n\to\infty} \frac{f(3^n x)}{3^n}$ exists $for\ each\ x \in X$ and defines a $fuzzy\ algebra\ derivation\ \delta: X \to Y$ such that

$$N(f(x) - \delta(x), t) \geq \frac{3(1-L)t}{3(1-L)t+\theta(x,y)}. \quad (4.18)$$

Proof. This Theorem can be readily demonstrated utilizing the same way to the Theorem 4.1.

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