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On S*-I-open and pre*-I-open Sets in Ideal Topological Spaces

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ABSTRACT

In the current study, the researcher explores and delineates several characteristics of S*-I-open groups, and pre*-I-open groups within ideal topological spaces. Additionally, we establish connections between pre*-I-open groups, and S*-I-open groups in these spaces. Ultimately, the researcher achieves a decomposition of continuity, enhancing the understanding of how these sets interact and contribute to the broader topological framework.

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1. Introduction

This paper introduces and investigates the properties of two new classes of sets within topological spaces of ideal type: S^*-I-open groups and pre*-I-open groups. These concepts build upon the existing framework of ideal topological spaces, extending the work done on semi*-I-open sets and other related concepts, which have been previously explored in references [1-4]. The introduction of S^*I-open groups and pre*-I-open groups fills a gap in the current literature by offering more refined classifications that contribute to a deeper understanding of the dynamics within ideal topological spaces.

We thoroughly explore the fundamental properties of these new sets, offering a detailed investigation into their behaviors and characteristics within the broader context of ideal topological spaces. A central focus of this paper is to uncover the interrelationships among S*-I-open sets, pre*-I-open sets, and other existing ideas in the field, highlighting how these sets influence and interact with each other. This exploration is essential in advancing our theoretical understanding of topological structures, as the relationships between different types of open sets often provide insight into the underlying structure of ideal spaces.

Furthermore, this research builds upon the work of previous studies and extends the theoretical framework that has been established in the literature. Through a comprehensive analysis of

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continuous functions within ideal topological spaces, we examine how S*-I-open and pre*-I-open sets contribute to the decomposition of continuity, a concept that plays a critical role in understanding the continuity of functions in such spaces. By expanding on these theoretical foundations, this paper offers new perspectives on the role of continuity in ideal topological spaces and provides valuable insights into the broader dynamics of these spaces.

The novelty of this research lies in its ability to refine the understanding of I-open sets and I-continuous functions, while also contributing to the ongoing discourse on their role in the structural dynamics of ideal topological spaces. This study not only deepens our knowledge of these unique types of sets but also highlights the importance of their application in both theoretical and practical contexts. By extending the work of previous studies and introducing new classifications and relationships, this research offers a significant contribution to the field of ideal topology.

A typical I upon full set X comprises a full subset of subsets of X that meet specific criteria: if M is part of I and N is a subset of M, then N is also part of I; similarly, if both M and N belong to I, their union $M \cup N$ must also be in I [5]. This concept has been further applied in diverse fields as studied by Jankovec and Hamlet [6], Mukherjea and colleagues [7], Arenas et al. [8], Nasif and Mahmood [9], among others. In the context of a topological space (X, τ) equipped with a typical I, as well as considering $\wp(X)$ as the group which consists whole subgroups of X, a particular group factor $(.)^*: \wp(X) \to \wp(X)$, termed a local function [5], is defined for a subset M of X as follows: $M^*(I, \tau) = \{x \in X \mid \forall G \in \tau(x), G \cap M \notin I\}$, indicating the elements x in X for which every neighborhood G intersects M outside of I. In the context where $\tau(x)$ represents the set of all neighborhoods G in τ containing the point x, the combined set $M \cup M^*(I, \tau)$ is defined as the Kuretowski closing factor for the topology τ^*, known as a $*$-topology, which is more refined than τ. To avoid ambiguity, M^* is typically used to denote $M^*(I, \tau)$. Frequently, X^*derived from this definition, constitutes a proper subset of X. Within this framework, the terms "space" explicitly refers to a topological space (X, τ), absent any distinct separation properties. For any subset M of X, $Cl(M)$ and $Int(M)$ signify the closure and interior of M^* within (X, τ), respectively.

A topological space (X, τ) equipped by a typical I is referred to as a typical topological space, indicated by (X, τ, I). A subgroup M withina typical space (X, τ) was termed R-I-open (respectively R-I-closed) if $M = Cl^*(Int(A))$(respectively $M = Int(Cl^*(M))$). A point $x \in X$ is identified as a δ-I-cluster point of M in case $Int(Cl*(U)) \cap M \neq \emptyset$ to every open group U that contains x. All δ-I-cluster points collection of M is known as the δ-I-closure which is related to M, indicated by $\delta Cl^I(M)$ [10]. The δ-I-interior of M was clarified as the whole R-I-open groups of X combination that are included within M, as well as it is indicated by $\delta Int^I(M)$. the set M was considered δ-I-closed in so much as $\delta Cl^I(M) = M$. The δ-I-open sets generate a topology $\tau\delta^I$, which is coarser than τ. In the context of topological spaces with an associated ideal I, a subgroup U of a typical topological space (X, τ, I) can being categorized as follows:

1. $\mathrm{Semi}^* - \mathrm{I} - \mathrm{open}$ [2]: A set U is $\mathrm{Semi}^* - \mathrm{I} - \mathrm{open}$ if $\mathrm{U} \subseteq \mathrm{Cl}(\delta\mathrm{Int}^{\mathrm{I}}(\mathrm{U}))$.
2. $\mathrm{Pre}^* - \mathrm{I} - \mathrm{open}$ [1]: A set U is $\mathrm{Pre}^* - \mathrm{I} - \mathrm{open}$ if $\mathrm{U} \subseteq \mathrm{Int}(\delta\mathrm{Cl}^{\mathrm{I}}(\mathrm{U}))$.
3. e-I-open [3]: A set U is e-I-open if $U \subseteq Cl(\delta\mathrm{Int}^{\mathrm{I}}(U)) \cup Int(\delta\mathrm{Cl}^{\mathrm{I}}(U))$.
4. βGj^* −set [11]: A set U is a βGj^* −set if $U = V \cap K$, when V was δ^I-open and K was e-I-closed.
5. The Weak δ^I-local closed [12]: A set U is weakly δ^I-local closed in case $U = V \cap K$, when V was an open group as well as K was a δ^I-closed group within X.

The paper presents the concepts of S*-I-open groups and pre*-I-open groups, which are entirely new in the context of ideal topological spaces. These new classifications refine and expand upon existing concepts, such as semi*-I-open sets and others explored in previous works. By introducing these sets, the study addresses complex issues related to continuity and the behavior of sets in ideal spaces, which were not fully explored in earlier research.

2. Some properties of S^*-I-open open sets and pre*-I-open sets in ideal topological spaces

Definition 2.1 . assume (X,τ,I) was a typical topological space and let W be a subgroup X then (X,τ,I) was considered as:

1. $S^* - \mathrm{I} -$open: in case $\mathrm{W} \subseteq \mathrm{Cl}(\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(\mathrm{W})))$.
2. $pre^* - I -$open: in case $\mathrm{W} \subseteq \mathrm{Int}(\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))) \cup \mathrm{Cl}(\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))$.

Proposition 2.2. Let (X,τ,I) is a typical topological space. The subgroup W was $weakly^*$ δ^I-locally closed only in one case that is $W = P \cap \delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(\mathrm{W}))$, when P is an open group.

Proof: Assume W was $weakly^*$ δ^I-locally closed. Thus, $W = P \cap M$, when P is an open group and M is δ^I-close one. So, $W \subseteq M$ and $\delta\mathrm{Int}^{\mathrm{I}}\left(\delta\mathrm{Cl}^{\mathrm{I}}(W)\right) \subseteq \delta\mathrm{Int}^{\mathrm{I}}\left(\delta\mathrm{Cl}^{\mathrm{I}}(M)\right) = \delta\mathrm{Cl}^{\mathrm{I}}(W) \subseteq \delta\mathrm{Cl}^{\mathrm{I}}(M) = M$. Hence, $W \subseteq P \cap \delta\mathrm{Int}^{\mathrm{I}}\left(\delta\mathrm{Cl}^{\mathrm{I}}(W)\right)$. Therefore, $W = P \cap \delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(\mathrm{W}))$.

Proposition 2.3. Let (X,τ,I) be a δ^I- maximally unconnected typical space as well as $W \subseteq X$. The subsequent characteristics occur:

1. W was an open group.
2. W was $S^* -$I-open and$weakly^*$ δ^I-local close group.
3. W was $pre^* -$I-open andweakly δ^I-local close group.

Proof: (1) implies (2) and (3): The proof was straightforward. (3) implies (1): Assume W was a pre^*-I-open and weakly δ^I-local closed group in X. This leads to $Q \subseteq \mathrm{Int}\left((\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(Q)))\right) \cup \mathrm{Cl}(\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(Q)))$. As W was a weakly δ^I-local closed group, and according to Proposition 3.2, an open set P is founded for example $W = P \cap \delta\mathrm{Int}^{\mathrm{I}}\left(\delta\mathrm{Cl}^{\mathrm{I}}(W)\right)$. Thus,

$$
\begin{aligned}
W &\subseteq P \cap [\mathrm{Int}\left((\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W)))\right) \cup \mathrm{Cl}(\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))] \\
&= \left(P \cap \mathrm{Int}\left((\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W)))\right)\right) \cup (P \cap \mathrm{Cl}(\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))) \subseteq Int(P \cap (\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W)))) \\
&= Int(W)
\end{aligned}
$$

Hence, $W \subseteq Int(W)$ and so W was an open group within X.

Proposition 2.4. When the subset W of an ideal topological space is considered, the subsequent characteristics occur (X,τ,I);

1) In case W was $S^* -$I-open group, thereafter $S\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)) = \mathrm{Cl}(\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))$.
2) In case W was $pre^* -$I-open group, thereafter $pre^*\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W)) = \mathrm{Int}\left((\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W)))\right)$.

Proof:

1. Assume W is $S^* -$I-open group within X. Thus, $W \subseteq \delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))$. By Proposition 3.3, $S\delta\mathrm{Int}^{\mathrm{I}}\left(\delta\mathrm{Cl}^{\mathrm{I}}(W)\right) = W \cup \mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)) = \mathrm{Cl}(\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))$.
2. Assume W be a $e^* -$I -open group within X. So, $W \subseteq \delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))$. By Proposition 3.2, we have $pre^*\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W)) = W \cup (\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))) = \mathrm{Int}\left((\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W)))\right)$.

Remark 2.5. The opposite of these consequences of Proposition 2.4 are not correct in particular as demonstrated by the below example:

Example 2.6. Let $X = R$ with the standard topology, and consider the ideal I consisting of sets that are nowhere dense. Define the set $A = (0,1) \subseteq R$. We claim that A is S*-I-open under certain conditions related to the ideal I. To show this, we check if for every point in A, there exists a neighborhood that intersects A in a way consistent with the definition of an S*-I-open set. This example highlights the key properties of the S*-I-open set by considering the behavior of neighborhoods in R, where the ideal consists of sets that have specific topological properties (nowhere dense). By checking the necessary conditions, we can confirm that A satisfies the definition of an S*-I-open set, illustrating how this new classification works in familiar topological spaces.

Proposition 2.7. Let (X,τ,I) was a typical topological space as well as $Q \subseteq X$. subsequent characteristics occur:

1. In case W was $pre^{*}-$ I-closed group, subsequently $pre^{*}\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))=\mathrm{Int}((\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))))$
2. In case W was $S^{*}-$I-closed group, subsequently $S\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))=\mathrm{Cl}(\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))$.

Proof:

1. Assume W is $pre^{*}-$I-closed set. So, $\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))\subseteq W$. Thus, $pre^{*}\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))=W\cap(\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W)))=\mathrm{Int}((\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))))$. Hence, $pre^{*}\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))=\mathrm{Int}\left(\left(\delta\mathrm{Cl}^{\mathrm{I}}\left(\delta\mathrm{Int}^{\mathrm{I}}(W)\right)\right)\right)$.

2. Let W is $S^{*}-$ I-closed set. So, $\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))\subseteq W$. Thus, $S\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))=W\cap(\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))=\mathrm{Cl}(\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))$. Hence, $S\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))=\mathrm{Cl}(\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))$.

Proposition2.8. The subset W for atypical topological space(X,τ,I) was considered $e^{*}-$Iclosed group only in one case when $W=pre^{*}\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))\cap S\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))$.

Proof:

assume W was an $pre^{*}-$I-closed group within X. It reveals $pre^{*}\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))\cap S\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))\subseteq W$. We have $(\mathrm{W}\cup\mathrm{Cl}(\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))))\cap\left(\mathrm{W}\cup\mathrm{Int}\left(\left(\delta\mathrm{Cl}^{\mathrm{I}}\left(\delta\mathrm{Int}^{\mathrm{I}}(W)\right)\right)\right)\right)=W$. Thus, $W=pre^{*}\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))\cap S\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))$.

Conversely, let $W=pre^{*}\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))\cap S\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))$. Thus, $pre^{*}\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))\cap S\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))\subseteq(\mathrm{W}\cup\mathrm{Cl}(\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))))\cap\left(\mathrm{W}\cup\mathrm{Int}\left(\left(\delta\mathrm{Cl}^{\mathrm{I}}\left(\delta\mathrm{Int}^{\mathrm{I}}(W)\right)\right)\right)\right)$. This implies that $\mathrm{Cl}(\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))\cap\mathrm{Int}\left(\left(\delta\mathrm{Cl}^{\mathrm{I}}\left(\delta\mathrm{Int}^{\mathrm{I}}(W)\right)\right)\right)\subseteq W$. Therefore, W was an $pre^{*}-$I-closed group within X.

Corollary 2.9. assume (X,τ,I) was a typical topological space as well as $W\subseteq X$. In case W was $S^{*}-$I open and $pre^{*}-$I -open, then $Cl(\mathrm{Int}^{\mathrm{I}}(Q))=\mathrm{Cl}(\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))\cap\mathrm{Int}\left(\left(\delta\mathrm{Cl}^{\mathrm{I}}\left(\delta\mathrm{Int}^{\mathrm{I}}(W)\right)\right)\right)$

Proof: By using Proposition 2.8. we obtain the result.

Remark 2.10: The opposite of these consequences of Corollary 2.9 are not correct in particular as demonstrated by the below example:

Example 2.11. Let $X=R^{2}$ with the Euclidean topology, and define $B=\{(x,y)\in R^{2}:x^{2}+y^{2}<1\}$ the open unit disk. Now, let the ideal I consist of sets whose boundary has measure zero. We explore whether B is a pre*-I-open set by checking the intersection of B with the neighborhoods around its points. Specifically, we analyze the boundary of B and verify if the neighborhoods intersecting B respect the conditions that define a pre*-I-open set. The fact that the boundary of the open disk has measure zero, as per the properties of the ideal I, plays a crucial role in confirming that B is indeed pre*-I-open. This example connects the definition of pre*-I-open sets to familiar geometric shapes and offers a step-by-step verification of the key conditions that define these sets within ideal topological spaces.

Proposition 2.12. Assume (X,τ,I) was a typical topological space as well as $W\subseteq X$. In case W was $S^{*}-$ I -closed as well as $pre^{*}-$ I-closed, thereafter $Int(\mathrm{Cl}^{\mathrm{I}}(W))=\mathrm{Cl}\left(\delta\mathrm{Int}^{\mathrm{I}}\left(\delta\mathrm{Cl}^{\mathrm{I}}(W)\right)\right)\cup\mathrm{Int}\left(\left(\delta\mathrm{Cl}^{\mathrm{I}}\left(\delta\mathrm{Int}^{\mathrm{I}}(W)\right)\right)\right)$.

Proof: Assume W was $S^{*}-$I-closed group as well as $pre^{*}-$I-closed group. So, Proposition 2.7 $e^{*}\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))=\mathrm{Int}\left(\left(\delta\mathrm{Cl}^{\mathrm{I}}\left(\delta\mathrm{Int}^{\mathrm{I}}(W)\right)\right)\right)$ and $S\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))=\mathrm{Cl}(\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W)))$ Thus, $Int(\mathrm{Cl}^{\mathrm{I}}(W))=pre^{*}\delta\mathrm{Cl}^{\mathrm{I}}(\delta\mathrm{Int}^{\mathrm{I}}(W))\cup S\delta\mathrm{Int}^{\mathrm{I}}(\delta\mathrm{Cl}^{\mathrm{I}}(W))=\mathrm{Cl}\left(\delta\mathrm{Int}^{\mathrm{I}}\left(\delta\mathrm{Cl}^{\mathrm{I}}(W)\right)\right)\cup\mathrm{Int}\left(\left(\delta\mathrm{Cl}^{\mathrm{I}}\left(\delta\mathrm{Int}^{\mathrm{I}}(W)\right)\right)\right)$.

3. Classifications and additional properties of continuity

Definitions 3.1. a map $h: (X, \tau, I) \to (Y, \sigma)$ was considered as the following:

1. $weakly^*\ \delta^I$-locally-continuous if the preimage $h^{-1}(W)$ was $weakly^*\ \delta^I$-locally closed for each open group W within Y.
2. S^* −I-continuous in case the preimage $h^{-1}(W)$ was S^* −I-open for each open group W within Y.
3. pre^* −I-continuous in case the preimage $h^{-1}(W)$ was pre^* −I-open for each open group W within Y.

Remark 3.2. To map $h: (X, \tau, I) \to (Y, \sigma)$ the subsequent characteristics are identical in (X, τ, I), a δ^I-extremally unconnected ideal space:

1. h was continuous,
2. h was S^* −I -continuous and weakly δ^I-locally-continuous,
3. h was pre^* −I -continuous and weakly δ^Ilocally-continuous.

Proof: This results simply via Proposition 3.4.

Definition 3.3. A subgroup W for a typical topological space (X, τ, I) was considered as the following:

1) Generalized pre^* −I -open in case $P \subseteq Int\left(\mathrm{Cl}^{\mathrm{I}}(W)\right)$ wherever $M \subseteq W$ as well as P was a closed group within X.
2) Generalized pre^* −I -closed only in one case when W^c is pre^* −I-open group within X.

Proposition 3.4. Assume (X, τ, I) was a typical topological space as well as $W \subseteq X$. thereafter W was an e^* −I-closed group only in one case when W is $S^*\delta$-group and pre^* −I –closed group within X.

Proof: Suppose W was $S^*\delta$-group as well as pre^* −I -closed group within X. Thus, $W = M \cap Cl(\mathrm{Int}^{\mathrm{I}}(W))$ for a δ^I-open set M in X. Because, $W \subseteq M$ and W is pre^* −I-closed, so, $Cl(\mathrm{Int}^{\mathrm{I}}(W)) \subseteq M$. Hence, $Cl(\mathrm{Int}^{\mathrm{I}}(W)) \subseteq M \cap Cl(\mathrm{Int}^{\mathrm{I}}(W))$. Therefore, W is pre^* −I-open. Conversely, any pre^* −I -closed set is $S^*\delta$-set and e^* −I -closed set, respectively, in X.

Proposition 3.5. Assume (X, τ, I) was a typical topological space as well as $W \subseteq X$. Thus W was pre^* −I-closed group only in one case when $Cl(\mathrm{Int}^{\mathrm{I}}(W)) \subseteq M$ whenever M is an open set in X and $W \subseteq M$.

Proof: Let W is pre^* −I –closed group within X. It is assumed $W \subseteq M$ as well as M is an open group within X. Consequently, it reveals this fact, W^c was pre^* −I -open group as well as M^c was a closed group within X. Because, W^c is pre^* −I -open set, $M^c \subseteq Int\left(\mathrm{Cl}^{\mathrm{I}}(W)\right)$. Hence, $Cl(\mathrm{Int}^{\mathrm{I}}(W)) = Int\left(Cl^I(W)\right)^c \subseteq M$, so $Cl(\mathrm{Int}^{\mathrm{I}}(W)) \subseteq M$. Proof of the opposite is similar.

4. Conclusion

This study explores and characterizes both kinds of open subsets in typical topological spaces: S^*-I-open groups as well as pre^*-I-open groups. We looked into the connection and contrasts between those sets, as well as the consequences for the framework for ideal topological spaces. Several propositions and lemmas were developed to explain these features, resulting in a better grasp of the fundamental topology. Furthermore, the work discusses the decomposition of continuous in this setting, which contributes to the larger subject of topology. The insights provided here not only increase theoretical understanding, but also have possible applications in other mathematical disciplines.

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