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# Exponentially Fitted - Diagonally Implicit Runge-Kutta Method for Direct Solution of Fifth-Order Ordinary Differential Equations

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ABSTRACT

In this paper, an Exponentially fitted - Diagonally implicit Runge-Kutta method is constructed, which can solve fifth-order ordinary differential equations (ODEs) directly. The order conditions are calculated using the expansion of the B-string theory and the colored tree theory to determine the ranking criteria of the Diagonally Implicit Runge-Kutta Method (DIRKF) approach. As a result, a five-degree, three-stage exponentially fitted - diagonally implicit Runge-Kutta method (EFDIRKFO5) is formulated. Comparing this method with existing implicit Runge-Kutta methods, numerical experiments show that the former is more accurate and requires fewer function evaluations.

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## Introduction

Ordinary differential equations are widely used in many scientific and engineering fields such as modeling the motion of objects (Jaleel & Fawzi,2023), studying vibrating systems (Hairer &Wanner,1991), heat and energy propagation, analysis of electrical circuits (Saleh, Fawzi & Hussain, 2023), fluid dynamics, structural mechanics (Fawzi & Jaleel, 2023), and biological processes such as biochemical reactions(Fawzi & Globe,2023). Differential equations are a powerful mathematical tool for providing a deep understanding of dynamic changes in various systems, and are an integral part of studying and interpreting the world around us (Butcher, 2016).

Implicit methods are significant due to their capacity to achieve high accuracy levels with a comparable number of stages, presenting an advantage that results in more precision than explicit approaches. This aids in resolving the challenges associated with the previously listed applications (Fawzi & Jaleel, 2023). Consequently, implicit Runge-Kutta methods are significant. Classifying physical and mathematical problems, such as differential algebraic equations, is essential (Hairer, Wanner & Lubich, 2006).

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Recently, numerous scholars have formulated exponentially-fitted implicit Runge-Kutta methods for addressing first-order and second-order ordinary differential equations. Vanden Berghe et al. (Ghawadri, Senu, Ismail & Ibrahim, 2018) formulated exponentially-fitted Runge-Kutta algorithms. (Simos,2000) expanded these exponentially-fitted Runge-Kutta methods for the numerical resolution of the Schrödinger equation and associated issues. (Kalogiratou, & Simos,2002) developed trigonometrically and exponentially fitted Runge-Kutta-Nyström algorithms for the numerical resolution of the Schrödinger equation and associated issues, attaining an eighth algebraic order. Simos et al. (Simos,2002) developed an exponentially-fitted Runge-Kutta-Nyström approach for numerically solving initial-value problems characterized by oscillatory solutions. (Berghe, Meyer, Daele & Hecke,2000) investigated an exponentially-fitted fourth-order explicit modified Runge-Kutta method for addressing third-order ordinary differential equations. (Fawzi & Jumma, 2022) devised Runge-Kutta methods for addressing third-order ordinary differential equations and first-order oscillatory issues.

Conventional numerical methods used to solve higher-order ordinary differential equations require transforming the higher-order differential equation into a system of first-order differential equations. This transformation takes time and effort, and the method may be inaccurate. In this paper, we will derive a direct numerical method to solve the fifth-order differential equation directly, without the need to transform it into a system of first-order differential equations. The method will be derived based on the colored trees theory and beta series theory to obtain the values of the phase conditions for the fifth-order method, exponential functions are included to obtain new values for the method coefficients. The purpose of this inclusion is to obtain an exponentially fitted - diagonally implicit Runge - Kutta method of order fifth (EFDIRKFO5) capable of direct solve a fifth-order differential equations in addition to dealing with stiff problems.

This work deals with exponentially-fitted explicit modified Runge-Kutta type methods for solving fifth-order ordinary differential equations (ODEs) of the form

$$\wp^{(5)}(q) = g(q,\wp(q)), \quad q \geq q_0 \qquad (1)$$

with initial conditions

$$\wp(q_0)=\wp_0 \;, \wp'(q_0)=\wp_0' \;, \wp''(q_0)=\wp_0'' \;, \wp'''(q_0)=\wp_0''' \;, \wp''''(q_0)=\wp_0''''$$

where a continuous-valued function $\wp(q) \in \mathcal{R}^d$, $f:\mathcal{R}^d \times \mathcal{R}^d \to \mathcal{R}^d$ is that does not include its first, second, third, or fourth derivatives.

## Derivation Of The Order Conditions For Efdirkfo5 Method

Problem (1.1) can be express by a system of first-order ODEs, as below:

$$\begin{pmatrix} \wp(q) \\ k(q) \\ l(q) \\ m(q) \\ w(q) \end{pmatrix}' = \begin{pmatrix} k(q) \\ l(q) \\ m(q) \\ w(q) \\ g(q,\wp(q)) \end{pmatrix} \qquad (2)$$

with initial conditions

$$\wp(q_n)=\wp_n \;, \; k(q_n)=\wp_n' \;, \; l(q_n)=\wp_n'' \;, m(q_n)=\wp_n''' \;, w(q_n)=\wp_n''''$$

The s-stage Runge-Kutta technique for solving first-order initial value problems (IVPs) $y' = g(q,\wp(q))$ define as:

$$\wp_{n+1} = \wp_n + h\sum_{i=1}^{s} b_i g(q_n + c_i h\,, \mathfrak{p}_i)\,,$$

$$\mathfrak{p}_i = \wp_n + h\textstyle\sum_{i,j=1}^{s} a_{ij} g\left(q_n + c_i h\,, \mathfrak{p}_j\right) \qquad (3)$$

We can be expressed The EFDIRKFO5 technique with an s-stage for solving equation (2) in the following general form:

$$\mathfrak{p}_i = \wp_n + h\textstyle\sum_{i,j=1}^{s} a_{ij}\mathfrak{p}_j' \qquad (4)$$

$$\mathfrak{p}_i' = \wp_n + h\textstyle\sum_{i,j=1}^{s} a_{ij}\mathfrak{p}_j'' \qquad (5)$$

$$\mathfrak{p}_i'' = \wp_n + h\sum_{i,j=1}^{s} a_{ij}\mathfrak{p}_j''' \quad (6)$$

$$\mathfrak{p}_i''' = \wp_n + h\sum_{i,j=1}^{s} a_{ij}\mathfrak{p}_j'''' \quad (7)$$

$$\mathfrak{p}_i'''' = \wp + h\sum_{i,j=1}^{s} a_{ij}\boldsymbol{f}\left(q_n + c_i h\,, \mathfrak{p}_j\right) \quad i = 1,2\ \dots \dots, s \quad (8)$$

$$\wp_{n+1} = \wp_n + h\sum_{i=1}^{s} \wp_i \mathfrak{p}_i' \quad (9)$$

$$\wp_{n+1}' = \wp_n' + h\sum_{i=1}^{s} \wp_i \mathfrak{p}_i'' \quad (10)$$

$$\wp_{n+1}'' = \wp_n' + h\sum_{i=1}^{s} \wp_i \mathfrak{p}_i''' \quad (11)$$

$$\wp_{n+1}''' = \wp_n' + h\sum_{i=1}^{s} \wp_i \mathfrak{p}_i'''' \quad (12)$$

$$\wp_{n+1}'''' = \wp_n' + h\sum_{i=1}^{s} \wp_i \boldsymbol{f}\left(q_n + c_i h\,, \mathfrak{p}_j\right) \quad (13)$$

If we ignoring $\mathfrak{p}_j', \mathfrak{p}_j'', \mathfrak{p}_j'''$ *and* $\mathfrak{p}_j''''$ from Eqs. (4) – (13), we conclude

$$\mathfrak{p}_i = \wp_n + h\sum_{j=1}^{s} a_{ij}\wp_n' + h^2\sum_{j,k=1}^{s} a_{ij}a_{jk}\wp_n'' + h^3 \sum_{j,k,l=1}^{s} a_{ij}a_{jk}a_{kl}\wp_n'''$$

$$+h^4\sum_{j,k,l,r=1}^{s} a_{ij}a_{jk}a_{kl}a_{lr}\wp_n'''' + h^5\sum_{j,k,l,r,q=1}^{s} a_{ij}a_{jk}a_{kl}a_{lr}a_{rq}\boldsymbol{f}\left(q_n + c_q h\,, \mathfrak{p}_q\right),\ i = 1,2\ \dots \dots s \quad (14)$$

$$\wp_{n+1} = \wp_n + h\sum_{i=1}^{s} \wp_i\wp_n' + h^2\sum_{i,j=1}^{s} \wp_i a_{ij}\wp_n'' + h^3 \sum_{i,j,k,=1}^{s} \wp_i a_{ij}a_{jk}\wp_n'''$$

$$+h^4\sum_{i,j,k,l=1}^{s} \wp_i a_{ij}a_{jk}a_{kl}\wp_n'''' + h^5\sum_{i,j,k,l,r=1}^{s} \wp_i a_{ij}a_{jk}a_{kl}a_{lr}\boldsymbol{f}(q_n + c_r h\,, \mathfrak{p}_r) \quad (15)$$

$$\wp_{n+1}' = \wp_n' + h\sum_{i=1}^{s} \wp_i\wp_n'' + h^2\sum_{i,j=1}^{s} \wp_i a_{ij}\wp_n''' + h^3 \sum_{i,j,k,=1}^{s} \wp_i a_{ij}a_{jk}\wp_n''''$$

$$+h^4\sum_{i,j,k,l=1}^{s} \wp_i a_{ij}a_{jk}a_{kl}\boldsymbol{f}(q_n + c_l h\,, \mathfrak{p}_l) \quad (16)$$

$$\wp_{n+1}'' = \wp_n'' + h\sum_{i=1}^{s} \wp_i\wp_n''' + h^2\sum_{i,j=1}^{s} \wp_i a_{ij}\wp_n'''' + h^3 \sum_{i,j,k=1}^{s} \wp_i a_{ij}a_{jk}\boldsymbol{f}(q_n + c_k h\,, \mathfrak{p}_k) \quad (17)$$

$$\wp_{n+1}''' = \wp_n''' + h\sum_{i=1}^{s} \wp_i\wp_n'''' + h^2\sum_{i,j=1}^{s} \wp_i a_{ij}\boldsymbol{f}\left(q_n + c_j h\,, \mathfrak{p}_j\right) \quad (18)$$

$$\wp_{n+1}'''' = \wp_n'''' + h\sum_{i=1}^{s} \wp_i a_{ij}\boldsymbol{f}(q_n + c_i h\,, \mathfrak{p}_i) \quad (19)$$

Suppose that

$$\sum_{j=1}^{s} a_{ij} = c_i,\quad \sum_{j,k=1}^{s} a_{ij}a_{jk} = \frac{1}{2}c_i^2,\quad \sum_{j,k,l=1}^{s} a_{ij}a_{jk}a_{kl} = \frac{1}{6}c_i^3,\quad \sum_{j,k,l=1}^{s} a_{ij}a_{jk}a_{kl}a_{lr} = \frac{1}{24}c_i^4,$$

$$\sum_{j=1}^{s} \wp_i = 1,\quad \sum_{j=1}^{s} \wp_i a_{ij} = \frac{1}{2},\quad \sum_{j=1}^{s} \wp_i a_{ij}a_{jk} = \frac{1}{6},\quad \sum_{j=1}^{s} \wp_i a_{ij}a_{jk}a_{kl} = \frac{1}{24},\qquad i = 1,2,\dots\dots s$$

and signify $\wp^T A = \wp'^T$ , $\wp^T A^2 = \wp''^T$ , $\wp^T A^3 = \wp'''^T$ , $\wp^T A^4 = \wp''''^T$ , $A^4 = \hat{A}$
i.e.
$\sum_{j=1}^{s} \wp_i a_{ij} = \wp_i'$, $\sum_{j,k=1}^{s} \wp_i a_{ij}a_{jk} = \wp_i''$, $\sum_{j,k,l=1}^{s} \wp_i a_{ij}a_{jk}a_{kl} = \wp_i'''$, $\sum_{j,k,l,r=1}^{s} \wp_i a_{ij}a_{jk}a_{kl}a_{lr} = \wp_i''''$, $\sum_{j,k,l,r,q=1}^{s} a_{ij}a_{jk}a_{kl}a_{lr}a_{rq} = \hat{a}_{ij}$, $i = 1, \dots\dots\ s$

Consequently, the specific fifth-order IVP (1), as indicated by the DIRKF approach, can be solved using the following direct integration method. DIRKF approach for solving the initial value problem (1) is represented by the following formula:

$$\wp_{n+1} = \wp_n + h\wp_n' + \frac{h^2}{2}\wp_n'' + \frac{h^3}{6}\wp_n''' + \frac{h^4}{24}\wp_n'''' + h^5\sum_{i=1}^{s} \wp_i\boldsymbol{f}(q_n + c_i h\,, \mathfrak{p}_i) \quad (20)$$

$$\wp_{n+1}' = \wp_n' + h\wp_n'' + \frac{h^2}{2}\wp_n''' + \frac{h^3}{6}\wp_n'''' + h^4\sum_{i=1}^{s} \wp_i'\boldsymbol{f}(q_n + c_i h\,, \mathfrak{p}_i) \quad (21)$$

$$\wp''_{n+1} = \wp''_n + h\wp'''_n + \frac{h^2}{2}\wp''''_n + h^3\sum_{i=1}^{s}\wp''_i \boldsymbol{f}(q_n + c_i h\,, \mathfrak{p}_i) \quad (22)$$

$$\wp'''_{n+1} = \wp'''_n + h\wp''''_n + h^2\sum_{i=1}^{s}\wp'''_i \boldsymbol{f}(q_n + c_i h\,, \mathfrak{p}_i) \quad (23)$$

$$\wp''''_{n+1} = \wp''''_n + h\sum_{i=1}^{s}\wp''''_i \boldsymbol{f}(q_n + c_i h\,, \mathfrak{p}_i) \quad (24)$$

$$\mathfrak{p}_i = \wp_n + hc_i\wp'_n + \frac{h^2}{2}c_i^2\wp''_n + \frac{h^3}{6}c_i^3\wp'''_n + \frac{h^4}{24}c_i^4\wp''''_n + h^5\sum_{i,j=1}^{s}\hat{a}_{ij}\boldsymbol{f}(q_n + c_i h\,, \mathfrak{p}_i) \quad (25)$$

All parameters $a_{ij}, \wp_i, \wp'_i, \wp''_i, \wp'''_i, \wp''''_i$ and $c_i$ where $i, j = 1,2 \ldots \ldots s$. are real numbers. Representation of the DIRKF method (20) – (25) in the Butcher tableau is as follows:

**Table 1 - the butcher tableau EFDIRKFO5 method**

| | | | | |
|---|---|---|---|---|
| $c_1$ | $a_{11}$ | $a_{12}$ | … | $a_{1s}$ |
| $c_2$ | $a_{21}$ | $a_{22}$ | … | $a_{2s}$ |
| $c_3$ | $a_{31}$ | $a_{32}$ | … | $a_{3s}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $c_s$ | $a_{s1}$ | $a_{s2}$ | … | $a_{ss}$ |
| | $b_1$ | $b_2$ | … | $b_s$ |
| | $b'_1$ | $b'_2$ | … | $b'_s$ |
| | $b''_1$ | $b''_2$ | … | $b''_s$ |
| | $b'''_1$ | $b'''_2$ | … | $b'''_s$ |
| | $b''''_1$ | $b''''_2$ | … | $b''''_s$ |

By expanding the EFDIRKFO5 method statement, using Taylor series expansion, the parameters of the new method given by (20) – (25) are derived, as this expansion is equivalent to the exact solution obtained by Taylor series expansion. The specific conditions of the new technique are determined by analyzing the direct truncation error at the local level. This concept is based on the development of criteria for determining the order of the RK approach, as mentioned in references.

**Definition 1** The EFDIRKFO5 method (20)– (25) has order p when problem (1) is considered with the assumption

$$\wp(q_0) = \wp_0\;, \wp'(q_0) = \wp'_0\;, \wp''(q_0) = \wp''_0\;, \wp'''(q_0) = \wp'''_0\;, \wp''''(q_0) = \wp''''_0$$

Therefore, the local truncation error (LTE) for the exact solution, as well as its first, second, third, and fourth derivatives, must be satisfied (Hussain, Ismail & Senu, 2016).

$$\wp(q_n + h) - \wp_{n+1} = O(h^{p+1}),$$

$$\wp'(q_n + h) - \wp'_{n+1} = O(h^{p+1}),$$

$$\wp''(q_n + h) - \wp''_{n+1} = O(h^{p+1}),$$

$$\wp'''(q_n + h) - \wp'''_{n+1} = O(h^{p+1}),$$

$$\wp''''(q_n + h) - \wp''''_{n+1} = O(h^{p+1}) \quad (26)$$

We use the next autonomous form of problem (1) to derive the order conditions for the EFDIRKFO5 method (20) – (25).

$$p^{(5)}(q) = g(p(q)) \tag{27}$$

with initial conditions

$$p(q_n) = p_n \;, p'(q_n) = p'_n \;, p''(q_n) = p''_n \,, p'''(q_n) = p'''_n \;, p''''(q_n) = p''''_n$$

Problem (1) can be reformulated as an equivalent autonomous problem by extending it with an additional one-dimensional vector $\omega = q$ as follows:

$$\omega^{(v)} = 0 \tag{28}$$

$$p^{(5)} = g(\omega, p) \tag{29}$$

$$\omega(q_n) = \omega_n = q_n, \qquad \omega'(q_n) = \omega'_n = 1, \qquad \omega''(q_n) = \omega''_n = 0 \;, \omega'''(q_n) = \omega'''_n = 0,$$

$$\omega''''(q_n) = \omega''''_n = 0 \tag{30}$$

$$p(q_n) = p_n \;, p'(q_n) = p'_n \;, p''(q_n) = p''_n \,, p'''(q_n) = p'''_n \;, p''''(q_n) = p''''_n \tag{31}$$

Applying EFDIRKFO5 method (20) – (25) to the scheme (27)– (30), we obtain:

$$\omega_i = \omega_n + hc_i\omega'_n + \frac{h^2}{2}c_i^2\omega''_n + \frac{h^3}{6}c_i^3\omega'''_n + \frac{h^4}{24}c_i^4\omega''''_n \tag{32}$$

$$\mathfrak{p}_i = p_n + hc_i p'_n + \frac{h^2}{2}c_i^2 p''_n + \frac{h^3}{6}c_i^3 p'''_n + \frac{h^4}{24}c_i^4 p''''_n + h^5\sum_{i,j=1}^{s}\hat{a}_{ij}\,g(\omega_j, \mathfrak{p}_j) \tag{33}$$

$$\omega_{n+1} = \omega_n + h\omega'_n + \frac{h^2}{2}\omega''_n + \frac{h^3}{6}\omega'''_n + \frac{h^4}{24}\omega''''_n \tag{34}$$

$$\omega'_{n+1} = \omega'_n + h\omega''_n + \frac{h^2}{2}\omega'''_n + \frac{h^3}{6}\omega''''_n \tag{35}$$

$$\omega''_{n+1} = \omega''_n + h\omega'''_n + \frac{h^2}{2}\omega''''_n \tag{36}$$

$$\omega'''_{n+1} = \omega'''_n + h\omega''''_n \tag{37}$$

$$\omega''''_{n+1} = \omega''''_n \tag{38}$$

$$p_{n+1} = p_n + hp'_n + \frac{h^2}{2}p''_n + \frac{h^3}{6}p'''_n + \frac{h^4}{24}p''''_n + h^5\sum_{i=1}^{s} b_i\,g(\omega_i, \mathfrak{p}_i) \tag{39}$$

$$p'_{n+1} = p'_n + hp''_n + \frac{h^2}{2}p'''_n + \frac{h^3}{6}p''''_n + h^4\sum_{i=1}^{s} b'_i\,g(\omega_i, \mathfrak{p}_i) \tag{40}$$

$$p''_{n+1} = p''_n + hp'''_n + \frac{h^2}{2}p''''_n + h^3\sum_{i=1}^{s} b''_i\,g(\omega_i, \mathfrak{p}_i) \tag{41}$$

$$p'''_{n+1} = p'''_n + hp''''_n + h^2\sum_{i=1}^{s} b'''_i\,g(\omega_i, \mathfrak{p}_i) \tag{42}$$

$$p''''_{n+1} = p''''_n + h\sum_{i=1}^{s} b''''_i\,g(\omega_i, \mathfrak{p}_i) \tag{43}$$

Substituting Equation (30) into the system of equations (32) – (43), we get

$$\omega_i = q_n + c_i h \tag{44}$$

$$\omega_{n+1} = q_n + h \tag{45}$$

$$\omega'_{n+1} = 1 \tag{46}$$

$$\omega''_{n+1} = 0 \tag{47}$$

$$\omega'''_{n+1} = 0 \tag{48}$$

$$\omega''''_{n+1} = 0 \tag{49}$$

$$p_{n+1} = p_n + hp'_n + \frac{h^2}{2}p''_n + \frac{h^3}{6}p'''_n + \frac{h^4}{24}p''''_n + h^5\sum_{i=1}^{s} b_i g(q_n + c_i h, \mathfrak{p}_i) \tag{50}$$

$$p'_{n+1} = p'_n + hp''_n + \frac{h^2}{2}p'''_n + \frac{h^3}{6}p''''_n + h^4\sum_{i=1}^{s} b'_i g(q_n + c_i h, \mathfrak{p}_i) \tag{51}$$

$$p''_{n+1} = p''_n + hp'''_n + \frac{h^2}{2}p''''_n + h^3\sum_{i=1}^{s} b''_i g(q_n + c_i h, \mathfrak{p}_i) \tag{52}$$

$$p'''_{n+1} = p'''_n + hp''''_n + h^2\sum_{i=1}^{s} b'''_i g(q_n + c_i h, \mathfrak{p}_i) \tag{53}$$

$$p''''_{n+1} = p''''_n + h\sum_{i=1}^{s} b''''_i g(q_n + c_i h, \mathfrak{p}_i) \tag{54}$$

$$\mathfrak{p}_i = p_n + hc_i p'_n + \frac{h^2}{2}c_i^2 p''_n + \frac{h^3}{6}c_i^3 p'''_n + \frac{h^4}{24}c_i^4 p''''_n + h^5\sum_{i,j=1}^{s} \hat{a}_{ij} g\left(q_n + c_i h, \mathfrak{p}_j\right) \tag{55}$$

We can conclude that Equations (50)– (55) are entirely analogous to the system of equations (20) – (25) was derived by using the EFDIRKFO5 method to problem (1). Hence, it is adequate to consider the numerical solutions for the autonomous form provided by Equation (27). Consequently, the EFDIRKFO5 method, as outlined in Equations (20) – (25), can be reformulated as follows:

$$p_{n+1} = p_n + hp'_n + \frac{h^2}{2}p''_n + \frac{h^3}{6}p'''_n + \frac{h^4}{24}p''''_n + h^5\sum_{i=1}^{s} b_i g(\mathfrak{p}_i),$$

$$p'_{n+1} = p'_n + hp''_n + \frac{h^2}{2}p'''_n + \frac{h^3}{6}p''''_n + h^4\sum_{i=1}^{s} b'_i g(\mathfrak{p}_i),$$

$$p''_{n+1} = p''_n + hp'''_n + \frac{h^2}{2}p''''_n + h^3\sum_{i=1}^{s} b''_i g(\mathfrak{p}_i),$$

$$p'''_{n+1} = p'''_n + hp''''_n + h^2\sum_{i=1}^{s} b'''_i g(\mathfrak{p}_i),$$

$$p''''_{n+1} = p''''_n + h\sum_{i=1}^{s} b''''_i g(\mathfrak{p}_i),$$

$$\mathfrak{p}_i = p_n + hc_i p'_n + \frac{h^2}{2}c_i^2 p''_n + \frac{h^3}{6}c_i^3 p'''_n + \frac{h^4}{24}c_i^4 p''''_n + h^5\sum_{i,j=1}^{s} \hat{a}_{ij} g\left(\mathfrak{p}_j\right) \tag{56}$$

The elementary differentials listed below are derived by applying the elementary differential notation to the analytical solution $p(q)$

$$p^{(1)} = p' \;, p^{(2)} = p'' \;, \; p^{(3)} = p'' \;, \; p^{(4)} = p'''' \;, \; p^{(5)} = g$$

$$p^{(6)} = g' p', \; p^{(7)} = g' p'' + g''(p', p')$$

$$p^{(8)} = g' p''' + 3g''(p', p'') + g'''(p', p', p')$$

$$p^{(9)} = g'p'''' + 4g''(p', p''') + 3g''(p'', p'') + 6g^{(3)}((p', p', p'') + g^{(4)}((p', p', p', p')$$

$$p^{(10)} =$$
$$g'g + 5g''(p', p''''') + 10g''(p'', p''') + 10g^{(3)}((p', p', p''') + 15g^{(3)}((p', p'', p'') + 10g^{(4)}((p', p', p', p'') + g^{(5)}((p', p', p', p', p') \quad (57)$$

These processes become more difficult very quickly as demand increases. An optimal way to overcome this challenge, according to (Hussain, Ismail & Senu, 2016), would be to use a graphical representation with some modifications of fifth-order ODEs, which are denoted by relevant colored trees. The five types of nodes in trees with related colors are "meager", "black ball", "white ball", "meager ball inside the white ball" and "star inside the white ball" and they are connected by arcs. In these trees:

1- The meager node is used to denote every $p'$.

2- The end black ball node to denote every $p''$.

3- The end white ball node to denote every $p'''$.

4- The end meager ball node to denote every $p''''$ .

5- The end star inside the white ball node to denote every $g$ , and every arc to denote every arc, leaving this node to represent the m-th derivative of $g$ with respect to $\omega$. In addition, $t_1, t_2, t_3, t_4$, and $t_5$ denote from first to fifth -order tree respectively. (see Fig.1)

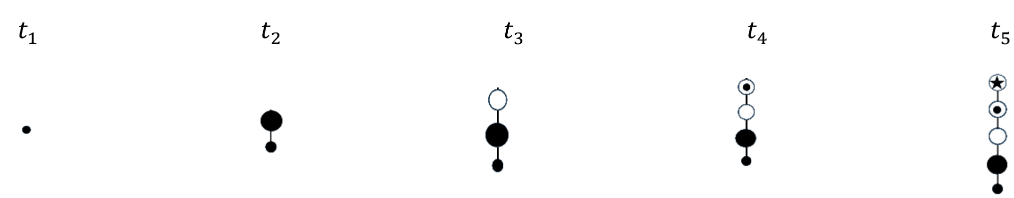



**Figure 1- the colored trees**

The following basic definitions of relevant-colored trees and their associated B series are necessary to support this work.

**Definition 2** The symmetry $\varrho(\Gamma)$ and order $(\Gamma)$ functions are defined recursively as follows:

1- $\Theta(t_1) = 1,\ \Theta(t_2) = 2,\ \Theta(t_3) = 3, \Theta(t_2) = 4,\ \Theta(t_3) = 5,$

2- $\varrho(t_1) = \varrho(t_2) = \varrho(t_3) = \varrho(t_3) = \varrho(t_4) = 1,$

3- If $\Gamma = [\Gamma_1, \Gamma_2, \dots, \Gamma_m]_5$ for each $\Gamma \in RT$, then $\Theta(\Gamma) = 5 + \sum_{i=1}^{m} \Theta(\Gamma_i)$ and $\varrho(\Gamma) = \prod_{i=1}^{m} \varrho(\Gamma_i)(v_1!\ v_2! \dots)$, where number of nodes of $\Gamma$ is $\Theta(\Gamma)$, $\forall\Gamma \in$ R and $v_1!\ v_2! \dots$ count equal trees among $\Gamma_1, \Gamma_2, \dots, \Gamma_m$. Then, defined the set $Sr$ which consists of every tree $RT$ of order.

**Lemma 1:** Let $\delta$ be a function $\delta: RT \cup \{\emptyset\} \to R^d$ with $\delta(\emptyset) = 1$. Thus $h^5 f(B(\delta, p)$ is also a B-series $h^5 g(B(\delta, p)) = B(\delta', p)$ where $\delta'(\emptyset) = \delta'(t_1) = \delta'(t_2) = \delta'(t_3) = \delta'(t_4) = 0$, $\delta'(t_5) = 1$ , and for $\Gamma\ [\Gamma_1, \Gamma_2, \dots, \Gamma_m]_5 \in RT$, $\delta'(\Gamma) = \delta(\Gamma_1), \cdots, \delta(\Gamma_m)$.

**Lemma 2:** If we assume that the analytic solution of Equation (27) is a B-series $B(\vartheta, q_0)$ which is defined on $RT \cup \{\emptyset\}$, with a real function $\vartheta$ , then.

$\varrho(t_1) = \varrho(t_3) = \varrho(t_3) = \varrho(t_4) = \varrho(t_5) = 1$

And $\Gamma = [\Gamma_1, \Gamma_2, \dots, \Gamma_m]_5 \in RT$, the following formula is obtained:

$$\varrho(\Gamma) = \frac{1}{\Theta(\Gamma)(\Theta(\Gamma)-1)(\Theta(\Gamma)-2)(\Theta(\Gamma)-3)(\Theta(\Gamma)-4)} \left(\varrho(\Gamma_1), \cdots, \varrho(\Gamma_m)\right).$$

**Proposition 1:** The density $\varrho(\Gamma)$ is the nonnegative integer factors defined on trees $RT$ , $\forall\Gamma \in RT$ satisfies:

1- $\varrho(t_1) = 1$, $\varrho(t_2) = 2$ , $\varrho(t_3) = 6$, $\varrho(t_4) = 24$ , $\varrho(t_5) = 120$
2- with $\Gamma = [\Gamma_1, \Gamma_2, \dots, \Gamma_m]_5$, this equation is obtained.
$\varrho(\Gamma) = \Theta(\Gamma)(\Theta(\Gamma)-1)(\Theta(\Gamma)-2)(\Theta(\Gamma)-3)(\Theta(\Gamma)-4)\left(\varrho(\Gamma_1). \dots . \varrho(\Gamma_m)\right)$.

**Proposition 2:** The non-negative integer $\varepsilon(\Gamma)$, $\forall\Gamma \in RT$ satisfy.

1- $q(t_1) = 1$, $q(t_2) = 1$, $q(t_3) = 1$, $q(t_4) = 1$, $q(t_5) = 1$
2- For the tree $\Gamma = \left[\Gamma_1^{v_1}, \dots, \Gamma_m^{v_m}\right]_5 \in RT$, with distinct $\Gamma_i$ , this from is obtained

$q(\Gamma) = (\Theta(\Gamma) - 5)!\ \prod_{i=1}^{m} \frac{1}{v_i} \left(\frac{q(\Gamma_i)}{q(\Gamma_i)!}\right)^{v_i}$ , where $v_i$ count similar tree of $\Gamma_i$, $i = 1, \cdots, m$.

**Theorem 1:** For the exact solution (27) the B-series is:

$$p(q_0 + h) = y_0 + \sum_{\Gamma \in RT} \frac{h^{\Theta(\Gamma)}}{\Theta(\Gamma)!} q(\Gamma) F(\Gamma)(p_0, p_0', p_0'', p_0''', p_0'''') \tag{58}$$

And from first to furth derivatives have the following B-series, respectively:

$$p'(q_0 + h) = \sum_{\Gamma \in RT} \frac{h^{\Theta(\Gamma)-1}}{(\Theta(\Gamma)-1)!} q(\Gamma) F(\Gamma)(p_0, p_0', p_0'', p_0''', p_0'''') \tag{59}$$

$$p''(q_0 + h) = \sum_{\Gamma \in RT/[t_1]} \frac{h^{\Theta(\Gamma)-2}}{(\Theta(\Gamma)-2)!} q(\Gamma) F(\Gamma)(p_0, p_0', p_0'', p_0''', p_0'''') \tag{60}$$

$$p'''(q_0 + h) = \sum_{\Gamma \in RT} \frac{h^{\Theta(\Gamma)-3}}{(\Theta(\Gamma)-3)!} q(\Gamma) F(\Gamma)(p_0, p_0', p_0'', p_0''', p_0'''') \tag{61}$$

$$p''''(q_0 + h) = \sum_{\Gamma \in RT/[t_1]} \frac{h^{\Theta(\Gamma)-4}}{(\Theta(\Gamma)-4)!} q(\Gamma) F(\Gamma)(p_0, p_0', p_0'', p_0''', p_0'''') \tag{62}$$

**Lemma 3:** The function $\eta_i(\Gamma) \in RT \backslash \{t_1, t_2, t_3, t_4\}$ can be calculated recursively as:

1- $\eta_i(t_5) = 1$,

2- For the tree $\Gamma = [t_1^{v_1}, t_2^{v_2}, t_3^{v_3}, . . . , \Gamma_m^{v_m}]_5 \in RT$, with distinct $\Gamma_i$, $i = 1, \cdots, m$ and different $t_1, t_2, t_3$ and $t_4$, $\eta_j = \frac{1}{2^{v_2} 6^{v_3} 24^{v_4}} c_j^{v_1+2v_2+3v_3+4v_4} \prod_{i=5}^{m} (\sum_{k=1}^{s} \hat{a}_{jk} \eta_k(\Gamma_i))^{v_i}$.

**Theorem 2:** When the DIRKFO5 method is applied to for the problem (27), it produces $\wp_{n+1}$ as the numerical solution and the numerical derivatives $\wp'_{n+1}, \wp''_{n+1}, \wp'''_{n+1}$ and $\wp''''_{n+1}$, with the following B-series;

$$\wp_{n+1} = \wp_n + h\wp'_n + \frac{1}{2}h^2\wp''_n + \frac{1}{6}h^3\wp'''_n + \frac{1}{24}h^4\wp''''_n + \sum_{\Gamma \in RT/\{t_1,t_2,t_3,t_4\}} \frac{h^{\Theta(\Gamma)}}{\Theta(\Gamma)!} q(\Gamma)\gamma(\Gamma)\vartheta(\Gamma).F(\Gamma)(\wp_n, \wp'_n, \wp''_n, \wp'''_n, \wp''''_n) \quad (63)$$

$$\wp'_{n+1} = \wp'_n + h\wp''_n + \frac{1}{2}h^2\wp'''_n + \frac{1}{6}h^3\wp''''_n + \sum_{\Gamma \in \Gamma \in RT/\{t_1,t_2,t_3,t_4\}} \frac{h^{\Theta(\Gamma)-1}}{\Theta(\Gamma)!} q(\Gamma)\gamma(\Gamma)\vartheta'(\Gamma).F(\Gamma)(\wp_n, \wp'_n, \wp''_n, \wp'''_n, \wp''''_n) \quad (64)$$

$$\wp''_{n+1} = \wp''_n + h\wp'''_n + \frac{1}{2}h^2\wp''''_n + \sum_{\Gamma \in \Gamma \in RT/\{t_1,t_2,t_3,t_4\}} \frac{h^{\Theta(\Gamma)-2}}{\Theta(\Gamma)!} q(\Gamma)\gamma(\Gamma)\vartheta''(\Gamma).F(\Gamma)(\wp_n, \wp'_n, \wp''_n, \wp'''_n, \wp''''_n) \quad (65)$$

$$\wp'''_{n+1} = \wp'''_n + h\wp''''_n + \sum_{\Gamma \in \Gamma \in RT/\{t_1,t_2,t_3,t_4\}} \frac{h^{\Theta(\Gamma)-3}}{\Theta(\Gamma)!} q(\Gamma)\gamma(\Gamma)\vartheta'''(\Gamma).F(\Gamma)(\wp_n, \wp'_n, \wp''_n, \wp'''_n, \wp''''_n) \quad (66)$$

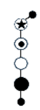
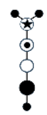
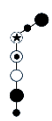
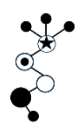
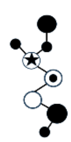
$$\wp''''_{n+1} = \wp''''_n + \sum_{\Gamma \in \frac{RT}{\{t_1,t_2,t_3,t_4\}}} \frac{h^{\Theta(\Gamma)-4}}{\Theta(\Gamma)!} q(\Gamma)\gamma(\Gamma)\vartheta''''(\Gamma).F(\Gamma)(\wp_n, \wp'_n, \wp''_n, \wp'''_n, \wp''''_n) \quad (67)$$

## ALGEBRAIC ORDER CONDITIONS

The main objective of this study is to achieve the order conditions of the EFDIRKFO5 method through Theorem 1 and Theorem 2 (Hussain, Ismail & Senu, 2016). the colored trees with ranks up to seventh are listed In Table 2, with the values of the associated functions.

**Table 2- elementary differentials, relevant-colored trees of up to eight orders, and related functions**

| Order $\Theta(t)$ | $t$ | tree | $q(t)$ | density | n(t) | elementary |
|---|---|---|---|---|---|---|
| 0 | Ø | Ø | 1 | 1 | | $\wp$ |
| 1 | $t_1$ |  | 1 | 1 | | $\wp'$ |
| 2 | $t_2$ |  | 1 | 2 | | $\wp''$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | $t_3$ |  | 1 | 6 | | $\mathscr{p}'''$ |
| 4 | $t_4$ |  | 1 | 24 | | $\mathscr{p}''''$ |
| 5 | $t_5$ |  | 1 | 120 | e | $\mathscr{g}$ |
| 6 | $t_6$ |  | 1 | 720 | c | $\mathscr{g}'\mathscr{p}$ |
| 7 | $t_{71}$ |  | 1 | 2520 | $c^2$ | $\mathscr{g}''(\mathscr{p}',\mathscr{p}')$ |
| | $t_{72}$ |  | 1 | 5040 | $\frac{1}{2}c^2$ | $\mathscr{g}'\mathscr{p}''$ |
| 8 | $t_{81}$ |  | 1 | 6720 | $c^3$ | $\mathscr{g}'''(\mathscr{p}',\mathscr{p}',\mathscr{p}')$ |
| | $t_{82}$ |  | 3 | 13440 | $\frac{1}{2}c^3$ | $\mathscr{g}''(\mathscr{p}',\mathscr{p}'')$ |
| | $t_{83}$ |  | 1 | 40320 | $\frac{1}{6}c^3$ | $\mathscr{g}'\mathscr{p}'''$ |

Based on Theorem 1, the order conditions for the EFDIRKFO5 method up to the seventh order can be written as follows:

Order 1: $b''''^T e = 1$

Order 2: $b''''^T c = \frac{1}{2}$ , $b'''^T e = \frac{1}{2}$

Order 3: $b''''^T c^2 = \frac{1}{3}$ , $b'''^T c = \frac{1}{6}$ , $b''^T e = \frac{1}{6}$

Order 4: $b''''^T c^3 = \frac{1}{4}$ , $b'''^T c^2 = \frac{1}{12}$ , $b''^T c = \frac{1}{24}$ , $b'^T e = \frac{1}{24}$

Order 5: $b''''^T c^4 = \frac{1}{5}$ , $b'''^T c^3 = \frac{1}{20}$ , $b''^T c^2 = \frac{1}{60}$ , $b'^T c = \frac{1}{120}$ , $b^T e = \frac{1}{120}$

Order 6: $b''''^T c^5 = \frac{1}{6}$ , $b'''^T c^4 = \frac{1}{30}$ , $b''^T c^3 = \frac{1}{120}$ , $b'^T c^2 = \frac{1}{360}$ , $b^T c = \frac{1}{720}$

$$b''''^T \hat{A} = \frac{1}{720}$$

Order 7: $b''''^T c^6 = \frac{1}{7}$ , $b'''^T c^5 = \frac{1}{42}$ , $b''^T c^4 = \frac{1}{210}$ , $b'^T c^3 = \frac{1}{840}$ , $b^T c^2 = \frac{1}{2520}$

$$b'''^T \hat{A} = \frac{1}{5040}, \quad b''''^T c\hat{A} = \frac{1}{840}, \quad b''''^T \hat{A}c = \frac{1}{5040}$$

## Exponentially-Fitted Method

To derive EFDIRKFO5 of Fifth order - three stages method, employed order conditions up to the fifth order. Consequently, we get system of equations contain of fifteen nonlinear equations that must be solved, letting $c_1 = 1$ and solving the system together yields the family of solution as follows:

$$c_2 = \frac{2}{5} - \frac{\sqrt{6}}{10}, c_3 = \frac{2}{5} + \frac{\sqrt{6}}{10}, b_1'''' = \frac{1}{9}, b_2'''' = \frac{4}{9} - \frac{\sqrt{6}}{36}, b_3'''' = \frac{4}{9} + \frac{\sqrt{6}}{36}, b_1''' = 0, b_2''' = \frac{1}{4} + \frac{\sqrt{6}}{36},$$

$$b_3''' = \frac{1}{4} - \frac{\sqrt{6}}{36}, b_1 = 0, b_2 = \frac{1}{240} + \frac{7\sqrt{6}}{4320}, b_3 = \frac{1}{240} - \frac{7\sqrt{6}}{4320}, b_1'' = 0, b_2'' = \frac{1}{12} + \frac{\sqrt{6}}{48},$$

$$b_3'' = \frac{1}{12} - \frac{\sqrt{6}}{48}, b_1' = -\frac{1}{40} - \frac{2}{5} b_3'\sqrt{6} + \frac{2}{5} b_3' + \frac{1}{90}\sqrt{6}, b_2' = \frac{2}{5} b_3'\sqrt{6} - \frac{7}{5} b_3' + \frac{1}{15} - \frac{1}{90}\sqrt{6}$$

The global error in eight free parameters given by

$$\begin{aligned}\left\|\Gamma_g^{(6)}\right\|_2 = \Big(&-\tfrac{7}{135000}\sqrt{6} + \tfrac{4}{81} a_{31}a_{32}\sqrt{6} - \tfrac{1}{18}\gamma a_{21}\sqrt{6} + \tfrac{1}{18}\gamma a_{31}\sqrt{6} + \tfrac{1}{18}\gamma a_{32}\sqrt{6} \\ &- \tfrac{18}{625} b_3'^2\sqrt{6} - \tfrac{2}{81} a_{21}^2\sqrt{6} + \tfrac{2}{81} a_{31}^2\sqrt{6} + \tfrac{2}{81} a_{32}^2\sqrt{6} + \gamma^2 + \tfrac{8}{9}\gamma a_{21} + \tfrac{8}{9}\gamma a_{31} \\ &+ \tfrac{131}{648} a_{21}^2 + \tfrac{125}{324} a_{21}a_{31} + \tfrac{125}{324} a_{21}a_{32} + \tfrac{131}{324} a_{31}a_{32} + \tfrac{131}{648} a_{31}^2 + \tfrac{131}{648} a_{32}^2 \\ &+ \tfrac{1}{12960} a_{21}\sqrt{6} - \tfrac{1}{12960} a_{31}\sqrt{6} - \tfrac{1}{12960} a_{32}\sqrt{6} - \tfrac{1}{810} a_{21} - \tfrac{1}{810} a_{31} - \tfrac{1}{810} a_{32} \\ &+ \tfrac{877}{6480000} + \tfrac{8}{9}\gamma a_{32} + \tfrac{63}{625} b_3'^2 - \tfrac{1}{360}\gamma - \tfrac{1}{150} b_3' + \tfrac{1}{375} b_3'\sqrt{6}\Big)^{1/2}\end{aligned} \tag{68}$$

By using minimize command in Maple for equation (68), we get

$$\gamma = -\frac{4}{10}, a_{21} = \frac{5}{10}, a_{31} = \frac{3}{10}, a_{32} = \frac{3}{10}, b_3' = \frac{2}{1000}, b_2' = \frac{479}{7500} - \frac{58\sqrt{6}}{5625}, b_1' = -\frac{121}{5000} + \frac{58\sqrt{6}}{5625}$$

and the global error is $\Gamma_g^{(6)} = 0.00250000$.

To construct the exponentially-Fitted Runge-Kutta type three-stage fifth-order method needs at each stage to integrate exactly the function $e^{wq}$ and $e^{-wq}$, therefore the following four equations are obtained (Jaleel & Fawzi,2023).

$$e^{\pm v} = 1 \pm v + \frac{v^2}{2} \pm \frac{v^3}{6} + \frac{v^4}{24} \pm v^5 \sum_{i=1}^{s} b_i e^{\pm c_i v} \tag{69}$$

$$e^{\pm v} = 1 \pm v + \frac{v^2}{2} \pm \frac{v^3}{6} + v^4 \sum_{i=1}^{s} b_i' e^{\pm c_i v} \tag{70}$$

$$e^{\pm v} = 1 \pm v + \frac{v^2}{2} \pm v^3 \sum_{i=1}^{s} b_i'' e^{\pm c_i v} \tag{71}$$

$$e^{\pm v} = 1 \pm v + v^2 \sum_{i=1}^{s} b_i''' e^{\pm c_i v} \tag{72}$$

$$e^{\pm v} = 1 \pm v \sum_{i=1}^{s} b_i'''' e^{\pm c_i v} \tag{73}$$

where $v = wq, w \in R$. By substitution $sinh(v) = \frac{e^v - e^{-v}}{2}$ and $cosh(v) = \frac{e^v + e^{-v}}{2}$ we have

$$cosh(v) = 1 + \frac{v^2}{2} + \frac{v^4}{24} + v^5 \sum_{i=1}^{s} b_i \, sinh(c_i v) \tag{74}$$

$$sinh(v) = v + \frac{v^3}{6} + v^5 \sum_{i=1}^{s} b_i \, cosh(c_i v) \tag{75}$$

$$cosh(v) = 1 + \frac{v^2}{2} + v^4 \sum_{i=1}^{s} b_i' cosh(civ) \tag{76}$$

$$sinh(v) = v + \frac{v^3}{6} + v^4 \sum_{i=1}^{s} b_i' \, sinh(c_i v) \tag{77}$$

$$cosh(v) = 1 + \frac{v^2}{2} + v^3 \sum_{i=1}^{s} b_i'' \, sinh(c_i v) \tag{78}$$

$$sinh(v) = v + v^3 \sum_{i=1}^{s} b_i'' \, cosh(c_i v) \tag{79}$$

$$cosh(v) = 1 + v^2 \sum_{i=1}^{s} b_i''' \, cosh(c_i v) \tag{80}$$

$$sinh(v) = v + v^2 \sum_{i=1}^{s} b_i''' \, sinh(c_i v) \tag{81}$$

$$cosh(v) = 1 + \sum_{i=1}^{s} b_i'''' \, sinh(c_i v) \tag{82}$$

$$sinh(v) = v + \sum_{i=1}^{s} b_i'''' \, cosh(c_i v) \tag{83}$$

Solving the eqs. (69) - (83) and using the above coefficient to find new $b_1, b_2, b_1', b_2', b_1'', b_2'', b_1''', b_2''', b_1''''$ and $b_2''''$, It is worth noting that we used the same coefficients $b_3, b_3', b_3'', b_3'''$ and $b_3''''$ approximately because they were not obtained from solving the equations mentioned above.

$$b_1 = \frac{cosh\left(\frac{1}{10}v\right) sinh\left(\frac{3}{5}v\right) - cosh\left(\frac{3}{5}v\right) * sinh\left(\frac{1}{10}v\right)}{1000\left(cosh(v)\, sinh\left(\frac{1}{10}v\right) - cosh\left(\frac{1}{10}v\right) sinh(v)\right)}$$

$$+ \frac{cosh\left(\frac{1}{10}v\right)v^4 - 4sinh\left(\frac{1}{10}v\right)v^3 + 12cosh\left(\frac{1}{10}v\right)v^2 + 24sinh(v)sinh\left(\frac{1}{10}v\right)}{24v^5\left(cosh(v)\, sinh\left(\frac{1}{10}v\right) - cosh\left(\frac{1}{10}v\right)sinh(v)\right)}$$

$$- \frac{24cosh\left(\frac{1}{10}v\right)cosh(v) - 24sinh\left(\frac{1}{10}v\right)v + 24cosh\left(\frac{1}{10}v\right)}{24v^5\left(cosh(v)\, sinh\left(\frac{1}{10}v\right) - cosh\left(\frac{1}{10}v\right)sinh(v)\right)}$$

$$b_2 = \frac{-1}{1000} \frac{cosh(v)sinh(\frac{3}{5}v) - cosh(\frac{3}{5}v) * sinh(v)}{cosh(v)sinh(\frac{1}{10}v) - cosh(\frac{1}{10}v)sinh(v)}$$
$$- \frac{1}{24} \frac{\left(cosh(v)v^4 - 4sinh(v)v^3 + 12cosh(v)v^2 + 24sinh(v)^2 - 24cosh(v)^2 - 24sinh(v)v + 24cosh(v)\right)}{v^5(\left(cosh(v)sinh\left(\frac{1}{10}v\right) - cosh\left(\frac{1}{10}v\right)sinh(v)\right)}$$

$$b_3=\frac{1}{10000}$$

$$b_1'=-\frac{1}{6}\frac{1}{v^4(\left(cosh(v)sinh\left(\frac{1}{10}v\right)-cosh\left(\frac{1}{10}v\right)sinh(v)\right)}\left(-\frac{3}{250}cosh\left(\frac{1}{10}v\right)sinh\left(\frac{3}{5}v\right)v^4+\frac{3}{250}cosh\left(\frac{3}{5}v\right)sinh\left(\frac{1}{10}v\right)v^4-cosh\left(\frac{1}{10}v\right)v^3+3sinh\left(\frac{1}{10}v\right)v^2+6cosh\left(\frac{1}{10}v\right)sinh(v)-6cosh\left(\frac{1}{10}v\right)v-6cosh(v)sinh\left(\frac{1}{10}v\right)+6sinh\left(\frac{1}{10}v\right)\right)$$

$$b_2'=\frac{1}{6}\frac{-\frac{3}{250}cosh(v)sinh\left(\frac{3}{5}v\right)v^4+\frac{3}{250}cosh\left(\frac{3}{5}v\right)sinh(v)v^4-cosh(v)v^3+3sinh(v)v^2-6cosh(v)v+6sinh(v)}{v^4\left(cosh(v)sinh\left(\frac{1}{10}v\right)-cosh\left(\frac{1}{10}v\right)sinh(v)\right)}$$

$$b_3'=\frac{1}{500}$$

$$b_1''=\frac{1}{2}\frac{1}{v^3\left(cosh(v)sinh\left(\frac{1}{10}v\right)-cosh\left(\frac{1}{10}v\right)sinh(v)\right)}\left(\frac{3}{50}cosh\left(\frac{1}{10}v\right)sinh\left(\frac{3}{5}v\right)v^3-\frac{3}{50}cosh\left(\frac{3}{5}v\right)sinh\left(\frac{1}{10}v\right)v^3+cosh\left(\frac{1}{10}v\right)v^2+2sinh(v)sinh\left(\frac{1}{10}v\right)-2cosh\left(\frac{1}{10}v\right)cosh(v)-2sinh\left(\frac{1}{10}v\right)v+2cosh\left(\frac{1}{10}v\right)\right)$$

$$b_2''=\frac{-\frac{3}{50}cosh(v)sinh\left(\frac{3}{5}v\right)v^3+\frac{3}{50}cosh\left(\frac{3}{5}v\right)sinh(v)v^3-cosh(v)v^2}{2v^3\left(cosh(v)sinh\left(\frac{1}{10}v\right)-cosh\left(\frac{1}{10}v\right)sinh(v)\right)}+\frac{-2sinh(v)^2+2cosh(v)^2+2sinh(v)v-2cosh(v)}{2v^3\left(cosh(v)sinh\left(\frac{1}{10}v\right)-cosh\left(\frac{1}{10}v\right)sinh(v)\right)}$$

$$b_3''=\frac{3}{100}$$

$$b_1'''=\frac{\frac{1}{10}cosh\left(\frac{1}{10}v\right)sinh\left(\frac{3}{5}v\right)v^2-\frac{1}{10}cosh\left(\frac{3}{5}v\right)sinh\left(\frac{1}{10}v\right)v^2-cosh\left(\frac{1}{10}v\right)sinh(v)}{v^2\left(cosh(v)sinh\left(\frac{1}{10}v\right)-cosh\left(\frac{1}{10}v\right)sinh(v)\right)}+\frac{cosh\left(\frac{1}{10}v\right)v+cosh(v)sinh\left(\frac{1}{10}v\right)-sinh\left(\frac{1}{10}v\right)}{v^2\left(cosh(v)sinh\left(\frac{1}{10}v\right)-cosh\left(\frac{1}{10}v\right)sinh(v)\right)}$$

$$b_2'''=\frac{-\frac{1}{10}cosh(v)sinh\left(\frac{3}{5}v\right)v^2+\frac{1}{10}cosh\left(\frac{3}{5}v\right)sinh(v)v^2-cosh(v)v+sinh(v)}{v^2(cosh(v)sinh\left(\frac{1}{10}v\right)-cosh\left(\frac{1}{10}v\right)sinh(v))}$$

$$b_3'''=\frac{1}{10}$$

$$b_1'''' = \frac{\frac{1}{2}cosh\left(\frac{1}{10}v\right)sinh\left(\frac{3}{5}v\right)v - \frac{1}{2}cosh\left(\frac{3}{5}v\right)sinh\left(\frac{1}{10}v\right)v + sinh(v)sinh\left(\frac{1}{10}v\right) - cosh\left(\frac{1}{10}v\right)cosh(v) + cosh\left(\frac{1}{10}v\right)}{v\left(cosh(v)sinh\left(\frac{1}{10}v\right) - cosh\left(\frac{1}{10}v\right)sinh(v)\right)}$$

$$b_2'''' = \frac{\frac{1}{2}cosh(v)sinh\left(\frac{3}{5}v\right)v - \frac{1}{2}cosh\left(\frac{3}{5}v\right)sinh(v)v + sinh(v)^2 - cosh(v)^2 + cosh(v)}{v\left(cosh(v)sinh\left(\frac{1}{10}v\right) - cosh\left(\frac{1}{10}v\right)sinh(v)\right)}$$

$$b_3'''' = \frac{1}{2}$$

These lead to our new method exponentially - fitted explicit modified Runge-Kutta type method three-stage fifth-order method denoted as (EFDIRKFO5), The corresponding Taylor series expansion of the solution is given by

$$b_1 = \frac{91}{162000} - \frac{7537}{113400000}v^2 + \frac{48431}{8100000000}v^4 - \frac{282091127}{561330000000000}v^6 + \frac{72261572521}{1741398750000000000}v^8 - \cdots,$$

$$b_2 = \frac{3107}{405000} - \frac{293}{40500000}v^2 + \frac{207131}{32400000000}v^4 - \frac{104301341}{204120000000000}v^6 + \frac{1428947715803}{34292160000000000000}v^8 - \cdots,$$

$$b_3 = \frac{1}{10000}$$

$$b_1' = \frac{19}{5400} - \frac{1181}{2835000}v^2 + \frac{1919131}{51030000000}v^4 - \frac{1057553}{334125000000}v^6 + \frac{1001667621599}{3831077250000000000}v^8 - \cdots,$$

$$b_2' = \frac{122}{3375} - \frac{701}{1417500}v^2 + \frac{1030807}{25515000000}v^4 - \frac{60290921}{18711000000000}v^6 + \frac{16098557194501}{61297236000000000000}v^8 - \cdots,$$

$$b_3' = \frac{1}{500}$$

$$b_1'' = \frac{1}{90} - \frac{223}{162000}v^2 + \frac{12121}{94500000}v^4 - \frac{206123}{18900000000}v^6 + \frac{3803564921}{4209975000000000}v^8 - \frac{189831825017}{2554051500000000000}v^{10} + \cdots,$$

$$b_2'' = \frac{113}{900} - \frac{607}{324000}v^2 + \frac{30737}{216000000}v^4 - \frac{10155713}{907200000000}v^6 + \frac{89100636299}{97977600000000000}v^8 - \frac{7488704442431}{100590336000000000000}v^{10} + \cdots,$$

$$b_3'' = \frac{3}{100}$$

$$b_1''' = \frac{2}{27} - \frac{37}{5400}v^2 + \frac{1879}{3150000}v^4 - \frac{2544497}{51030000000}v^6 + \frac{577162879}{140332500000000}v^8 - \frac{28773957683}{85135050000000000}v^{10} + \cdots,$$

$$b_2''' = \frac{44}{135} - \frac{11}{1350}v^2 + \frac{3977}{6300000}v^4 - \frac{5162123}{102060000000}v^6 + \frac{18535394467}{4490640000000000}v^8 - \frac{614393965693}{1816214400000000000}v^{10} + \cdots,$$

$$b_3''' = \frac{1}{10}$$

$$b_1'''' = \frac{1}{6} - \frac{2}{270}v^2 + \frac{107}{405000}v^4 - \frac{1039}{47250000}v^6 + \frac{2322721}{1275750000000}v^8 - \frac{63080431}{420997500000000}v^{10} + \frac{18708811133}{1520268750000000000}v^{12} - \cdots,$$

$$b_2'''' = \frac{1}{3} - \frac{1}{216}v^2 + \frac{1933}{6480000}v^4 - \frac{13747}{604800000}v^6 + \frac{599656019}{326592000000000}v^8 - \frac{12946277431}{86220288000000000}v^{10} + \frac{25557313566049}{20756736000000000000}v^{12} - \cdots,$$

$$b_3'''' = \frac{1}{2}$$

## NUMERICAL RESULTS

In this paper, the new method was compared with the DIRKFO5 method and Radua I method to prove the effectiveness of the new method. We used the following equations (Butcher, 2016) to compare the numerical methods:

**Problem1:** $p^{(5)} = \cos(q)$, $p(0) = 0$, $p'(0) = 1$, $p''(0) = 0$, $p'''(0) = -1$, $p''''(0) = 0$.

Exact: $p(q) = \sin(q)$.

**Problem2:** $p^{(5)} = -p$, $p(0) = 1$, $p'(0) = -1$, $p''(0) = 1$, $p'''(0) = -1, p''''(0) = .1$

Exact: $p(q) = e^{-q}$.

**Problem 3:** $p^{(5)} = -120\, p^6$, $p(0) = 1$, $p'(0) = -1$, $p''(0) = 2$, $p'''(0) = -6$, $p''''(0) = 24$.

Exact: $p(q) = \frac{1}{q+1}$

**Problem 4:** $p^{(5)} = 32\, p + \cos q - 32\, \sin q$, $p(0) = 1$, $p'(0) = 3$, $p''(0) = 4$, $p'''(0) = 7$, $p''''(0) = 16$.

Exact: $p(q) = \sin q + e^{2q}$

**Table 3. Numerical comparisons for problem 1,2,3, and 4**

| problem | H | Methods | Maxerror | Function call |
|---|---|---|---|---|
| 1 | 0.05 | EFDIRKFO5 | 1.867090e-003 | 256 |
| | | Radua I | 9.987503e-001 | 384 |
| | | DIRKFO5 | 7.976149e-003 | 256 |
| | 0.025 | EFDIRKFO5 | 1.361409e-003 | 504 |
| | | Radua I | 9.996875e-001 | 756 |
| | | DIRKFO5 | 7.086568e-003 | 504 |
| | 0.00625 | EFDIRKFO5 | 1.056264e-003 | 2016 |
| | | Radua I | 9.999805e-001 | 3024 |
| | | DIRKFO5 | 6.849297e-003 | 2016 |
| | 0.000125 | EFDIRKFO5 | 9.408668e-004 | 12567 |
| | | Radua I | 1.000000e+00 | 150804 |
| | | DIRKFO5 | 6.684801e-003 | 12567 |
| 2 | 0.05 | EFDIRKFO5 | 1.917723e-004 | 160 |
| | | Radua I | 9.512294e-001 | 240 |
| | | DIRKFO5 | 7.188142e-004 | 160 |
| | 0.025 | EFDIRKFO5 | 1.435004e-004 | 320 |
| | | Radua I | 9.753099e-001 | 480 |
| | | DIRKFO5 | 6.771130e-004 | 320 |
| | 0.00625 | EFDIRKFO5 | 1.062120e-004 | 1288 |
| | | Radua I | 9.937695e-001 | 1932 |
| | | DIRKFO5 | 6.638844e-004 | 1288 |

| | | | | |
|---|---|---|---|---|
| | 0.000125 | EFDIRKFO5 | 8.919178e-005 | 64000 |
| | | Radua I | 9.998750e-001 | 96000 |
| | | DIRKFO5 | 6.331451e-004 | 64000 |
| 3 | 0.05 | EFDIRKFO5 | 8.278770e-003 | 160 |
| | | Radua I | 9.070295e-001 | 240 |
| | | DIRKFO5 | 4.732103e-002 | 160 |
| | 0.025 | EFDIRKFO5 | 7.394370e-003 | 320 |
| | | Radua I | 9.518144e-001 | 480 |
| | | DIRKFO5 | 4.606572e-002 | 320 |
| | 0.00625 | EFDIRKFO5 | 6.711436e-003 | 1288 |
| | | Radua I | 9.876162e-001 | 1932 |
| | | DIRKFO5 | 4.604044e-002 | 1288 |
| | 0.000125 | EFDIRKFO5 | 6.222574e-003 | 64000 |
| | | Radua I | 9.997500e-001 | 96000 |
| | | DIRKFO5 | 4.428127e-002 | 64000 |
| 4 | 0.05 | EFDIRKFO5 | 1.427496e-002 | 160 |
| | | Radua I | 1.531841e+001 | 240 |
| | | DIRKFO5 | 4.182804e-002 | 160 |
| | 0.025 | EFDIRKFO5 | 9.736498e-003 | 320 |
| | | Radua I | 1.531841e+001 | 480 |
| | | DIRKFO5 | 3.860521e-002 | 320 |
| | 0.00625 | EFDIRKFO5 | 6.388726e-003 | 1288 |
| | | Radua I | 1.549903e+001 | 1932 |
| | | DIRKFO5 | 3.739140e-002 | 1288 |
| | 0.000125 | EFDIRKFO5 | 4.988286e-003 | 64000 |
| | | Radua I | 1.531841e+001 | 96000 |
| | | DIRKFO5 | 3.535101e-002 | 64000 |

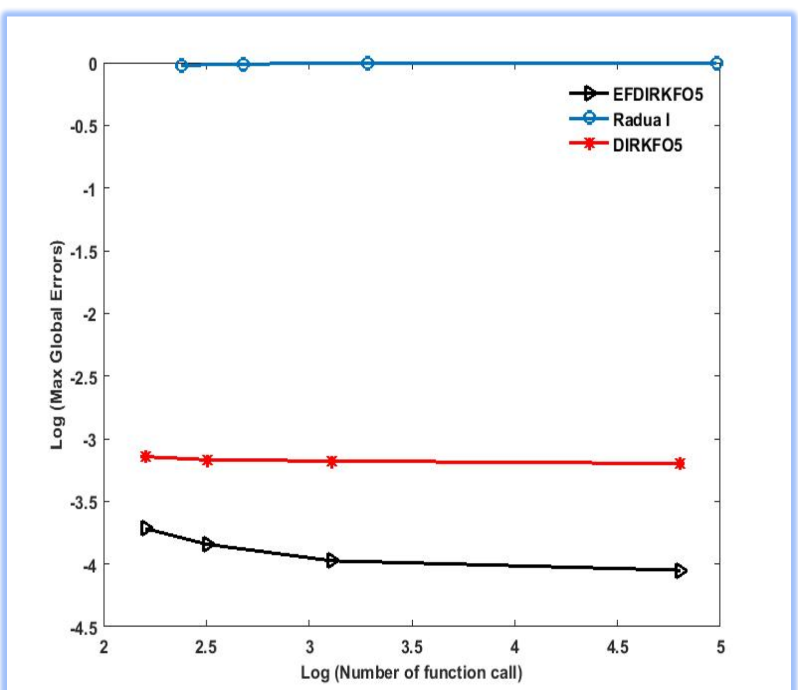



**Figure 2- competence graphs for problem 1**

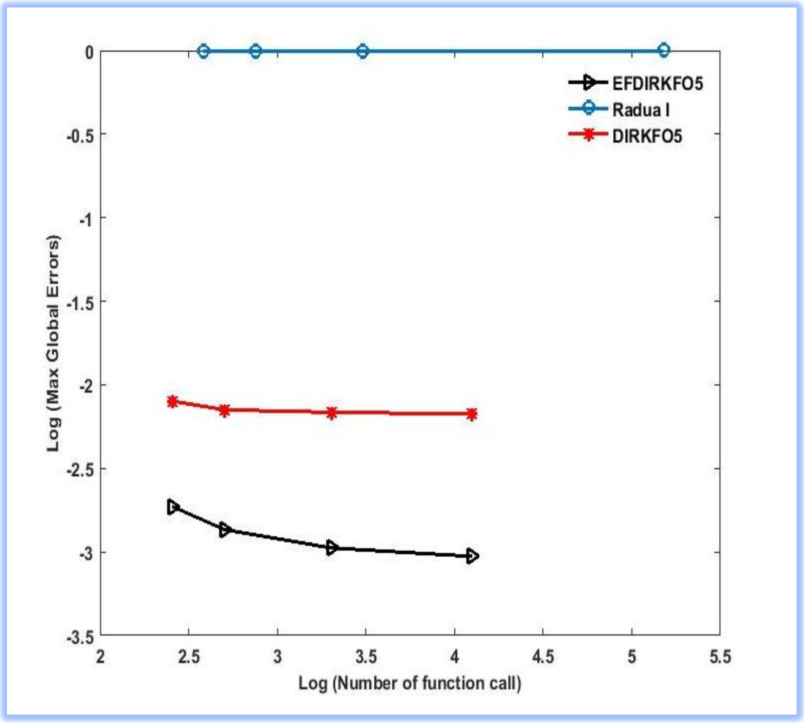



**Figure 3- competence graphs for problem 2**

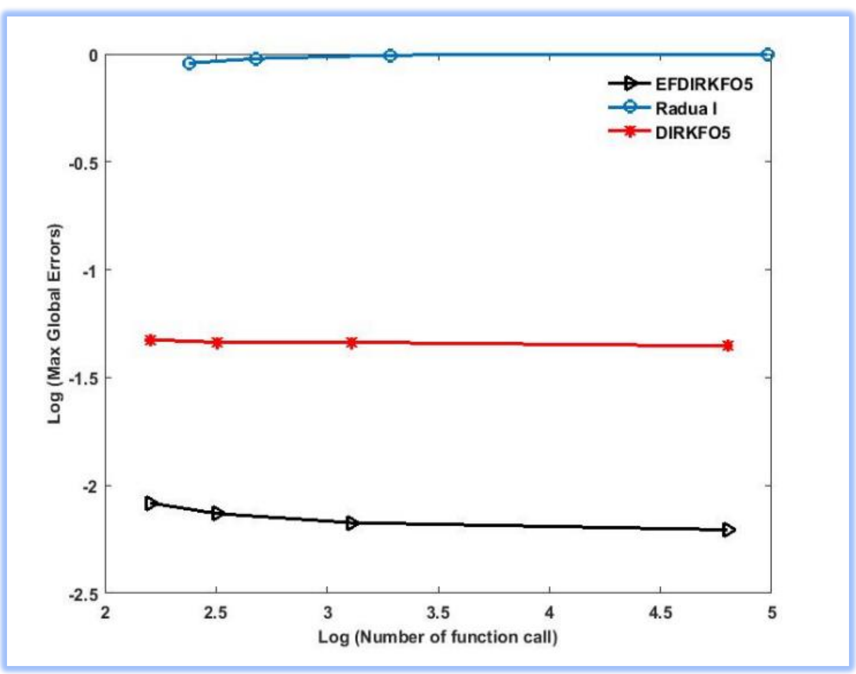



**Figure 4- competence graphs for problem 3**

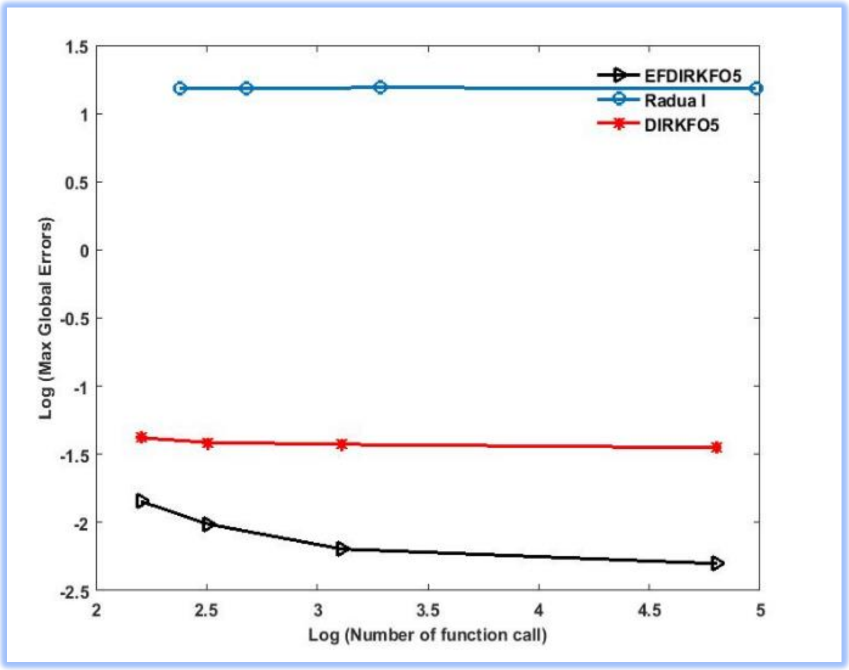



**Figure 5- competence graphs for problem 4**

## Conclusion

This study introduces an exponential implicit Runge-Kutta method for solving fifth-order ordinary differential equations represented as $p^{(5)}(q) = g(q,p)$. Consequently, we develop a three-stage, fourth-order exponential implicit approach known as the EFDIRKFO5 method, which is directly utilized to solve differential equations. The numerical findings in table 3 and figures 1,2,3 and 4 indicate that the global inaccuracy of the new approach is reduced and it demonstrates much greater efficiency compared to the Radua I and DIRKFO5 methods. The reason for this is due to the high accuracy of the implicit methods compared to the conventional methods, in addition to including exponential functions in the derivation of our new method. Consequently, our novel approach is more precise and efficient for addressing fifth-order ordinary differential equations represented as $p^{(5)}(q) = g(q,p)$.

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