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Exponentially Semi E-preinvexity: Properties and Applications

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ABSTRACT

Generalized exponential convexity has been one of the most important concept in generalized convex analysis, not only because of its structural properties used to relax convexity assumptions, but also because of its various applications in applied fields, especially in mathematics and optimization theory. In this paper, a class of two types of generalized convex functions is introduced, namely exponential semi E-preinvex functions and exponentially quasi semi E-preinvex functions. Several insights are presented, starting with providing a necessary and a sufficient condition for a function to be semi E-preinvex function. Also, several general algebraic properties of these functions are estalished. Moreover, various characterizations and conditions are provided to relate these functions to their levels and graph sets. Finally, some optimality properties for nonlinear optimization problems with exponentially E-preinvex functions, exponentially semi E-preinvex functions and exponentially quasi semi E-preinvex functions are shown. Two examples are presented to illustrate the new functions developed in this work.

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1. Introduction and Preliminaries

Convexity and generalized convexity of sets and functions are considered to be the most important effective tools for dealing with many concepts in applied mathematics, especially in optimization [18, 22, 24, 27, 37, 39]. Dealing with convex sets and functions to create optimization problems leads to some limitations in application [9]. Therefore, extensive research is being carried out to overcome these limitations by relaxing the convexity assumptions through many interesting generalized convex sets and functions. The class of preinvex functions, first introduced in [4, 10] and so named in [14], is considered as one of the most important generalizations of the class of convex functions by relaxing the convexity condition through the action of a mapping $\rho: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Preinvex functions are usually defined on a generalized convex set, which is called an invex set. This is first defined on subsets of the real numbers [40] and then extended to subsets of the n-dimensional space [26]. Further research has explored preinvexity and its applications in optimization theory. For example, see [5, 16, 17, 20,28, 29] and the references therein.

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On the other hand, the class of E-convex sets and E-convex functions is introduced as another class of generalized convex sets and functions [41]. In the class of E-convexity, a mapping $E: \mathbb{R}^n \to \mathbb{R}^n$ is introduced as the main tool to relax the convexity assumption. E-convexity motivated Youness and his collaborators and many other researchers to study this class in depth and extensively with its applications to optimization problems (e.g., [11, 19, 25, 38, 42]). For example, the class of E-convex functions and its applications have been generalized to a new class that includes semi E-convex, quasi semi E-convex and pseudo semi E-convex functions [6].

By combining preinvex and E-convex functions, the class of E-preinvex functions on E-invex sets is defined in [8]. Following [8] and Chen [6], many researchers have further studied semi E-convexity and E-preinvexity then introduced semi E-preinvex functions and their generalizations [1, 7, 12, 15, 21, 43].

Recently, a new class of exponentially convex functions has been developed as a further generalization of convex functions. The functions of this class have various applications in applied fields such as information theory, data analysis, machine learning, and statistics [3, 36]. The applications of this class in mathematics and optimization problems are presented in [13, 30-34]. The study of exponentially convex functions motivates researchers to generalize this class into new classes, such as the class of exponentially preinvex functions [35], the class of exponentially E-convex functions [2], and more recently the class of exponentially E-preinvex functions [2]. In these exponentially generalized convex function classes, various general and optimality properties for single and multi-objective optimization problems have been investigated.

The aim of this work is to continue the ongoing research on exponentially generalized convexity. Thus, the class of exponentially semi and quasi semi E-preinvex functions defined on E-invex set is introduced and various properties and characterizations for these functions are obtained. As an application to optimization problems, some optimality properties of nonlinear optimization problems are discussed. The paper is organized as follows. In the remainder of this section, we recall introductory concepts related to some generalized convex sets and convex functions. In section two, we introduce the definition of exponentially semi and quasi semi E-preinvex functions and construct two examples to illustrate the new functions. We also show necessary and sufficient conditions for a function to be semi E-preinvex. Later in the second section, we discuss some algebraic properties related to the new class. In addition, a number of properties and characterizations of exponentially semi and quasi E-preinvex functions are studied in detail using various of their level sets and epigraph sets. Section three is devoted to the application of this class of functions and the class of exponentially E-preinvex functions [2] to obtain some optimality conditions for nonlinear optimization problems.

We begin by fixing some needed preliminaries. Let $K \subseteq \mathbb{R}^n$ be a non-empty set and $f: K \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, Assume also that $E: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ where $E(k)$ is written as Ek and $\rho: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are two given mappings. The identity mapping is defined as $I: \mathbb{R} \longrightarrow \mathbb{R}$.

Definition (1.1) [26]. A set K is called invex with respect to ρ (for short, K is invex set), if for each $k_1, k_2 \in K$ and for each $\lambda \in [0,1]$, we have $k_2 + \lambda\rho(k_1, k_2) \in K$.

Definition (1.2) [41]. The set K is said to be an E-convex set if $\lambda Ek_1 + (1-\lambda)Ek_2 \in K$, for each $k_1, k_2 \in K$ and $\lambda \in [0,1]$.

Definition (1.3) [8]. The set K is said to be E-invex set with respect to ρ (for short, K is E-invex set), if $Ek_2 + \lambda\rho(Ek_1, Ek_2) \in K$, for each $k_1, k_2 \in K$ and $\lambda \in [0,1]$.

Definition (1.4) [8]. Let K_1 and K_2 be two subsets of $\mathbb{R}^n$. Then, K_1 is called slack E-invex with respect to K_2 if, for each $k_1, k_2 \in K_1 \cap K_2$ and every $\lambda \in [0,1]$ such that $Ek_2 + \lambda\rho(Ek_1, Ek_2) \in K_2$ we get $Ek_2 + \lambda\rho(Ek_1, Ek_2) \in K_1$.

Proposition (1.5) [8, 41]. If a set K is either E-convex or E-invex. Then, $E(K) \subseteq K$.

Definition (1.6) [2]. The function f is called exponentially E-convex on the E-convex set K , if for every $k_1, k_2 \in K$ and $\lambda \in [0,1]$

$$e^{f(\lambda Ek_1 + (1-\lambda)Ek_2)} \leq \lambda e^{f(Ek_1)} + (1-\lambda)e^{f(Ek_2)}.$$

Definition (1.7) [23]. The function f is named as exponentially E-preinvex on the E-invex set K with respect to ρ (for short, exponentially E-preinvex function), if for every $k_1, k_2 \in K$ and $\lambda \in [0,1]$

$$e^{f(Ek_2+\lambda\rho(Ek_1,Ek_2))} \leq \lambda e^{f(Ek_1)} + (1-\lambda)e^{f(Ek_2)}.$$

Definition (1.8) [37]. The epigraph set of a function f is given by $epif = \{(k,\gamma) \in K \times \mathbb{R}: e^{f(k)} \leq \gamma\}$.
In a similar manner, we define the epigraphs associated with the mapping E as follows.

Definition (1.9).

i. $E - epi\ f = \{(k,\gamma) \in K \times \mathbb{R} : e^{f(Ek)} \leq \gamma\}$.
ii. $epi^E\ f = \{(Ek,\gamma) \in E(K) \times \mathbb{R} : e^{f(k)} \leq \gamma\}$.
iii. $epi_E\ f = \{(Ek,\gamma) \in E(K) \times \mathbb{R} : e^{f(Ek)} \leq \gamma\}$.

With each epigraph defined in Definition 1.8 and Definition 1.9(i-ii), γ-level sets are associated, respectively, as follows.

Definition (1.10). Let $\gamma \in \mathbb{R}$. Then,

i. $L_\gamma = \{k \in K: e^{f(k)} \leq \gamma\}$. [35]

ii. $E\text{-}L_\gamma = \{k \in K: e^{f(Ek)} \leq \gamma\}$.

iii. $L_\gamma^E = \{Ek \in E(K): e^{f(k)} \leq \gamma\}$.

2. Exponentially semi E-preinvexity

In this section, a class of generalized convex functions is defined that includes exponentially semi and quasi semi E-preinvex functions. Two examples of an exponentially semi E-preinvex function and an exponentially quasi semi E-preinvex but not exponentially semi E-preinvex function are illustrated. Many properties are given for this class, ranging from algebraic properties to various conditions and characterizations of exponentially semi and quasi E-preinvex functions, using the certain assumptions, level sets, and epigraph sets of these functions.

Definition (2.1). Let K be E-invex set. Then, the function f is called

1. Exponentially semi E-preinvex function on K with respect to ρ (for short, exponentially semi E-preinvex function), if for each $k_1, k_2 \in K$, and each $\lambda \in [0,1]$

$$e^{f(Ek_2+\lambda\rho(Ek_1,Ek_2))} \leq \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}.$$

2. Exponentially quasi semi E-preinvex on K with respect to ρ (for short, exponentially quasi semi E-preinvex function), if for every $k_1, k_2 \in K$ and $\lambda \in [0,1]$

$$e^{f(Ek_2+\lambda\rho(Ek_1,Ek_2))} \leq \max\{e^{f(k_1)}, e^{f(k_2)}\}.$$

Let us start with an example of exponentially semi E-preinvex function.

Example (2.2). Suppose that $K \subseteq \mathbb{R}^n$ is E-invex set contains zero vector and $f : K \to \mathbb{R}$ such that $f(x_1, \dots, x_n) = ax_2$ where $a > 0$. If $E: \mathbb{R}^n \to \mathbb{R}^n$ such that $E(x_1, \dots, x_n) = (x_1 + 2, x_2, \dots, x_n)$ and $\rho: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined as $\rho(x,y) = x - y\ \ \forall x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$. To confirm that f is exponentially semi E-preinvex, let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in K$ then $Ey + \lambda\rho(Ex,Ey) \in K$ and

$$e^{f\left(Ey+\lambda\rho(Ex,Ey)\right)} = e^{f(\lambda x_1+(1-\lambda)y_1+2,\lambda x_2+(1-\lambda)y_2+\dots+\lambda x_n+(1-\lambda)y_n)}$$

$$= e^{a\lambda x_2 + a(1-\lambda)y_2} \leq \lambda e^{a x_2} + (1-\lambda)e^{a y_2}$$

$$= \lambda e^{f(x)} + (1-\lambda)e^{f(y)}$$

The above inequality holds for any $p, q > 0$ and $\lambda \in [0,1]$. i.e., $p^{\lambda} q^{1-\lambda} \leq \lambda p + (1-\lambda)q$. Hence, f is exponentially semi E-preinvex function on K.

Exponentially quasi semi E-preinvex function is not necessarily exponentially semi E-preinvex function as the following example shows.

Example (2.3). Let $f, E: \mathbb{R} \to \mathbb{R}$ and $\rho: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are defined as follows

$$f(x) = \begin{cases} -3 & if\ x = 0 \\ 1 & if\ x \neq 0 \end{cases}, Ex = \begin{cases} 0 & if\ x = 0 \\ 4 & if\ x \neq 0 \end{cases} \text{ and } \rho(x,y) = \begin{cases} 0, & x = y = 0 \\ y - x, & \text{otherwise} \end{cases}$$

Let $x, y \in \mathbb{R}$ then $Ey + \lambda\rho(Ex, Ey) \in \mathbb{R}$. First, let us prove the exponentially quasi semi E-preinvexity of f by considering the following possibilities:

Case 1: If $x = y = 0$, then $e^{f\left(Ey + \lambda\rho(Ex,Ey)\right)} = e^{f(0)} = e^{-3} = \max\left\{e^{f(x)}, e^{f(y)}\right\}$.

Case 2: If $x \neq 0, y \neq 0$, then $e^{f\left(Ey + \lambda\rho(Ex,Ey)\right)} = e^{f(4)} = e^{1} = \max\left\{e^{f(x)}, e^{f(y)}\right\}$.

Case 3: If $x = 0, y \neq 0$, then $e^{f\left(Ey + \lambda\rho(Ex,Ey)\right)} = e = \max\{e^{-3}, e^{1}\}$.

Case 4: If $x \neq 0, y = 0$, then $e^{f\left(Ey + \lambda\rho(Ex,Ey)\right)} = e^{f(4)} = e = \max\{e^{1}, e^{-3}\}$.

In all cases, $e^{f\left(Ey + \lambda\rho(Ex,Ey)\right)} = \max\left\{e^{f(x)}, e^{f(y)}\right\}$, and hence the exponentially quasi semi E-preinvexity of f is confirmed. The function f is not exponentially semi E-preinvex function, take $x = 0, y \neq 0$. Then,

$e^{f\left(Ey + \lambda\rho(Ex,Ey)\right)} = e^{f(-4)} = e > \lambda e^{f(x)} + (1-\lambda)e^{f(y)} = \lambda(e^{-3} - e) + e$ as required.

Remark (2.4). For the rest of this section and unless otherwise stated, the set K is E-invex.

A necessary condition for a function to be exponentially semi E-preinvex is given below.

Proposition (2.5). Let f is exponentially semi E-preinve.Then, $e^{f(Ek)} \leq e^{f(k)}$, for each $k \in K$.

Proof. From the assumption of f, we have

$$e^{f\left(Ek_2 + \lambda\rho(Ek_1, Ek_2)\right)} \leq \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}$$

for arbitrary $k_1, k_2 \in K$, Choose $\lambda = 0$, we get $e^{f(Ek_2)} \leq e^{f(k_2)}$. ∎

For an exponentially E-preinvex function, the necessary condition in the above proposition becomes sufficient as it is provided below.

Proposition (2.6). Assume that f is exponentially E-preinvex. Then, f is exponentially semi E-preinvex if and only if $e^{f(Ek)} \leq e^{f(k)}$, for each $k \in K$.

Proof. From Proposition 2.5, the direct implication is satisfied. Conversely, assume that f is exponentially E-preinvex and $e^{f(Ek)} \leq e^{f(k)}$, for $k \in K$. Then, for any $k_1, k_2 \in K$, we have

$$e^{f\left(Ek_2 + \lambda\rho(Ek_1, Ek_2)\right)} \leq \lambda e^{f(Ek_1)} + (1-\lambda)e^{f(Ek_2)}$$

$$\leq \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}.$$

Hence, f is exponentially semi E-preinvex. ∎

The reverse implication of the proceeding proposition and the below example yield that exponentially E-preinvexity of a function is not exponentially semi E-preinvexity, in general.

Example (2.7). Consider K, f, and ρ are defined as in Example 2.2. Let $E: \mathbb{R}^n \to \mathbb{R}^n$ such that $E(x_1, \dots, x_n)$ $=(x_1, x_2+2, \dots, x_n)$. Then, f is exponentially E-preinvex and not semi exponentially E-preinvex function. Indeed, let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in K$ then $Ey + \lambda\rho(Ex, Ey) \in K$ and

$$e^{f\left(Ey+\lambda\rho(Ex,Ey)\right)} = e^{f(\lambda x_1+(1-\lambda)y_1,\lambda(x_2+2)+(1-\lambda)(y_2+2)+\cdots+\lambda x_n+(1-\lambda)y_n)}$$

$$= e^{a\lambda(x_2+2)+a(1-\lambda)(y_2+2)} \leq \lambda e^{a(x_2+2)} + (1-\lambda)e^{a(y_2+2)}$$

$$= \lambda e^{f(Ex)} + (1-\lambda)e^{f(Ey)}$$

The above inequality holds for any $p, q > 0$ and $\lambda \in [0,1]$.

i.e., $p^\lambda q^{1-\lambda} \leq \lambda p + (1-\lambda)q$. Thus, f is exponentially E-preinvex, however, it is not exponentially semi E-preinvex function. To show this, let $k = (0, \dots, 0) \in K \subseteq \mathbb{R}^n$ be a zero vector in $\mathbb{R}^n$ then $e^{f(E(k))} = e^{2a} > e^{f(k)} = e^0$. Thus, from Proposition 2.5, f is not exponentially semi E-preinvex function.

The relation between exponentially semi (resp., quasi semi) E-preinvex functions and the level set L_γ are shown next.

Proposition (2.8). For any $\gamma \in \mathbb{R}$,

1. The level set L_γ of the exponentially semi E-preinvex function f is E-invex set.
2. L_γ is E-invex set if and only if the function f is exponentially quasi semi E-preinvex.

Proof. To show (1), choose arbitrary $k_1, k_2 \in L_\gamma$ then $e^{f(k_1)} \leq \gamma$ and $e^{f(k_2)} \leq \gamma$. From the assumptions, $Ek_2 + \lambda\rho(Ek_1, Ek_2) \in K$ and

$$e^{f\left(Ek_2+\lambda\rho(Ek_1,Ek_2)\right)} \leq \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}$$

$$\leq \lambda\gamma + (1-\lambda)\gamma = \gamma.$$

Then, $Ek_2 + \lambda\rho(Ek_1, Ek_2) \in L_\gamma$. Next, we show (2), since K is E-invex set then, for any $k_1, k_2 \in K$, $Ek_2 + \lambda\rho(Ek_1, Ek_2) \in K$. Let $\gamma = \max\{e^{f(k_1)}, e^{f(k_2)}\}$, then $k_1, k_2 \in L_\gamma$. Now, assume that L_γ is E-invex set then $e^{f\left(Ek_2+\lambda\rho(Ek_1,Ek_2)\right)} \leq \gamma = \max\{e^{f(k_1)}, e^{f(k_2)}\}$. The last inequality yields the exponentially quasi semi E-preinvexity of f. Conversely, let f is exponentially quasi semi E-preinvex. Then, for any $k_1, k_2 \in L_\gamma$, $Ek_2 + \lambda\rho(Ek_1, Ek_2) \in K$, $e^{f(k_1)} \leq \gamma$, $e^{f(k_2)} \leq \gamma$, and

$e^{f\left(Ek_2+\lambda\rho(Ek_1,Ek_2)\right)} \leq \max\{e^{f(k_1)}, e^{f(k_2)}\} \leq \gamma$. Thus, $Ek_2 + \lambda\rho(Ek_1, Ek_2) \in L_\gamma$ and the set L_γ is E-invex. ■

In the next proposition, the relationship between exponentially semi E-preinvex and its level set L_γ^E are provided.

Proposition (2.9). Assume that $E(K)$ is E-invex set with respect to $E \circ \rho$. If f is exponentially semi E-preinvex and E is idempotent and linear. Then,

1. L_γ^E is E-invex set with respect to $E \circ \rho$, for all $\gamma \in \mathbb{R}$.

2. L_γ^E is slack-E-invex with respect to $E(K)$, for all $\gamma \in \mathbb{R}$.

Proof. Let $\gamma \in \mathbb{R}$. To prove part (1), let $Ek_1, Ek_2 \in L_\gamma^E$, i.e., $k_1, k_2 \in K$ such that $e^{f(k_1)} \leq \gamma$ and $e^{f(k_2)} \leq \gamma$. Since $Ek_1, Ek_2 \in E(K)$ and $E(K)$ is E-invex with respect to $E \circ \rho$, we have

$$E^2k_2 + \lambda(E \circ \rho)(E^2k_1, E^2k_2) \in E(K).$$

Since E is idempotent and linear, we get

$$E\left(Ek_2 + \lambda\rho(Ek_1, Ek_2)\right) \in E(K). \qquad (1)$$

Since f is exponentially semi E-preinvex, then

$$e^{f\left(Ek_2+\lambda\rho(Ek_1,Ek_2)\right)} \leq \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}$$

$$\leq \lambda\gamma + (1-\lambda)\gamma = \gamma. \qquad (2)$$

From (1) and (2), we get the required conclusion. Now, we prove part (2), let $Ek_1, Ek_2 \in L_\gamma^E \cap E(K)$ such that

$$E^2k_2 + \lambda(E \circ \rho)(E^2k_1, E^2k_2) \in E(K)$$

Using the same procedure of part (1), we get

$$E^2k_2 + \lambda(E \circ \rho)(E^2k_1, E^2k_2) \in L_\gamma^E.$$

This shows L_γ^E is slack-E-invex with respect to $E(K)$. ■

Two characterizations of the exponentially quasi semi E-preinvexity of f and $f \circ E$ using L_γ^E and E-L_γ respectively, are shown next.

Proposition (2.10). Let $E(K)$ is E-invex with respect to $E \circ \rho$ and E is idempotent and linear mapping. Then, L_γ^E is E-invex set with respect to $E \circ \rho$ for all $\gamma \in \mathbb{R}$ if and only if f is exponentially quasi semi E-preinvex.

Proof. To show the direct implication, let $k_1, k_2 \in K$ and by setting $\gamma = \max\{e^{f(k_1)}, e^{f(k_2)}\}$. Then $Ek_1, Ek_2 \in L_\gamma^E$ and $e^{f(k_1)} \leq \gamma$ and $e^{f(k_2)} \leq \gamma$. From the E-invexity of L_γ^E with respect to $E \circ \rho$, we get

$$E^2k_2 + \lambda(E \circ \rho)(E^2k_1, E^2k_2) \in L_\gamma^E \quad \text{for any } \gamma \in \mathbb{R}.$$

The assumptions on E yield $E(Ek_2 + \lambda\rho(Ek_1, Ek_2)) \in L_\gamma^E$.

Applying now the definition of L_γ^E to get

$$e^{f\left(Ek_2+\lambda\rho(Ek_1,Ek_2)\right)} \leq \gamma = \max\{e^{f(k_1)}, e^{f(k_2)}\}.$$

Then, f is exponentially quasi semi E-preinvex. Now, we prove the reverse direction. Let $\gamma \in \mathbb{R}$ and $Ek_1, Ek_2 \in L_\gamma^E$ then $Ek_1, Ek_2 \in E(K)$ and $e^{f(k_1)} \leq \gamma$ and $e^{f(k_2)} \leq \gamma$. From the invexity of $E(K)$ with respect to $E \circ \rho$, we get

$$E^2k_2 + \lambda(E \circ \rho)(E^2k_1, E^2k_2) = E\left(Ek_2 + \lambda\rho(Ek_1, Ek_2)\right) \in E(K), \qquad (3)$$

where in (3), the assumptions on E are used. From the exponentially quasi semi E-preinvexity of f and the fact that $Ek_2 + \lambda\rho(Ek_1, Ek_2) \in K$, then

$$e^{f(Ek_2+\lambda\,\rho(Ek_1,Ek_2)} \leq \max\{e^{f(k_1)}, e^{f(k_2)}\} \leq \gamma \qquad (4)$$

From (3) and (4), we get the required conclusion. ■

Proposition (2.11). The function $f \circ E$ is exponentially quasi semi E-preinvex function if and only if E-L_γ is E-invex set for all $\gamma \in \mathbb{R}$.

Proof. First, we show that E-L_γ is E-invex set. For all $\gamma \in \mathbb{R}$ and $k_1, k_2 \in E$-L_γ then $e^{f(Ek_1)} \leq \gamma$ and $e^{f(Ek_2)} \leq \gamma$, i.e., $e^{(f\circ E)(k_1)} \leq \gamma$ and $e^{(f\circ E)(k_2)} \leq \gamma$. Since $k_1, k_2 \in K$ and K is E-invex set, we have

$$Ek_2 + \lambda\rho(Ek_1, Ek_2) \in K \qquad (5)$$

From the exponentially quasi semi E-preinvex of $f \circ E$, we get

$$e^{(f\circ E)(Ek_2+\lambda\rho(Ek_1,Ek_2))} \leq \max\{e^{(f\circ E)(k_1)}, e^{(f\circ E)(k_2)}\} \leq \gamma. \qquad (6)$$

Then by (5) and (6), we get $Ek_2 + \lambda\rho(Ek_1, Ek_2) \in E$-$L_\gamma$. Therefore, E-L_γ is E-invex set for all $\gamma \in \mathbb{R}$. Let us now show that $f \circ E$ is exponentially quasi semi E-preinvex. Let $k_1, k_2 \in K$. Since K is E-invex set, we have

$Ek_2 + \lambda\rho(Ek_1, Ek_2) \in K$. Set $\gamma = \max\{e^{(f\circ E)(k_1)}, e^{(f\circ E)(k_2)}\}$. Since E-L_γ is E-invex, we get

$$e^{(f\circ E)(Ek_2+\lambda\rho(Ek_1,Ek_2))} \leq \gamma = \max\left\{e^{(f\circ E)(k_1)}, e^{(f\circ E)(k_2)}\right\}$$

This shows $f \circ E$ is exponentially quasi semi E-preinvex. ■

For the rest of this section, some characterizations and properties of exponentially semi E-preinvex functions using the epigraph sets defined in Definitions 1.9 -1.10 are shown. First, a characterization of exponentially semi E-preinvex function using the $E \times I$ invexity of $epif$ is presented below.

Proposition (2.12). The function f is exponentially semi E-preinvex if and only if $epif$ is $E \times I$-invex on $K \times \mathbb{R}$ with respect to $\rho \times \rho_\circ$ where $\rho_\circ: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as $\rho_\circ(\eta, \omega) = \eta - \omega$ for all $\eta, \omega \in \mathbb{R}$.

Proof. Let $(k_1, \eta), (k_2, \omega) \in epif$ such that $e^{f(k_1)} \leq \eta$, $e^{f(k_2)} \leq \omega$. Since f is exponentially semi E-preinvex function. Then,

$$e^{f\left(Ek_2+\lambda\rho(Ek_1,Ek_2)\right)} \leq \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}$$
$$\leq \lambda\eta + (1-\lambda)\omega,$$

This implies,

$$(Ek_2 + \lambda\rho(Ek_1, Ek_2), \lambda\eta + (1-\lambda)\omega) \in epif \qquad (15)$$

Now, $\lambda\eta + (1-\lambda)\omega = \omega + \lambda(\eta - \omega) = I(\omega) + \lambda\rho_\circ\big(I(\eta), I(\omega)\big)$.

where $\rho_\circ(\eta, \omega) = \eta - \omega$. Thus, we re-write (15) as

$(Ek_2 + \lambda\rho(Ek_1, Ek_2), I(\omega) + \lambda\rho_\circ\big(I(\eta), I(\omega)\big) \in epi\, f$. Thus, $epi\, f$ is $E \times I$ invex on $K \times \mathbb{R}$ with respect to $\rho \times \rho_\circ$. Conversely, let $epi\, f$ is $E \times I$-invex set on $K \times \mathbb{R}$ with respect to $\rho \times \rho_\circ$ and $k_1, k_2 \in K$, then $\left(k_1, e^{f(k_1)}\right), \left(k_2, e^{f(k_2)}\right) \in epi\, f$ and

$$(Ek_2 + \lambda\rho(Ek_1, Ek_2), I\left(e^{f(k_2)}\right) + \lambda\rho_\circ\left(I\left(e^{f(k_1)}\right) - I\left(e^{f(k_2)}\right)\right) \in epi\, f$$

i.e., $(Ek_2 + \lambda\rho(Ek_1, Ek_2), e^{f(k_2)} + \lambda\rho_\circ\left(e^{f(k_1)} - e^{f(k_2)}\right) \in epi\, f$

Using the definition of $\rho_\circ$, the last expression yields

$$(Ek_2 + \lambda\rho(Ek_1, Ek_2), \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}) \in epi\, f$$

which implies that $e^{Ek_2+\lambda\rho(Ek_1,Ek_2)} \leq \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}$. Hence, f is exponentially semi E-preinvex. ■

Next, a characterization is proposed for the exponentially E-preinvex function f to be semi E-preinvex functions by employing the relation between $epi\ f$ and E-$epi\ f$.

Proposition (2.13). Let f is exponentially E-preinvex function. Then, $epi\ f \subset E - epi\ f$ if and only if f is exponentially semi E-preinvex function.

Proof. Assume that $k_1, k_2 \in K$ and $(k_1, e^{f(k_1)}), (k_2, e^{f(k_2)}) \in epi\ f \subset E - epi\ f$, thus $(k_1, e^{f(k_1)}), (k_2, e^{f(k_2)}) \in E - epi\ f$. i.e., $e^{f(Ek_1)} \leq e^{f(k_1)}$ and $e^{f(Ek_2)} \leq e^{f(k_2)}$. Since f is exponentially E-preinvex function, then we have

$e^{f\left(Ek_2+\lambda\rho(Ek_1,Ek_2)\right)} \leq \lambda e^{f(Ek_1)} + (1-\lambda)e^{f(Ek_2)} \leq \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}$. Hence, f is exponentially semi E-preinvex function. To show the reverse direction, assume that f is exponentially semi E-preinvex function and we show that $epi\ f \subset E - epi\ f$. Let $(k, \alpha) \in epi\ f$ then $k \in K$ and $e^{f(k)} \leq \alpha$. Since f is exponentially semi E-preinvex function then from Proposition 2.6 and the definition of $epi\ f$, we have $e^{f(Ek)} \leq e^{f(k)} \leq \alpha$ for all $k \in K$. Thus, $(k, \alpha) \in E - epi\ f$, hence, $epi\ f \subset E - epi\ f$. ■

In what follows, different sufficient conditions are introduced to have f exponentially semi E-preinvex functions using different epigraph sets.

Proposition (2.14). If $E(K)$ be invex set with respect to ρ, $epi\ f \subset E - epi\ f$, and $epi_E\ f$ is invex set with respect to $\rho \times \rho_\circ$ where $\rho_\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\rho_\circ(\eta, \omega) = \eta - \omega$ for all $\eta, \omega \in \mathbb{R}$. Then, f is exponentially semi E-preinvex function.

Proof. L et $k_1, k_2 \in K$ and $\left(k_1, e^{f(k_1)}\right), \left(k_2, e^{f(k_2)}\right) \in epi\ f \subset E - epi\ f$. Thus, $e^{f(Ek_1)} \leq e^{f(k_1)}$, $e^{f(Ek_2)} \leq e^{f(k_2)}$ which yields $\left(E k_1, e^{f(k_1)}\right)$, $\left(Ek_2, e^{f(k_2)}\right) \in epi_E\ f$. From the invexity of $E(K)$ and $epi_E\ f$, we get

$Ek_2 + \lambda\rho(E k_1, Ek_2) \in E(K) \subseteq K$ and

$$(Ek_2 + \lambda\rho(E k_1, Ek_2), e^{f(k_2)} + \lambda\rho_\circ\left(e^{f(k_1)}, e^{f(k_2)}\right)) \in epi_E\ f .$$

Then,

$$e^{f\left(Ek_2+\lambda\rho(E k_1,Ek_2)\right)} \leq e^{f(k_2)} + \lambda\rho_\circ\left(e^{f(k_1)}, e^{f(k_2)}\right) = \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}.$$

Hence, f is exponentially semi E-preinvex function. ■

Proposition (2.15). If $e^{f(Ek)} \leq e^{f(k)}$ $\forall k \in K$ and $epi^E f$ is invex set with respect to $\rho \times \rho_\circ$. Then, f is exponentially semi E-preinvex function.

Proof. Let $k_1, k_2 \in K$ and $\left(E k_1, e^{f(k_1)}\right), \left(Ek_2, e^{f(k_2)}\right) \in epi^E\ f$ which is invex set with respect to $\rho \times \rho_\circ$

$$\left(Ek_2 + \lambda\rho(E k_1, Ek_2), e^{f(k_2)} + \lambda\rho_\circ\left(e^{f(k_1)}, e^{f(k_2)}\right)\right) \in epi^E f \subseteq E(K) \times \mathbb{R}.$$

Since $Ek_2 + \lambda\rho(E k_1, Ek_2) \in E(K)$ then there exists $w \in K$ such that $Ew = Ek_2 + \lambda\rho(E k_1, Ek_2) \in E(K)$ and

$$e^{f(w)} \leq e^{f(k_2)} + \lambda\rho_\circ\left(e^{f(k_1)}, e^{f(k_2)}\right) = \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}. \quad (8)$$

From the assumption and the inequality in (8)

$$e^{Ek_2+\lambda\rho(E k_1,Ek_2)} = e^{f(Ew)} \leq e^{f(w)} \leq \lambda e^{f(k_1)} + (1-\lambda)e^{f(k_2)}.$$

Hence, f is exponentially semi E-preinvex function on K. ■

Proposition (2.16). Assume that $E(K)$ is invex set with respect to ρ, $e^{f(Ek)} \leq e^{f(k)} \quad \forall k \in K$, and $epi\ f$ is slack invex with respect to $E(K) \times \mathbb{R}$. Then, f is exponentially semi E-preinvex function.

Proof. Let $k_1, k_2 \in K$, and $\left(E\,k_1, e^{f(E\,k_1)}\right), \left(Ek_2, e^{f(Ek_2)}\right) \in epi\ f \cap E(K) \times \mathbb{R}$

Since $E(K)$ is invex set with respect to ρ, hence $Ek_2 + \lambda\rho(E\,k_1, Ek_2) \in E(K)$. Then

$$\left(Ek_2 + \lambda\rho(E\,k_1, Ek_2), \lambda e^{f(E\,k_1)} + (1-\lambda)e^{f(Ek_2)}\right) \in E(K) \times \mathbb{R},$$

which yields, from the slackness assumption, that

$$\left(Ek_2 + \lambda\rho(E\,k_1, Ek_2), \lambda e^{f(E\,k_1)} + (1-\lambda)e^{f(Ek_2)}\right) \in epi\ f$$

Thus, $e^{f\left(Ek_2+\lambda\rho(E\,k_1,Ek_2)\right)} \leq \lambda e^{f(\,k_1)} + (1-\lambda)e^{f(k_2)}$, where the right inequality yields from the assumption $e^{f(Ek)} \leq e^{f(k)} \quad \forall k \in K$. Hence, f is exponentially semi E-preinvex function on K. ■

Proposition (2.17). Let $E - epi\ f$ is $E \times I$- invex set with respect to $\rho \times \rho_\circ$ and $epi\ f \subset E - epi\ f$. If the mapping E is idempotent and linear. Then, f is exponentially semi E-preinvex function.

Proof. Let $k_1, k_2 \in K$ such that $\left(\,k_1, e^{f(\,k_1)}\right), \left(k_2, e^{f(k_2)}\right) \in epif \subset E - epi\ f$.

Since $E - epi\ f$ is $E \times I$-invex set with respect to $\rho \times \rho_\circ$, thus $\left(Ek_2 + \lambda\rho(E\,k_1, Ek_2)\,, e^{f(k_2)} + \lambda\rho_\circ\left(e^{f(\,k_1)}, e^{f(k_2)}\right)\right) \in E - epi\ f$ which yields

$$e^{f\left(E\left(Ek_2+\lambda\rho(E\,k_1,Ek_2)\right)\right)} \leq e^{f(k_2)} + \lambda\rho_\circ\left(e^{f(\,k_1)}, e^{f(k_2)}\right) = \lambda e^{f(\,k_1)} + (1-\lambda)e^{f(k_2)}.$$

From the assumptions on E, the left hand side of the last inequality becomes $e^{f\left(E\left(Ek_2+\lambda\rho(E\,k_1,Ek_2)\right)\right)} = e^{f\left((EoE)k_2+\lambda\rho((EoE)\,k_1,(EoE)k_2\right)}$,

$$= e^{f\left(Ek_2+\lambda\rho(E\,k_1,Ek_2\right)}$$

$$\leq \lambda e^{f(\,k_1)} + (1-\lambda)e^{f(k_2)}.$$

Hence, f is exponentially semi E-preinvex function. ■

3. Applications of Exponentially E-Preinvexity and Semi E-Preinvexity in Nonlinear Optimization Problems

In this section, we consider some applications of exponentially E-preinvex, semi exponentially E-preinvex, and exponentially quasi semi E-preinvex functions in optimization programming problem (P_E) defined below. Some optimality properties are satisfied under different conditions for the objective function foE. To start, consider the nonlinear constrained optimization problem (P_E) as follows.

$$\min\ (f \circ E)(k)$$

$$\text{subject to}\ \ k \in K,$$

where K, f and E are defined as in section one.

Definition (3.1). The set of all global minima (or optimal solutions) of problem (P_E) is denoted by $argmin_K f \circ E$ and is defined as

$$argmin_K f \circ E = \{k^* \in K : (f \circ E)(k^*) \leq (f \circ E)(k) \quad \forall k \in K\}.$$

A global minimum $k^* \in K$ is said to be strict when

$$(f \circ E)(k^*) < (f \circ E)(k) \quad \text{for all } x \in K\,, k^* \neq k\,.$$

A point $k^* \in \mathbb{R}^n$ is called a local minimizer for problem (P_E) if there is exists $\delta > 0$ such that $(f \circ E)(k^*) \leq (f \circ E)(k) \quad \forall k \in B(k^*, \delta) \cap K$.

Remark (3.2). In all results of this section. If the function f in the optimization problem (P_E) is considered to be exponentially (or strictly) E-preinvex or strictly exponentially quasi semi E-preinvex on K, then it is assumed that the constraint set K is E-invex and invex and the mapping E is linear.

Under various assumptions on the functions f or $f \circ E$, every local minimum of the problem (P_E) is a global minimum, as we show below.

Theorem (3.3).

1. If f is exponentially E-preinvex. Then, every local minimum of problem (P_E) is a global minimum.
2. If $f \circ E$ is exponentially semi E-preinvex on the E-invex set K and k^* be a local minimum of problem (P_E) such that $k^* = Ek^*$. Then k^* is a global minimum.

Proof. To show (1), assume that f is exponentially E-preinvex on K and k^* is local minimum of problem (P_E) then, $k^* \in K$ and there exists $\delta > 0$ such that $e^{f(Ek^*)} \leq e^{f(Ek)}$ for all $k \in K \cap B(k^*, \delta)$. Suppose that k^* is not a global minimum, hence, there exists $k^o \in K$ such that $f(Ek^o) \leq f(Ek^*)$. i.e., $e^{f(Ek^o)} \leq e^{f(Ek^*)}$. Since $k^o, k^* \in K$ and K is invex set then $k^o + \lambda\rho(k^*, k^o) \in K$ and

$$Ek^o + \lambda\rho(Ek^*, Ek^o) = E\big(k^o + \lambda\rho(k^*, k^o)\big) \in \mathrm{K},$$

where we used, in the last expression, the linearity of E, the invexity and E-invexity of K as it is stated in Remark 3.2. From the assumptions on f and E with the fact that $e^{f(Ek^o)} \leq e^{f(Ek^*)}$, we have

$$\begin{aligned} e^{f\left(E(k^o+\lambda\rho(k^*,k^o))\right)} = e^{f(\,Ek^o+\lambda\rho(Ek^*,Ek^o)} &\leq \lambda e^{f(Ek^*)} + (1-\lambda)e^{f(Ek^o)} \\ &\leq \lambda e^{f(Ek^*)} + (1-\lambda)e^{f(Ek^*)} \\ &= e^{f(Ek^*)}. \qquad (9) \end{aligned}$$

Now, for sufficiently small $\lambda \in (0,1]$ then $k^o + \lambda\rho(k^*, k^o)$ will be close enough to k^*. i.e., there exists $\delta > 0$ such that $k^o + \lambda\rho(k^*, k^o) \in B(k^*, \delta) \cap K$. From the local minimality of k^*, we get $f(Ek^*) \leq f(\,Ek^o + \lambda\rho(Ek^*, Ek^o)$. i.e., $e^{f(Ek^*)} \leq e^{f(\,Ek^o+\lambda\rho(Ek^*,Ek^o)}$ which contradicts (9). Thus, k^* is a global minimum. Next, we prove (2), since $k^* = Ek^*$ is a local minimum, then there exists $\delta > 0$ such that

$$(f \circ E)(k^*) \leq (f \circ E)(k) \ \ \forall k \in K \cap B(k^*, \delta) = D.$$

Now, to show $(f \circ E)(k^*) \leq (f \circ E)(k)$, for any $k \in K/D$. On contrary, suppose there exists $k_0 \in K$ and $k_0 \notin B(k^*, \delta)$ where

$$(f \circ E)(k^*) > (f \circ E)(k_0). \qquad (10)$$

Since K is E-invex and $f \circ E$ is exponentially semi E-preinvex, then $Ek^* + \lambda\rho(Ek_0. Ek^*) \in K$, and

$$e^{(f\circ E)\,(Ek^*+\lambda\rho(Ek_0,Ek^*))} \leq \lambda e^{(f\circ E)(k_0)} + (1-\lambda)e^{(f\circ E)(k^*)}$$

Applying (10),

$$\begin{aligned} e^{(f\circ E)\,(Ek^*+\lambda\rho(Ek_0,Ek^*))} &< \lambda e^{(f\circ E)(k^*)} + (1-\lambda)e^{(f\circ E)(k^*)} \\ &= e^{(f\circ E)(Ek^*)} = e^{(f\circ E)(k^*)} \qquad (11) \end{aligned}$$

If $\rho(Ek_0, Ek^*) = 0$, then (11) becomes $e^{(f\circ E)(Ek^*)} < e^{(f\circ E)(k^*)}$, i.e., $(f \circ E)(Ek^*) < (f \circ E)(k^*)$ which contradicts the assumption. If now $\rho(Ek_0, Ek^*) \neq 0$, take λ small enough such that $Ek^* + \lambda\rho(Ek_0, Ek^*) \in B(k^*, \delta)$. From (11), this contradicts the local minimality of k^*. Hence, $(f \circ E)(k^*) \leq (f \circ E)(k)$ for any $k \in K \,/\, D$ as required. ■

Next result discusses the uniqueness of optimal solution of problem (P_E) under different three hypothesis.

Theorem (3.4).

1. If f is strictly exponentially E-preinvex on K. Then, the global minimum of problem (P_E) is unique.
2. If $f \circ E$ is strictly exponentially semi E-preinvex on the E-invex set K. Then the global optimal solutions of problem (P_E) is unique.
3. If f is strictly exponentially quasi semi E-preinvex on the E-invex set K and $Ek^* = k^*$ for any global optimal solution k^*. Then, the global minimum of (P_E) is unique.

Proof. Let us first proceed the proof of part (1). Let k_1, k_2 be two minima of problem (P_E) such that $k_1 \neq k_2$, thus $f(Ek_1) = f(Ek_2)$. i.e, $e^{f(Ek_1)} = e^{f(Ek_2)}$. From Remark 3.2, K is E-invex, K is invex, and E is linear, then

$$Ek_2 + \lambda\rho(E\,k_1, Ek_2)) = E\big(k_2 + \lambda\rho(\,k_1, k_2)\big) \in E(K) \subseteq K,$$

thus, there exists $z \in K$ such that $z = k_2 + \lambda\rho(\,k_1, k_2) \in K$ where $z \neq k_1$ and $z \neq k_2$. Since f is exponentially strictly E-preinvex on K, then

$$e^{f(Ez)} = e^{f\left(Ek_2+\lambda\rho(E\,k_1,Ek_2)\right)} < \lambda e^{f(E\,k_1)} + (1-\lambda)e^{f(Ek_2)}$$

$$= e^{f(E\,k_1)}$$

This means, $f(Ez) < f(Ek_1)$, hence z is a global minimum. This is contradiction. Thus, the global minimum is singleton. Next, we proof part (2), Let k_1, k_2 be two minima of problem (P_E) such that $k_1 \neq k_2$, thus $f(Ek_1) = f(Ek_2)$. i.e, $e^{f(Ek_1)} = e^{f(Ek_2)}$. From the E-invexty of K and the assumption on $f \circ E$, for each $\lambda \in (0,1)$, we get $z = Ek_2 + \lambda\rho(Ek_1, Ek_2) \in K$ where $z \neq k_1(\neq k_2)$ and

$$e^{(f\circ E)(z)} < \lambda e^{(f\circ E)(k_1)} + (1-\lambda)e^{(f\circ E)(k_2)}$$

$$= \lambda e^{(f\circ E)(k_1)} + (1-\lambda)e^{(f\circ E)(k_1)} = e^{(f\circ E)(k_1)},$$

i.e., $(f \circ E)(z) < (f \circ E)(k_1)$ which contradicts the optimality of k_1. Thus, the global mimimum of problem (P_E) is unique. Lastly, we show (3), Let Let $k_1, k_2 \in K$ be two different global minima then $e^{f(Ek_1)} = e^{f(Ek_2)}$. Since E is fixed with respect to the global minima, then $f(Ek_1) = f(k_1) = f(k_2) = f(Ek_2)$. Since f is strictly exponentially quasi semi E-preinvex, then

$$e^{f\left(Ek_2+\lambda\rho(k_1,Ek_2)\right)} < \max\left\{ e^{f(k_1)}, e^{f(k_2)}\right\} = e^{f(k_1)} = e^{f(Ek_1)}.$$

Since E is linear, the left-hand side of the inequality above can be written as

$$e^{f\left(E(k_2+\lambda\rho(k_1,k_2))\right)} = e^{f\left(Ek_2+\lambda\rho(Ek_1,Ek_2)\right)} < e^{f(Ek_1)}. \qquad (12)$$

The expression above is a contradiction. This is because K is invex and K is E-invex, then $E(k_2 + \lambda\rho(k_1, k_2)) \in K$. thus, there exists $z = (k_2 + \lambda\rho(k_1, k_2) \in K$ where $z \neq k_1$ and $z \neq k_2$. Using the last conclusion with the inequality in (12), we obtain that $f(Ez) = f\big(E(k_2 + \lambda\rho(k_1, k_2))\big) < f(Ek_1)$. i.e,, w is a global minimum which is a contradiction. Thus, $k_1 = k_2$. ■

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