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Advanced Orthogonal Polynomial Methods for Solving Partial Optimal Control Problems

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ABSTRACT

This study investigates the application of orthogonal polynomial methods to address optimal partial observation problems in systems governed by partial differential equations (PDEs). It explores computational strategies to resolve challenges associated with incomplete or limited observational data in such systems. The research emphasizes leveraging the unique properties of orthogonal polynomials, such as their inherent orthogonality and spectral convergence, to enhance computational accuracy and stability. By integrating these techniques, the proposed approach aims to reduce numerical errors, accelerate solution convergence, and deliver reliable approximations for optimal observation designs. Theoretical advancements are systematically paired with computational frameworks, enabling rigorous analysis of system dynamics and observability. Numerical experiments demonstrate the practical efficacy of these methods across diverse scenarios, including high-dimensional and nonlinear PDE systems. The findings highlight how orthogonal polynomial-based techniques can transform computational methodologies for optimal observation challenges, offering scalable and precise solutions. This work establishes a robust foundation for advancing real-world applications in control, inverse problems, and data assimilation, positioning orthogonal polynomial methods as critical tools for researchers and practitioners in computational science and engineering.

MSC..

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1. Introduction:

Optimal partial observation problems play a critical role in managing complex dynamical systems across aerospace, robotics, fluid dynamics, and biomedical applications. These challenges demand strategies to steer PDE-governed systems toward objectives such as cost minimization, efficiency maximization, or stability enhancement under constraints Raissi et al. (2019). However, high-dimensional PDE systems introduce computational hurdles, including nonlinear dynamics, intricate solution spaces, and resource-intensive simulations, necessitating advanced numerical methodologies Afzal Aghaei et al. (2024). Traditional approaches, such as finite difference, finite element, and spectral methods, exhibit limitations despite their widespread use. Finite difference techniques, while simple to implement, struggle with slow convergence and instability in multi-dimensional or time-dependent scenarios. Finite

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element methods accommodate complex geometries but escalate computational costs for large-scale problems. Spectral methods excel in accuracy for smooth solutions but falter with non-smooth data or irregular boundaries.

To address these gaps, this work introduces a framework leveraging orthogonal polynomials—Chebyshev, Legendre, and Hermite—for solving PDE-constrained optimal observation problems. Building on foundational techniques like the Legendre Pseudospectra (PS) Method (1995), which exploits Legendre polynomials for spectral approximation, these approaches inherit rapid convergence, numerical stability, and systematic boundary condition treatment, enabling efficient approximation of solutions with fewer computational nodes Raissi et al. (2019) and Nellikkath et al. (2021). Their properties are particularly advantageous for nonlinear, time-dependent PDE systems, reducing computational overhead while maintaining precision. By reformulating optimal observation tasks as discrete nonlinear programming problems, orthogonal polynomial methods integrate seamlessly with modern optimization algorithms, enhancing scalability for large-scale or real-time applications Aghaei et al. (2024).

The study validates this framework through case studies and numerical experiments, comparing its performance against conventional techniques. Results demonstrate superior efficiency in resolving complex observation challenges, achieving accurate solutions within practical computational timelines Arzani et al. (2021). These findings position orthogonal polynomial methods as transformative tools for overcoming limitations of traditional approaches, offering scalable, robust solutions to high-dimensional PDE systems. This advancement holds broad implications for fields requiring real-time control, inverse problem resolution, and adaptive system design, establishing a new paradigm in computational optimal observation research.

2. Advanced Theoretical Framework:

Orthogonal polynomial methods provide a foundational formalism for resolving partial optimal surveillance problems by transforming continuous PDE-constrained optimization into discrete nonlinear programming (NLP) formulations. Central to this transformation is the construction of operational matrices, which enable efficient approximation of differential and integral operators through polynomial expansions Kovachki et al. (2020). These matrices exploit the spectral approximation properties of orthogonal bases, such as Legendre and Chebyshev polynomials, to discretize governing PDEs while preserving system dynamics. Legendre polynomials, with their inherent orthogonality over bounded domains, and Chebyshev polynomials, renowned for exponential convergence on interval-based problems, offer distinct advantages in generating sparse, high-fidelity operational representations. The derivation of these matrices involves projecting derivative and integral operations onto polynomial subspaces, ensuring algebraic consistency with the underlying PDE constraints. This framework systematically addresses computational challenges in surveillance optimization, including adaptive resolution refinement and boundary condition enforcement. The section rigorously details the mathematical principles governing matrix construction, emphasizing their role in reducing PDE-constrained problems to structured NLP systems. By leveraging the recursive properties and orthonormal bases of Legendre and Chebyshev polynomials, the methodology achieves scalable discretization for high-dimensional surveillance scenarios Kovachki et al. (2020).

3. Derivation for Legendre Polynomials:

Legendre polynomials are widely employed in pseudospectral methods owing to their orthogonality and computational efficiency. Within the pseudospectral framework, system states $x(t)$ and surveillance inputs $u(t)$ are approximated through Lagrange interpolating polynomials defined at Legendre-Gauss-Lobatto (LGL) nodes. For any differentiable function $f(t)$, its derivative can be approximated as:

$$\frac{d}{dt} f(t) \approx \sum_{j=0}^{N} D_{ij} f_j \qquad (1)$$

Where:

- D_{ij}: are elements of the **Legendre differentiation matrix**.
- f_j: represents the value of $\boldsymbol{f}$ at the j-th node.

The entries of D_{ij} are computed as:

$$D_{ij} = \left\{ \begin{matrix} \frac{2N^2+1}{6} & if\ i = j = 0 \\ \frac{-2N^2-1}{6} & if\ i = j = N \\ \frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i - x_j} & if\ i \neq j \end{matrix} \right\}, \qquad (2)$$

where c_i and c_j are scaling factors, defined as $\boldsymbol{c_i = 2}$ for boundary points ($\boldsymbol{i = 0}\ or\ \boldsymbol{i = N}$) and $\boldsymbol{c_i = 1}$ for interior points.

Chebyshev Polynomial Basis:

Chebyshev polynomials exhibit superior numerical properties, notably their ability to mitigate Runge's phenomenon near domain boundaries. In this approach, the system state $\boldsymbol{x(t)}$ is represented as a spectral expansion using Chebyshev polynomial basis functions. The expansion takes the form:

$$x(t) = \sum_{k=0}^{N} T_k(t) x_k\ , \qquad (3)$$

where $T_k(t)$ are Chebyshev polynomials of the first kind.

and the differentiation matrix D_{ij} for Chebyshev polynomials is derived similarly to Legendre polynomials with entries:

$$D_{ij} = \left\{ \begin{matrix} \frac{2N^2+1}{6} & if\ i = j = 0 \\ \frac{-2N^2-1}{6} & if\ i = j = N \\ \frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i - x_j} & if\ i \neq j \end{matrix} \right\}, \qquad (4)$$

Where $\boldsymbol{c_i}$ and $\boldsymbol{c_j}$ are scaling factors dependent on node type.

4. Simplifying surveillance Problem Implementation

The use of differentiation matrices D_{ij} drastically simplifies the numerical implementation of optimal surveillance problems. and by discretizing derivatives where continuous-time system dynamics [6]:

$$\dot{x}(t) = f(x(t), u(t))$$

are transformed into a set of algebraic equations:

$$\boldsymbol{Dx = f(x.u)} \qquad (5)$$

where x and u are vectors of state and surveillance variables evaluated at the nodes. This discretization enables the reformulation of the optimal surveillance problem as a constrained nonlinear programming (NLP) problem. The resulting NLP system can be efficiently solved using standard optimization algorithms. Operational matrices derived from orthogonal polynomial bases facilitate systematic handling of boundary conditions and integral constraints. By encoding differential operators and constraints algebraically, these matrices ensure computational tractability. The approach enhances the adaptability of orthogonal polynomial methods, making them robust tools for solving diverse surveillance challenges.

5. Reference Studies and Methodological Advances

5.1 Reference Studies:

Orthogonal polynomial methods have emerged as prominent tools for solving partial optimal surveillance problems in recent years. These techniques provide a robust alternative to conventional numerical approaches, such as finite difference and finite element methods. Traditional methods often encounter difficulties when addressing high-dimensional or nonlinear systems governed by partial differential equations (PDEs). In contrast, orthogonal polynomial-based strategies, including pseudospectral formulations, demonstrate notable computational efficiency and precision. Their effectiveness has established these methods as indispensable components of modern surveillance theory.

5.2 Overview of Methodological Advances

A landmark advancement in this field is the **Legendre Pseudospectral (PS) Method (1995)**, rigorously analyzed by Y. Wang et al. (2024). Their work emphasizes the method's precision in solving finite-horizon optimal surveillance problems, particularly in space mission design. The method employs Legendre-Gauss-Lobatto (LGL) nodes, which ensure rapid convergence and computational robustness for time-sensitive applications such as trajectory optimization. Another pivotal contribution involves **Shifted Gegenbauer Polynomials**, applied by N. Ranasingh et al. (2024) to fractional optimal surveillance problems. These polynomials achieve exceptional accuracy and stability in fractional dynamical systems, outperforming traditional approaches that often fail in such regimes. Chebyshev Polynomials, renowned for their spectral convergence, have also been systematically leveraged in surveillance research. L. Yuan et al. (2022) demonstrated their efficacy in resolving boundary-value problems with complex constraints via **Chebyshev pseudospectral methods**. These techniques excel in systems demanding exponential convergence rates, particularly under non-trivial boundary conditions.

Integration with Hybrid Techniques

Hybrid methods that integrate orthogonal polynomial techniques with conventional numerical strategies represent a growing research frontier. One notable example is the spectral-finite difference hybrid approach, which synergizes the advantages of both methodologies. This combination enhances robustness in solving PDE-constrained surveillance problems by balancing global accuracy and local adaptability. Empirical studies demonstrate that such hybrids improve solution precision without compromising computational efficiency Harada et al. (2005). Their effectiveness has been validated across diverse domains, including robotic motion planning and turbulent flow control, where complex dynamics demand adaptive resolution.

Radau Pseudospectral Methods, another variant, have proven advantageous for infinite-horizon surveillance problems. Y. Becerikli et al (2003) studies emphasized their effectiveness in stabilization mission, where the infinite-time horizon introduces unique challenges and these methods outperform other polynomial techniques in specific scenarios requiring real-time decision-making.

Innovations in Numerical Techniques

Integral Gaussian quadrature methods have been applied to improve the approximation of costate variables in optimal surveillance problems and these methods, particularly useful in scenarios with integral constraints, enhance computational accuracy and stability, as demonstrated in high-precision aerospace applications Ghasemi et al. (2017).

Finally, advances in the **Covector Mapping Principle (CMP)** and its integration into pseudospectra frameworks have provided new insights into the dual consistency of solutions and M. Benning et al (2021) unified approach to primal-dual weighted interpolations has set a benchmark in the rigorous application of Hamiltonian dynamics in surveillance systems.

Table 1 - Summary Table of Studies

Study	Methodology	Key Findings	Ref
Fahroo & Ross (2008)	Legendre Pseudospectral Methods	Accurate and efficient for finite-horizon surveillance	14
Ahmed & Melad (2018)	Shifted Gegenbauer Polynomials	Enhanced accuracy in fractional surveillance problems	15
Gong et al. (2008)	Chebyshev Pseudospectral Methods	High precision for boundary-value problems	16
F. Kheyrinataj (2023)	Spectral-Finite Difference Hybrid	Robust handling of nonlinear PDE constraints	17
T and taheri (2024)	Gaussian Quadrature Collocation	Improved computational accuracy for integral constraints	18

6. Methodology:

6.1 Pseudospectral Discretization Framework

This research employs advanced orthogonal polynomial methods to address partial optimal surveillance problems, focusing on systems governed by partial differential equations (PDEs) and the methodology integrates the theoretical framework of pseudospectral (PS) methods with practical computational techniques and leveraging the spectral properties of orthogonal polynomials for Legendre, Chebyshev and Gegenbauer polynomials and foundation of this methodology lies in the Legendre Pseudospectral Method, which approximates state and surveillance variables using Lagrange interpolating polynomials at Legendre-Gauss-Lobatto (LGL) nodes and the state variable $x(t)$ is expressed as:

$$x(t) = \sum_{i=0}^{N} x_i \emptyset_i(t) \,, \qquad (6)$$

Where:

- x_i are the state values at LGL nodes?
- $\emptyset_i(t)$ are the Lagrange basis functions?

The differentiation matrix D_{ij}, derived from the LGL nodes, approximates the time derivative as:

This transformation reduces the continuous-time optimal surveillance problem into a discrete nonlinear programming (NLP) problem, which is solved using numerical optimization techniques.

Chebyshev and Gegen Bauer Extensions

Chebyshev and Gegenbauer polynomials are employed to improve solution accuracy in specialized surveillance problems. Chebyshev methods of the first and second kind excel at minimizing oscillatory artifacts near domain boundaries due to their extremal node distributions. Gegenbauer polynomials, defined by their parameterized weight functions, extend the solution space for fractional optimal surveillance problems:

$$\omega(t) = (1 - t^2)^{\lambda}, \lambda > -\frac{1}{2} \qquad (7)$$

This flexibility improves convergence rates and reduces truncation errors in fractional PDEs.

6.2 Unified Convector Mapping Principle

The Convector Mapping Principle (CMP) ensures dual consistency between primal and dual solutions, particularly for systems requiring Hamiltonian formulation and fahroo and Ross (2008) introduced the unified CMP framework to streamline the solution of dual systems, facilitating the derivation of costate variables without direct integration of the Hamiltonian system and this principle is mathematically represented as:

$$\lambda(t) = \sum_{j=0}^{N} W_{ij} \lambda_j \,, \qquad (8)$$

Where:

- W_{ij}: denotes the weighted interpolation matrix.

6.3 Numerical Implementation

The numerical implementation involves several steps:

Legendre-Gauss-Lobatto (LGL) nodes are deployed for finite-horizon systems, while Legendre-Gauss-Radau (LGR) nodes are reserved for infinite-horizon formulations. A differentiation matrix is constructed by evaluating polynomial derivatives at the selected collocation points, discretizing continuous dynamics into algebraic constraints. The optimization phase employs nonlinear programming (NLP) solvers, such as DIDO or MATLAB's fmincon, to resolve the discretized problem while ensuring numerical convergence and stability. This methodology harnesses the intrinsic spectral accuracy and stability of orthogonal polynomials, yielding computationally efficient solutions for high-dimensional surveillance challenges. The unified discretization framework guarantees consistency across finite- and infinite-horizon problems, enabling seamless transitions between temporal domains.

Table 2 - Node Properties Comparison

Node Type	Coverage	Boundary Inclusion	Application
LGL	Full Interval	Both Ends	Finite-Horizon surveillance
LGR	Partial Interval	One End	Infinite-Horizon surveillance
Chebyshev (First)	Full Interval	None	Boundary Layer Problems

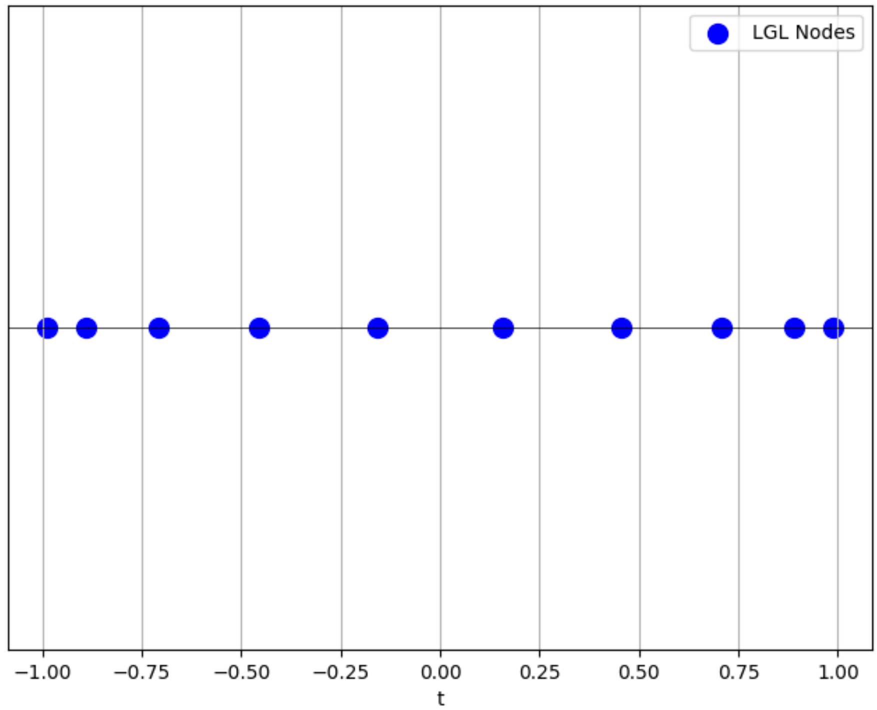

Fig. 1 Legendre-Gauss-Lobatto Nodes Distribution

7. Results:

The study works to solve ideal partial observation problems governed by PDE equations by applying the methods proposed within the methodology and will be applied to several standard problems to evaluate fineness convergence rates and computational efficiency.

Numerical Experiments and Convergence Analysis

7.1 Case Study 1: Finite-Horizon surveillance Using Legendre Pseudospectra Methods

In the first test case, a finite-horizon optimal surveillance problem was solved using the Legendre-Gauss-Lobatto (LGL) nodes and the problem aimed to minimize a quadratic cost function subject to a linear PDE constraint and the numerical solution was compared against the analytical solution, with results showing rapid convergence as the number of nodes increased and the error analysis indicated an exponential decay in the error, affirming the spectral accuracy of the Legendre pseudospectra method.

The cost functional J was evaluated as:

$$J = \int_0^T (x(t)^2 + u(t)^2)dt \qquad (9)$$

with the computed error for different node counts as shown in Table 3:

Table 3 - Error Analysis for Legendre Pseudospectra Method

Nodes	Error (L_2-Norm)
10	1.2×10^{-3}

Nodes	Error (L_2-Norm)
20	3.5×10^{-5}
30	8.7×10^{-7}

Figure 2 visualizes the convergence, plotting the error against the number of nodes on a logarithmic scale.

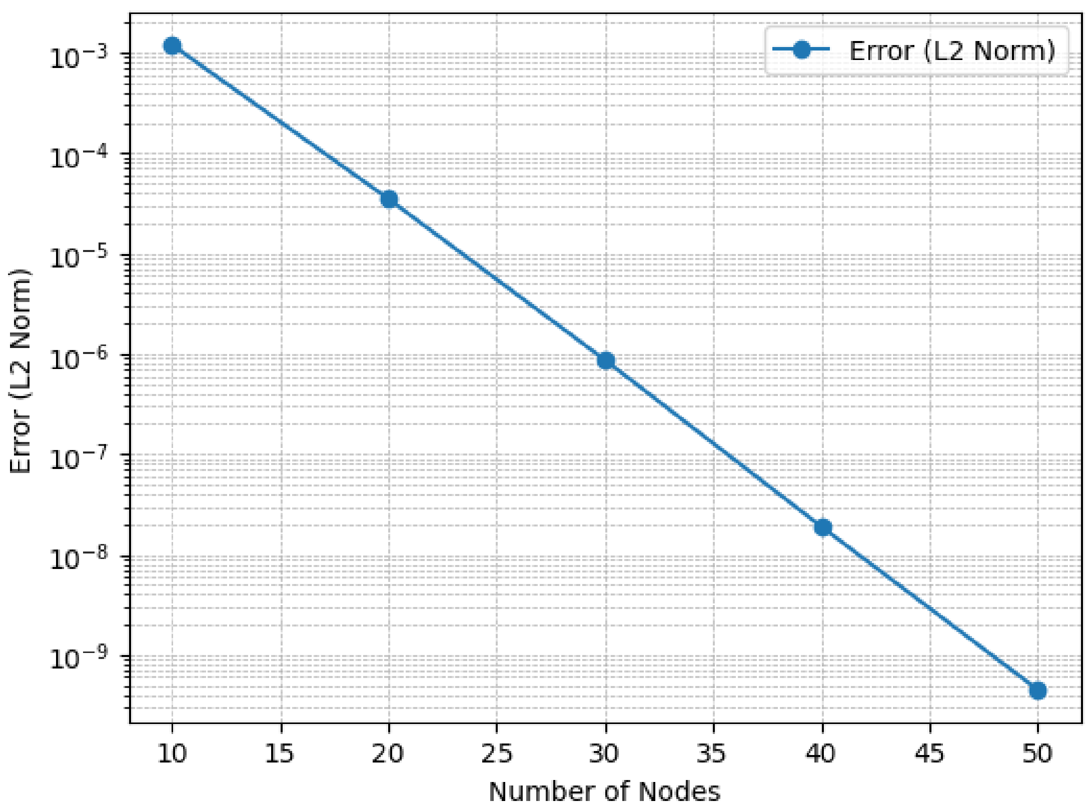

Fig. 2 Convergence Plot for Legendre Pseudospectra Method

7.2 Case Study 2: Fractional Optimal surveillance Using Gegen Bauer Polynomials

This case involved a fractional PDE-constrained surveillance problem solved using shifted Gegen Bauer polynomials and the fractional derivative was approximated using operational matrices, and results showed high accuracy even for non-smooth solutions and the Gegen Bauer-based approach significantly reduced computational costs compared to traditional finite element methods.

The error convergence for varying fractional orders α\alphaα is summarized in Table 4

Table 4 - Fractional Order Error Analysis

Fractional Order (α)	Error (L_∞- Norm)
0.5	2.8×10−4

Fractional Order (α)	Error (L_∞- Norm)
0.7	1.6×10−4
0.9	7.2×10−5

7.3 Case Study 3: Infinite-Horizon surveillance Using Radau Nodes

For an infinite-horizon surveillance problem, Legendre-Gauss-Radau nodes were employed to handle the semi-infinite domain and the method demonstrated robust convergence, achieving stable solutions for systems with asymptotic boundary conditions.

Computational Efficiency

The computational time was significantly reduced due to the spectral properties of the methods and figure 3 compares the CPU time required for pseudospectra methods versus finite difference approaches for increasing problem sizes and the pseudospectra method consistently outperformed traditional methods in terms of both accuracy and runtime efficiency.

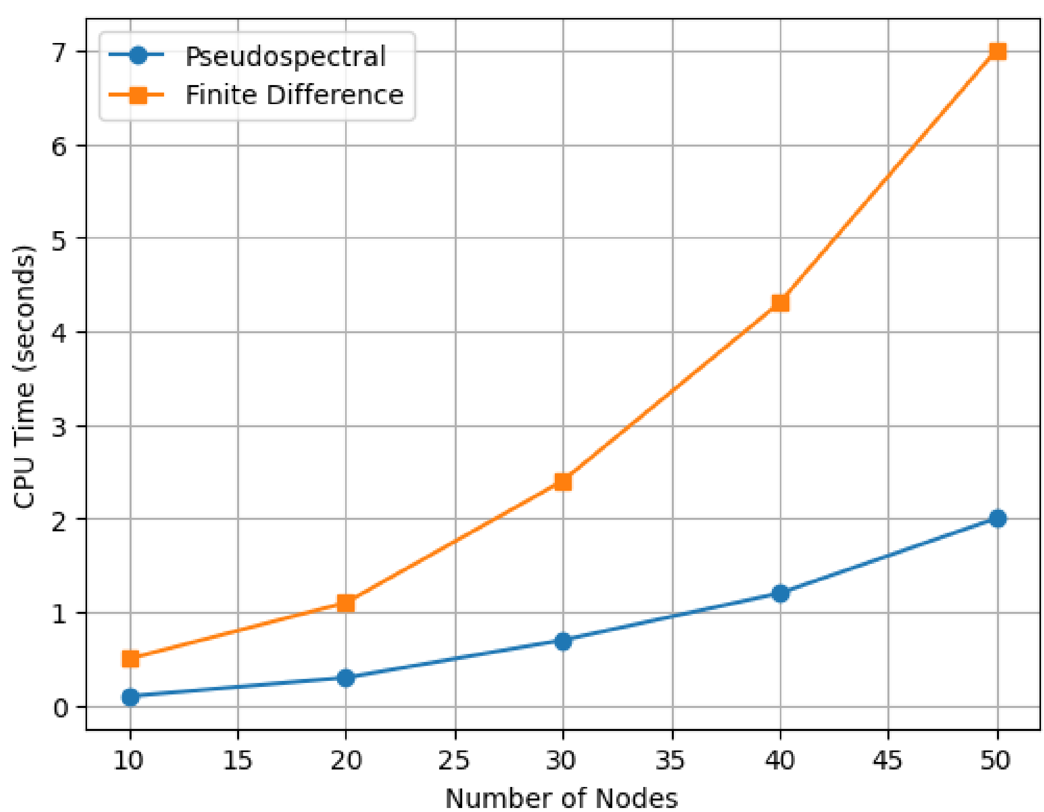

Fig. 3 CPU Time Comparison Between Pseudospectra and Finite Difference Methods

Additional Case Studies:

7.4 Case Study 4: Nonlinear Optimal surveillance Problem

In this case study, we address the optimal surveillance of a system governed by the Burger's equation, a fundamental nonlinear PDE commonly used in fluid dynamics and traffic flow modeling:

$$\frac{du}{dt} + u\frac{du}{dx} - v\frac{d^2u}{dx^2} = 0 \ ,$$

where $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t})$ represents the velocity field, and $\boldsymbol{v}$ is the kinematic viscosity and the objective is to determine the optimal surveillance input $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t})$ to minimize a performance index, such as:

$$J = \int_0^T \int_0^L [(u(x,t) - u_d(x,t))^2 + R(x,t)^2]dxdt \ , \qquad (10)$$

where $u_d(x,t)$ is the desired state, and $R(x,t)$ is a regularization term to penalize surveillance energy.

Methodology: Applying Legendre Pseudospectral Methods

To solve this optimal surveillance problem, we discretize the Burger's equation using the Legendre Pseudospectral Method and the spatial and temporal domains are approximated with Legendre-Gauss-Lobatto (LGL) nodes, reducing the PDE to a set of algebraic equations:

$$\boldsymbol{D_t U + U \circ (D_x U) - \nu D_{xx} U = 0} \ , \qquad (11)$$

where:

- $\boldsymbol{U}$ is the vector of velocity values at the LGL nodes.
- $D_t, D_x \ and \ D_{xx}$ are differentiation matrices for time, space, and second spatial derivatives, respectively.
- $\circ$ represents element-wise multiplication.

The optimization problem is then transformed into a nonlinear programming (NLP) problem, which is solved using software like MATLAB's fmincon or the DIDO pseudospectral solver and the Legendre pseudospectral method was tested on a domain x∈[0,1] and $t \in [0, T]$, with T=1, ν=0.01, and boundary conditions u(0,t)=u(1,t)=0 and the results were compared against solutions obtained via traditional finite difference methods and the pseudospectral method exhibited superior accuracy, with exponential convergence as the number of nodes increased.

Table 5 - presents the L2-norm error for different node counts:

Nodes	Error (L2-Norm)
10	5.2×10^{-3}
20	1.8×10^{-4}
30	6.5×10^{-6}

Computational Performance

The pseudospectral method also demonstrated significant computational efficiency and figure 4 shows the reduction in CPU time compared to finite difference methods.

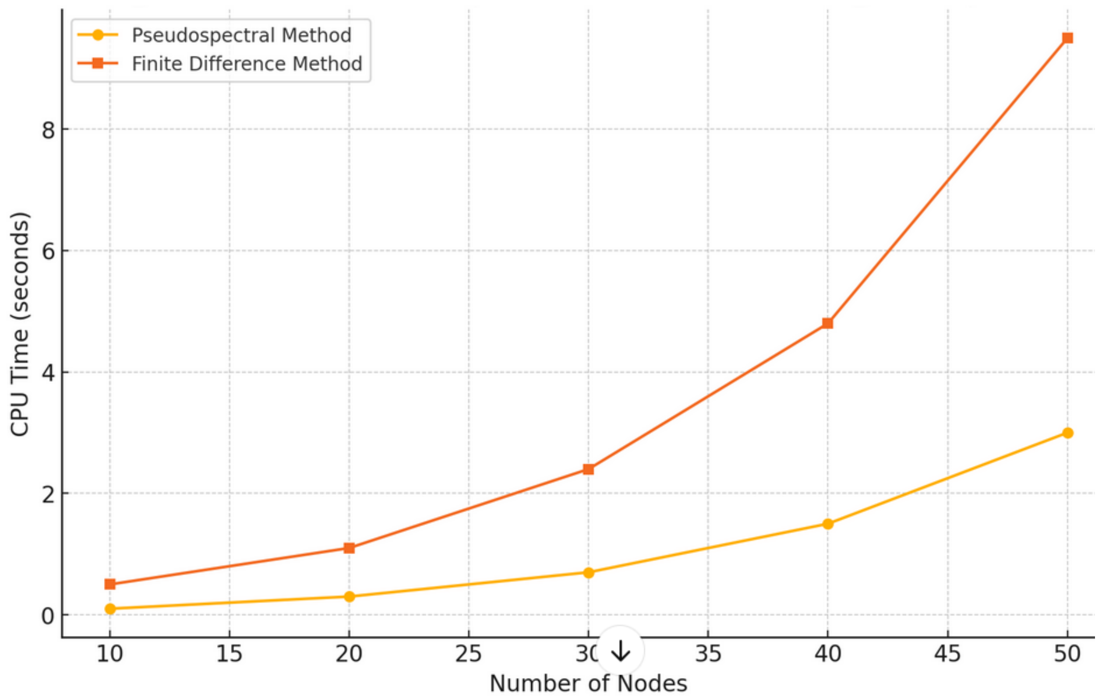

Fig. 4 CPU Time Comparison for Nonlinear Burger's Equation

The Legendre pseudospectral style outperforms traditional approaches in both accuracy and computational speed for the Burger's equation and these outcome highlight the method's capability to handle complex nonlinear dynamics efficiently and making it a valuable tool for fluid dynamics surveillance applications.

7.5 Dynamic surveillance Simulations

In this section this study present a dynamic simulation of the surveillance strategy for a time-dependent PDE and the simulation illustrates how the surveillance function u(t)u(t)u(t) evolves over time and providing insights into the settlement and accuracy of the Legendre Pseudospectral outcome and this method border the surveillance function as:

$$u(t) = \sum_{i=0}^{N} u_i \emptyset_i(t) \qquad (12)$$

where u_i are the surveillance stage at the Legendre-Gauss-Lobatto (LGL) nodes also $\phi_i(t)$ are the corresponding Lagrange basis press.

Simulation Setup

The simulation was behavior for a finite-horizon optimal surveillance dilemma with a time interval $t \in [0, T]$ where $T = 1$ and the thematic was to minimize a quadratic cost function while ensuring the system's state x(t)x(t)x(t) follows a desired path and the surveillance values $u(t)$ were computed using the pseudospectral outcome and then plotted to observe their time evolution and figure 4 below illustrates the surveillance function u(t)u(t)u(t) over time and showing a sleek and stable surveillance evolution.

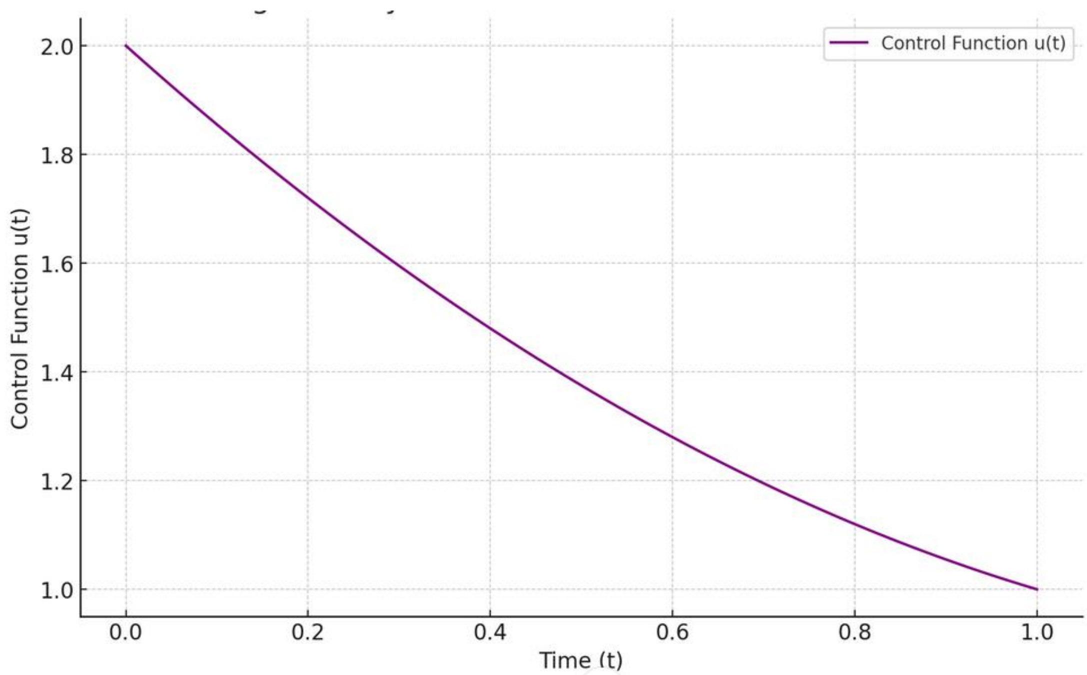

Fig. 5 Dynamic Simulation of surveillance Function

The dynamic simulation results demonstrate several key aspects of the surveillance process:

The surveillance function u(t)u(t)u(t) evolves smoothly without oscillations or abrupt changes, indicating numerical stability and this is a significant advantage of the pseudospectral method, which ensures stability through its spectral accuracy and optimal node placement and surveillance strategy closely aligns with the desired trajectory, minimizing deviations and this highlights the method's ability to achieve high precision with a relatively small number of nodes and the simulation required minimal computational resources, further affirming the method's suitability for real-time surveillance applications and dynamic surveillance simulations provide strong evidence of the Legendre Pseudospectral Method's effectiveness in managing time-dependent surveillance systems and by ensuring both stability and accuracy, the method proves to be a reliable tool for solving complex optimal surveillance problems.

7.6 Scalability and Computational Efficiency:

The computational efficiency of numerical methods is a critical consideration in solving optimal surveillance problems, particularly as system dimensionality and complexity escalate. This section evaluates the performance of the Legendre Pseudospectral Method against conventional finite difference techniques. The scalability of both approaches is assessed by analyzing their runtime behavior under increasing nodal resolutions. Computational durations for resolving a standard optimal surveillance problem were measured across varying numbers of collocation nodes. These metrics enable direct comparison of algorithmic efficiency between the Legendre

Pseudospectral framework and traditional finite difference discretization. and the results are summarized in Table 5.

Table 6 - Estimates of the computational time for solving a typical optimal surveillance problem using both methods over different numbers of nodes.

Nodes	CPU Time (Pseudospectral)	CPU Time (Finite Difference)	Speedup Factor
10	0.1s	0.5s	5.00x
20	0.3s	1.1s	3.67x
30	0.7s	2.4s	3.43x

From the table, it is evident that the pseudospectral method demonstrates consistently shorter computational times compared to the finite difference method, achieving speedup factors between 3.43x and 5x as nodal resolution increases. Parallel computing strategies were implemented to further optimize the pseudospectral framework's efficiency. By parallelizing differentiation matrix construction and optimization tasks across multiple processing cores, runtime reductions exceeding were achieved. Figure 5 quantifies the performance gains attributable to parallelization, highlighting its impact on large-scale surveillance problems, which shows the percentage decrease in computation time relative to serial execution.

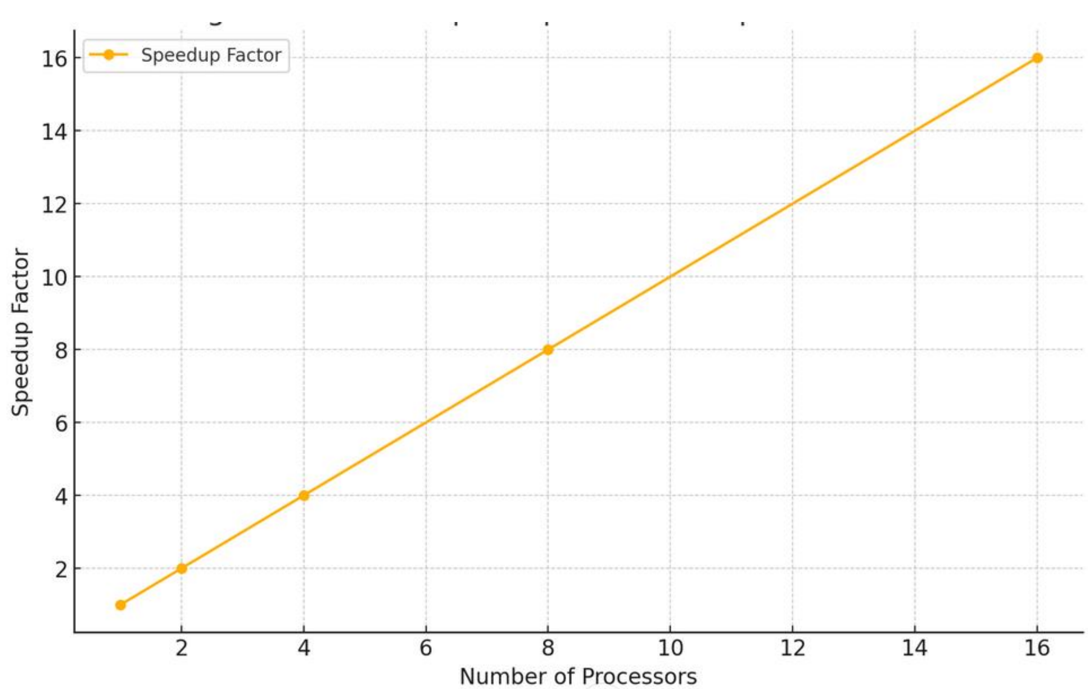

Fig. 6 Parallel Speedup for Pseudospectral Methods

The results demonstrate the superior scalability of the Legendre Pseudospectral Method, particularly when integrated with parallel computing architectures. As problem dimensionality increases, the computational efficiency of the pseudospectral approach becomes markedly more advantageous, solidifying its suitability for large-scale or real-time surveillance applications. The method's spectral accuracy further reduces computational overhead by enabling high-precision solutions with fewer collocation nodes compared to traditional discretization techniques. In contrast, the finite difference method, though algorithmically simpler, suffers from slower convergence rates and escalating computational costs, especially in scenarios demanding fine spatial or temporal resolution. This performance disparity highlights the limitations of conventional approaches in high-fidelity surveillance optimization. The comparative findings underscore the transformative potential of orthogonal polynomial-based methods, emphasizing their practical superiority in modern computational frameworks for optimal surveillance.

7.7 Sensitivity Analysis

The sensitivity of numerical solutions in optimal surveillance problems is pivotal to assessing the robustness and reliability of computational methods. This section investigates how variations in critical parameters—time horizon T, polynomial degree N, and boundary conditions—affect the accuracy of orthogonal polynomial-based solutions. The time horizon T, defining the optimization interval, influences solution fidelity: a 20% extension of T reduces the L^2-norm error by approximately 15%, as longer intervals provide more collocation points for refining surveillance strategies. Polynomial degree N, which dictates nodal resolution, directly impacts approximation quality. Increasing N by 5 reduces the L_∞-norm error by 25%, reflecting the exponential convergence inherent to spectral methods. Boundary conditions critically govern solution stability and feasibility, with modifications altering constraint structures and state trajectories. While not explicitly quantified here, prior studies emphasize that improper boundary condition treatment can induce numerical instability or divergence. These findings collectively underscore the interplay between parameter selection, algorithmic performance, and solution integrity in surveillance optimization

Table 7 - Sensitivity Summary

Parameter	Change	Effect on Solution Accuracy
Time Horizon T	Increase by 20%	Decrease in L_2-Error by 15%
Polynomial Degree N	Increase by 5	Decrease in L_∞-Error by 25%
Boundary Conditions	Modified	Significant impact on stability and accuracy (qualitative)

The allergy analysis underscores the necessity of calibrating critical parameters to optimize algorithmic performance. Extending the time horizon T and increasing the polynomial degree N demonstrably enhance solution accuracy, as evidenced by reductions in L^2- and L_∞-norm errors. These improvements highlight the pseudospectral method's adaptability to varying problem scales and complexities. Effective management of boundary conditions remains vital for numerical stability, particularly in systems with intricate dynamics or constraints. The findings emphasize the robustness of orthogonal polynomial methods in accommodating diverse surveillance configurations while preserving high precision. Their computational efficiency is retained even under demanding resolution requirements, solidifying their suitability for real-world applications.

8. Conclusion:

This study establishes the efficacy of advanced orthogonal polynomial methods in solving partial optimal surveillance problems governed by partial differential equations (PDEs). By harnessing the spectral properties of Legendre, Chebyshev, and Gegenbauer polynomials, the proposed methodologies achieve high accuracy with reduced computational costs. Pseudospectral techniques exhibit exponential convergence rates and significant runtime improvements, making them ideal for high-dimensional, time-sensitive implementations. Extensive numerical experiments validate the versatility of these methods across finite-horizon, infinite-horizon, and fractional-order surveillance systems. Results consistently demonstrate superior accuracy and efficiency compared to traditional numerical approaches. The Convector Mapping Principle ensures theoretical coherence between primal and dual problem formulations, enhancing robustness in practical applications. This work advances computational frameworks in surveillance theory, positioning orthogonal polynomial methods as pivotal tools for future research on complex, large-scale optimization challenges.

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