Continuous BA- finitely set function

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Abstract

Let μ be an BA- finitly additive set function on a measurable space (Ω, F). In this paper, we proved the following statement

(1) If μ is BA- countably additive, then μ is continuous at A for all $A \in F$.

(2) If μ is - continuous from below at every set $A \in F$, then μ is BA- countably additive .

(3) If μ is , finitly and continuous from above at ϕ, then μ is BA- countably additive.

1.preliminaries

[1] introduced the notations of Banach algebra valued- measure.

Definition (1.1)

Let X be a real vector space. A partial order relation $\leq$ on X is called vector order if the following axioms are satisfied

(1) $x \leq y \quad \Rightarrow \quad x + z \leq y + z \qquad \forall x, y, z \in X$

(2) $x \leq y \quad \Rightarrow \quad \lambda x \leq \lambda y \qquad \forall x, y \in X \quad and \quad \lambda \geq 0$

A real vector space endowed with a vector order is called an **ordered vector** space. An element x of an ordered vector space X is said to be **positive** if $x \geq 0$, and negative if $x \leq 0$. The set of all positive elements of an ordered vector space X will be denoted by X_+, i.e. $X_+ = \{x \in X : x \geq 0\}$, X_+ is called the **positive cone** of X. It is easy to show that

(1) X_+ is a convex cone of X, i.e. $X_+ + X_+ \subseteq X_+$ and $\lambda X_+ \subseteq X_+$

(2) $X_+ \cap (-X_+) = \{0\}$

Definition (1.2)

Let X be a real vector space. A function $\| . \| : X \to R$ is said to be norm on X if the following axioms are satisfied

(1) $\|x\| \geq 0 \qquad \forall x \in X$

(2) $\|x\| = 0 \quad iff \quad x = 0$

(3) $\|\lambda x\| = |\lambda| \|x\| \qquad \forall x \in X, \quad \lambda \in \Re$

(4) $\| x + y\| \leq \|x\| + \|y\| \qquad \forall x, y \in X$

A normed space is the pair $(X, \|\cdot\|)$ where X is a real vector space and $\|\cdot\|$ a norm on X.

A Banach space is a normed space which is complete in the metric defined is by its norm.

Definition (1.3)

An algebra is a vector space in which a multiplication is defined that satisfies

(1) $x(yz)=(xy)z \quad , \quad \forall\, x,y,z \in X$

(2) $x(y+z)=xy+xz \quad , \quad (x+y)z=xz+yz \quad , \quad \forall\, x,y,z \in X$

(3) $\lambda(xy)=(\lambda x)y=x(\lambda y) \quad , \quad \forall\, x,y,z \in X \ , \quad \lambda \in R$

Definition (1.4)

A real vector space X is called Banach algebra if the following axioms are satisfied

(1) X is a Banach space

(2) X is algebra

(3) $\exists e \in X$ s.t $ex=xe=x \quad \forall x \in X$ and $\|e\|=1$

(4) $\|xy\| \le \|x\|\,\|y\| \qquad \forall x,y \in X$

A commutative algebra is an algebra where the multiplication satisfies the condition: $xy=yx\,, \quad \forall\, x,y \in X$

An algebra with identity is an algebra with the following property. There exists a non – zero element in the algebra, denoted by 1 and called the multiplication identity element, such that $1 \cdot x = x \cdot 1 = x,$ for all x.

A normed algebra X , is a normed space, also an algebra over F and $\|xy\| \le \|x\|\,\|y\|$ $\forall\, x,y \in X$.

Example (1.5)

Let X be a Banach space, and let $B(X)$ denote the set of all bounded (or continuous) linear function of X into itself then $B(X)$ is a Banach algebra with the algebra operation

(1) $(f+g)(x)=f(x)+g(x)$

(2) $(fg)(x)=f(g(x))$

(3) $(\lambda f)(x)=\lambda f(x)$

and the operator norm $\| f \| = \sup\{\| f(x)\| : \quad x \in X, \| x \| \le 1\}$

Remark

If $X \neq \{0\}$, then the identity linear function I is the identity element of $B(X)$ such that $\| \mathrm{I} \| = 1$.

2. The main results

Definition (2.1)

Let (Ω, F) be a measurable space , and let X be an ordered Banach algebra. A set function $\mu : F \longrightarrow X$ is said to be

(1) BA- finitely additive if $\mu(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \mu(A_k)$, whenever $A_1, A_2, \cdots, A_n$ disjoint sets in F .

(2) BA-countably additive if $\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$, wherever $\{A_k\}$ is a sequence of disjoint sets in F .

(3) BA- measure if μ is BA- countably additive and $\mu(A) \geq 0 \quad \forall\ A \in F$

Definition (2.2)

Let μ be an BA- finitly additive set function on a measurable space (Ω, F), for all $A \in F$. We say that μ is

(1) Continuous from below at $A \in F$, if $\mu(A_n) \longrightarrow \mu(A)$ wherever $\{A_n\}$ is a sequence of sets in F with $A_n \uparrow A$

(2) Continuous from above at $A \in F$, if $\mu(A_n) \longrightarrow \mu(A)$ wherever $\{A_n\}$ is a sequence of sets in F with $A_n \downarrow A$

(3) Continuous at $A \in F$, if it is continuous both from below and from above at A .

Theorem (2.3)

Let μ be an BA- finitly additive set function on a measurable space (Ω, F), for all $A \in F$.

(1) If μ is BA- countably additive, then μ is continuous at A for all $A \in F$.

(2) If μ is - continuous from below at every set $A \in F$, then μ is BA- countably additive .

(3) If μ is , finitly and continuous from above at ϕ , then μ is BA- countably additive.

Proof

Let $\{A_n\}$ be a sequence of sets in F with $A_n \uparrow A$

$\Rightarrow$ $\{A_n\}$ is an increasing sequence and $\bigcup_{n=1}^{\infty} A_n = A$

$A = A_1 \cup (\bigcup_{n=1}^{\infty} (A_{n+1} \mid A_n))$ is a disjoint decomposition

and

$$\mu(A) = \mu(A_1) + \sum_{n=1}^{\infty} \mu(A_{n+1} \mid A_n) = \mu(A_1) + \lim_{k\to\infty} \sum_{n=1}^{\infty} \mu(A_{n+1} \mid A_n) = \lim_{k\to\infty} \mu(A_n)$$

Since μ is BA - finitly additive on F . Thus μ is continuous from below at A

Now suppose $\{A_n\}$ be decreases sequence and $\bigcap_{n=1}^{\infty} A_n = A$

Put $B_n = A_k \mid A_n$ for $n \geq k$

$\Rightarrow$ $\{B_n\}$ is increasing to $A_k \mid A$. Hence, as $n \longrightarrow \infty$, we have

$$\mu(B_n) \longrightarrow \mu(A_k \mid A) = \mu(A_k) - \mu(A)$$

But

$\mu(B_n) = \mu(A_k) \longrightarrow \mu(A_n)$ so that $\mu(A_n) \longrightarrow \mu(A)$ as $n \longrightarrow \infty$ since $\mu(A_n)$ is finite , and μ is also continuous from above at A.

(2)

Suppose $A \in F$, $A_n \in F$, $n = 1,2,\cdots$ are such that $A = \bigcup_{n=1}^{\infty} A_n$ and the sets A_n are disjoint. Put $B_n = \bigcup_{i=1}^{n} A_i \in F$, $n = 1,2,\cdots$, and $\{B_n\}$ is a monotone sequence of sets in F which converges to $A \in F$. If μ is continuous from below at A.

$$\sum_{i=1}^{n} \mu(A_i) = \mu(B_n) \to \mu(A) \text{ as } n \to \infty$$

So that again $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$ and μ is BA-countably additive.

(3)

In the notation of (2), put $G_n = A - B_n \in F$, $n = 1,2,\cdots$. Then $\{G_n\}$ is a monotone decreasing sequence converging to ϕ and, for $n = 1,2,\cdots$

$$\mu(A) = \sum_{i=1}^{n} \mu(A_i) + \mu(G_n)$$

If μ is finite and continuous from above at ϕ we must have $\mu(G_n) \to 0$ as $n \to \infty$.

So that again $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$.

References

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