On Equivalent Martingale Measures on L^p-space

By

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Abstract

We apply the no-arbitrage and no-free lunch definitions of Kreps (1981). Arbitrage is a linear algebraic notion, while free lunch is a topological notion. The notion of free lunch, unlike that of arbitrage, is somewhat ambiguous because it depends on a topological choice that is often tacit, i.e. there is source of problem when we use the topology in our work.

Our main point in this paper is that how the choice of a topology which the space are free lunches. We consider a number of topological spaces depending on:

(1) not a free lunch. (does not converge).

(2) a free lunch (converge to a positive limit).

(3) not a free lunch (converge to a non-positive limit).

1.Introducation

In recent years the mathematical theory of stochastic integration (stochastic process) has become of interest because of its several areas of mathematical stochastic integrals with respect to martingales which were first discussed by Winer (1923). The extension of this definition is due of square martingale.

The relation between the theorem of price asset and arbitrage were introduced from (Arrow-Debreu)(1959) model, formula of Black and Scholes (1973), linear price model of Cox and Ross (1976).

In their fundamental paper, Harrison and Kreps (1979), discussed the fundamental theorem and introduced the concepts of equivalent martingale measure. Absence of arbitrage alone was not sufficient to obtain an equivalent martingale measure for the stochastic process. . Different solutions have been introduced to relate the topological conditions of arbitrage [see Back and pliska (1991), Dalank, Morton.

The triple (Ω,F,P) is called probability space, where Ω be a non –empty set ,F *is a* σ*-field on* Ω, and P is a probability measure. i.e P(Ω)=1 where P be a measure on (Ω,F) ,the process X, Sometimes denoted $(X_t)_{t\in I}$ is supposed to be $\Re$ -valued,
i.e the function X:$\Omega\rightarrow\Re$ is called **random variable** if X is measurable function. This mean the function X: $\Omega\rightarrow\Re$ is random variable if and only if $\{X\leq a\} \in F$ for all $a\in\Re$.

2. L^p-spaces

We recall all definitions and concepts which we need in this paper. Also, we recall if (Ω,F,P) is a probability space and p is a real number with $p \geq 1$, the space of all real-valued measurable functions f such that $|f|^p$ is P-integrable has many important properties. In order to fully develop these properties, it will introduced all definitions in this subject.

Definition (2.1), [6]:

Let X,Y be topological spaces and let F(X,Y) be the set of all mappings from X into Y. Let μ be set of all finite subsets of X,τ_μ is called of **simple convergence** (or pointwise convergence) on X. If μ is the set of all bounded subsets of X, τ_μ is called **the topology of bounded convergence**. If μ consist of all compact subsets of X,τ_μ is called **topology of compact convergence**.

Definition (2.2), [6]: A subset E of X is called **weakly compact** if every sequence in E contains a subsequence which is weakly convergent in E.

Definition (2.3), [6]: A set A is called balance set if $\lambda A \subset A$ for all real number λ with $|\lambda| \leq 1$

Definition (2.4), [7]:

The uniform convergence topology on the set of all weakly compact, convex and balanced subsets μ of X is called **Mackey topology**.

Definition (2.5), [3]:

Let (Ω, F, μ) be a measure space and let p be a real number with $(1 \leq p < \infty)$. We define $L^p(\Omega, F, \mu)$ as follows:

$L^p = L^p(\Omega, F, \mu) = \{f : f : \Omega \to \Re$ is μ-measurable, $\|f\|_p < \infty\}$.

Where

$$\|f\|_p = \left[\int_\Omega |f|^p \, d\mu \right]^{\frac{1}{p}}$$

for all $f \in L^p$.

If $\|f\|_p < \infty$ for some p with $(1 \leq p < \infty)$, then f is said to be p-integrable.

Definition (2.6), [6]

If p and q are positive real numbers, such that $\frac{1}{p} + \frac{1}{q} = 1$, p and q are called a **pair of conjugate exponents**.

Theorem (2.7) (Hőlder's Inequality), [6]

Let p and q be conjugate exponents. If $f, q \in L^p$ then $fg \in L^p$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$ i.e.

$$\int_\Omega |fg| \, d\mu \leq \left[\int_\Omega |f|^p \, d\mu \right]^{\frac{1}{p}} \left[\int_\Omega |g|^q \, d\mu \right]^{\frac{1}{q}}$$

Theorem (2.8) "(Minkowski's Inquality), [6]

If $f, g \in L^p$ $(1 \leq p < \infty)$, then $f+g \in L^p$ and

$$||f+g||_p \le ||f||_p+||g||_p$$

i.e.

$$\left[\int_\Omega |f+g|^p \, d\mu\right]^{\frac{1}{p}} \le \left[\int_\Omega |f|^p \, d\mu\right]^{\frac{1}{p}} + \left[\int_\Omega |g|^p \, d\mu\right]^{\frac{1}{p}}$$

by Minkowski's inequality and the fact $||\lambda f||_p=|\lambda| \, ||f||_p$ for all $f\in L^p$, L^p $(1 \le p<\infty)$ is a vector space over the real field.

If a statement holds for every $t\in\Omega$ except for a subset of measure zero, then it is said to hold almost every where (a.e.) on Ω. The space X is usually L^p $(\Omega,\mathcal{F}, P)$, for $1 \le p<\infty$.

Now, we construct L^∞ as follows:

firstly, we define the essential supermum of real-valued measurable function g on $(\Omega,\mathcal{F},\mu)$ as:

$$\textbf{ess sup } g=\inf \{\alpha\in\Re^* : \mu\{\omega : g(\omega) >\alpha\}=0\} \qquad [6]$$

that is, the smallest number α such that $g\le \alpha$ a.e.[μ].

If f is real-valued measurable function on $(\Omega,\mathcal{F},\mu)$, we define

$$||f||_\infty=\textbf{ess sup } |f|$$

i.e. $||f||_\infty=\inf \{\alpha\in\Re^*: \mu\{\omega: |f(\omega)|>\alpha\}=0\}$

$=\inf \{\alpha\in\Re^*: |f(\omega)|\le\alpha$ for μ-almost every $\omega\in\Omega\}$

$L^\infty(\Omega,F,\mu)=\{f: f:\Omega\to\Re$, f is μ-measurable and $||f||_\infty<\infty\}$

$=\{f: f:\Omega\to\Re$, f is measurable and there exists a finite constant α such that $|f(\omega)| \le \alpha$ for μ-a.e. $\omega\in\Omega\}$.

Thus $f\in L^\infty$, f is essential bounded, that is, bounded outside a set of measure 0 [i.e., there is a positive real α such that $|f(\omega)| \le \alpha$ for μ-a.e. $\omega\in\Omega$].

Now, we have

$$||f+g||_\infty \le ||f||_\infty+||g||_\infty$$

In particular $f,g\in L^\infty$ implies $f+g\in L^\infty$.

Thus L^∞ is a vector space over the real field. $||.||_\infty$ is a semi norm on L^∞.

We now wish to give metric structures to the space of all p-integrable functions and to the spaces of all essentially bounded measurable functions on Ω. Minkowski's inequality shows that if we let $d_p(f,g)=||f+g||_p$ with $(1 \le p \le\infty)$, then d_p satisfies all the properties of a metric except that $d_p(f,g)=0$.

$\Rightarrow$ f=g at all point of Ω.

We can, in effect change $||.||_\infty$ into norm by passing to equivalence classes as follows:

If $f,g\in L^p$, define $f\sim g$ if and only if $\mu\{t\in\Omega : f(t)\neq g(t)\}=0$. It is easily seen that if f is a non-negative measurable function on L^p then $f\sim 0$ if and only if $\int_\Omega f d\mu =0$.If we identify two p-integrable or essentially bounded functions which belong to the same equivalence class, then d_p is indeed a metric on these spaces of functions.

The set of all equivalence classes of p-integrable (resp., essentially bounded) function is denoted by $L^p(\Omega,F,\mu)$ (resp. $L^\infty(\Omega,F,\mu)$), that is L^p $(1 \le p<\infty)$ is a normed space. Hence, L^p is locally convex since any normed space is locally convex.

3. Free lunch in τp-norm topology

Let (Ω,F,P) be a probability Lebesgue measure when $\Omega=[0,1]$, $F=\beta([0,1])$, and P is a probability space, and let $X=L^p(\Omega,F,P)$. It is clear L^p $(1 \le p <\infty)$ is topological space, in which case τ is usually either

(1) τ_p, the L^p norm topology.

(2) τ_m, the topology of convergence in P-measure.

Let $M_0=\{ \{\delta_\lambda\}_{\lambda\in(0,\lambda]}\cup B \}$ where $B:\Omega\rightarrow\Re$ a function by $B(\omega)=1$ for all $\omega\in\Omega$, and $\delta_\lambda:\Omega\rightarrow\Re$ defined by $\delta_\lambda=I_\lambda$,

i.e.

$$\delta_\lambda=\begin{cases} 1 , & \omega=\lambda \\ \end{cases}$$

$-\ 0 \quad , \quad \omega\neq\lambda$

It is clear to show that the the indicter function δ_λ is a measurable function for all $\lambda\in(0, 1]$,and B is a measurable function.

Define $\pi_0:M_0\to\Re$ by:

$$\pi_0(m_0)=\int_\Omega m_0(w)dp(w) \quad \text{for all } m_0\in M_0.$$

$$\pi_0(B)=\int_\Omega B(w)dp(w)=\int_\Omega 1\,dp=P(\Omega)=1.$$

Let K=positive part of X

i.e. $K=X_+ \mid \{0\}$ and $M=\text{span}\{M_0\}$.

Now,

define $\pi:M\to\Re$ as follows:

$$\pi(m)=\sum_{i=1}^{m}\lambda_i\pi_0(m_i)$$

for all $m=\lambda_i m_i\in M$ where $m_i\in M_0$ for all $i=1,2,\ldots,m$.

Then by definition, $m\in M$ if and only if:

$$m=\alpha_0B+\sum_{i=1}^{n}\alpha_i\delta_i \quad \text{for some } n\geq 0.\text{Then}$$

$$m(\omega)=\begin{cases}\alpha_0+\alpha_\omega & \text{if } \omega\leq n\\ \alpha_0 & \text{if } \omega>n\end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\pi(m)=\int\alpha_0\pi_0(B)dP(\omega)+\sum_{i=1}^{n}\alpha_i\int\delta_i dP(\omega)$$

$$=\alpha_0+\sum_{i=1}^{n}\alpha_i\Gamma_i \qquad \text{where } \Gamma_i=\int\delta_i dP(\omega)$$

$$= \sum_{i=1}^{n} (\alpha_0 + \alpha_i) + \alpha_0(1- \sum_{i=1}^{n} \Gamma_i) \text{ ....}(2)$$

$$\pi(m)=E[my] + \alpha_0(1- \sum_{i=1}^{n} \Gamma_i) \quad \text{.................................... } (3)$$

where $\quad y(\omega)=d\Gamma_\omega/dp_\omega$

$$E[my]= \int m(\omega)y(\omega)dp_\omega = \int m(\omega)d\Gamma_\omega$$

$$E[y] = \Gamma_\omega \quad \forall\ \omega\in\Omega.$$

If $m\in K$

for all $\omega\in\Omega$, $m(\omega) \geq 0$. That is

$\alpha_0 \geq 0$ and $\alpha_0+\alpha_\omega \geq 0$ for all $\omega \leq n$.

By definition, there are no arbitrage opportunities whenever $\pi(m) > 0$ for all m. From (2) we see this condition holds if and only if:

$$\delta_i > 0 \ \forall\ i,\ E[y] \leq 1 \quad \text{....................................}(4)$$

Theorem (3.1):

The price system (M,π) in (L^p,τ_p) $(1 \leq p < \infty)$ is no-free lunch.

Proof:

Next let $(X,\tau)=(L^p,\tau_p)$ where $1\leq p<\infty$, in which case $X^*=L^q$, where $q=p/(p-1)$.

In such spaces, every price system $\psi\in\Psi$ has a Riesz Representation z such that:

$$\psi(x)= \int_\Omega x(\omega)z(\omega)\,dP(\omega)$$

$$=E[xz] \quad \text{....................................}(5)$$

Where $\forall\ \omega, z(\omega)>0$ and $z\in X^*$.

Comparing (3) with (5), we see that if ψ extends π, then we must have:

(a) $z=y$,

(b) $E[y] =1$, and

(c) $y=\dfrac{d\Gamma}{dP} \in X^*$.

The latter condition means that (for p>1)

$$(\|y\|_q)^q = \int_\Omega |y(\omega)|^q \, dP(\omega)$$

$$= \int_\Omega \left|\frac{d\Gamma_\omega}{dP_\omega}\right|^q dP_\omega < \infty.$$

For p=1, $y \in X^*$ requires that y be bounded.

If π cannot be extended, there are free lunches; this happens in two cases:

first, extension fails whenever E[y] <1, in which case have $\psi(B) \neq \pi_0(B) = 1$.

Consider

$$z_n = \frac{1}{\Gamma}\int \delta_i(\omega) \, dP(\omega) - B \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6)}$$

Define

$$z_\infty = (1/E[y]-1)B$$

and that for all $n, z_n \in M$,

$$\pi(z_n)=0, \text{ and } \lim_{n\to\infty} (\|z_n - z_\infty\|)^p = \int_\Omega \left|\frac{1}{\Gamma}\int \delta_i(\omega)\, dP(\omega) - 1/E[y]B\right|^p dP_\omega$$

$$= \int_\Omega \left| \frac{1}{\Gamma} - (1/\Gamma)B \right|^p dP_\omega$$

therefore $z_n \to z_\infty > 0$.

Second, assuming E[y]=1, extension fails (and there are free lunches) whenever $y \notin X^*$.

when this happens, there exists an element $\bar{x}$ such that $E[\bar{x}\, y]=\infty$.Define $x_n = \sum \bar{x}\delta_i \in M$, and consider the sequence $z_n \in M$, where:

$$z_n = B - \frac{x_n}{E[x_n y]}.$$

Then, $\pi(z_n) = 0$ for all $n \in N$, and

$$\lim_{n\to\infty} \|z_n - B\|_p = \lim_{n\to\infty} \frac{\|x_n\|}{E[x_n y]} = 0.$$

Therefore $z_n \to B > 0$, which shows that $\{z_n\}$ is free lunch. ●

Theorem (3.2):

Let $X=L^{\infty}$ with the Mackey topology τ_{μ} The price system (M,π) in (L^{∞},τ_{μ}) is free lunch.

Proof:

The representation for $\psi(x)$ is given by (5).

Since $y\in L^1$, ψ (as represented by y) extends π to all of X provided only that $E[y]=1$. If this condition is not satisfied there are free lunches, an example being provided by the sequence in (6).

A more general representation does exist: for $x\in L^{\infty}$, we have

$\psi(x)=E[xy]+\int x d\Phi$ .Where

$$\int m d\Phi=\alpha_0(1-E[y]) \text{ for all } m\in M. \qquad \bullet$$

Theorem (3.3):

Let $(X,\tau)=(L^p(P),\tau_p)$, $(1\leq p\leq\infty)$. Assume $y(\omega)>0$ for all $\omega\in\Omega$ and $E[y]=1$. Then there exists equivalent measure on L^p-space.

Proof:

Let Q: $\mathcal{F}\rightarrow\Re$ by

$$Q[A]=\int y(\omega)dP(\omega) \quad \text{for all } \omega\in\Omega, \text{ for all } A\in\mathcal{F}.$$

(1) $Q(A)\geq 0$ since $dP_{\omega}\geq 0$, $y(\omega)=d\Gamma_{\omega}/dP_{\omega}>0$

(2) $Q\{\bigcup_{i=1}^{\infty}A_i\}=\int_{\bigcup A_i} y(\omega)dP_{\omega} \qquad \omega\in\bigcup A_i$, and $\bigcap A_i=\phi$

$$=\int_{A_1} y(\omega)dP_{\omega}+\int_{A_2} y(\omega)dP_{\omega}+\ldots$$

$$=\sum_{i=1}^{n}\int_{A_i} y(\omega)dP$$

If $Q(A)=0 \Rightarrow \int_A y(\omega)dP_\omega=0 \Rightarrow P(A)=0$ since $y\neq 0$, that is Q is equivalent measure of P. To prove there exists $\psi\in\Psi$ extending to π.

Since L^p is lattice $\Rightarrow$ M is lattice.

If $m\in M \Rightarrow |m|\in M$, where

$|m|=\{x\in\Omega:m(x)\geq 0\}$.

Since Q~P

Then $E_p[xy]=\int x(\omega)y(\omega)dP_\omega$

$E_p[xy]=\int x(\omega)dQ_\omega=E_Q(x)$

Then $E_p[|m|y]=E_Q(|m|)$

Hence $\pi(|m|)=E_Q(|m|)+\alpha_0(1-E[y])$

Since $E[y]=1$

Then $\pi(|m|)=E_p[|m|y]=E_Q(|m|)$. ●

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