On paracompact Space Using Semi-Open Sets

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Abstract. This work consists of two sections. In section one we recall the definition concerning semi-open sets. In section two we recall the definition of paracompact space and we introduce similar definition by using semi open sets also we proof some results about it.

Introduction. N. Levine (1963) in [6] gives the definition of semi open set ($s-$open) and studies the properties of it. The concept of paracompact is due to Dieudonne and was studied in [8] and [10]. In this work we will give similar definition $s-$paracompact space using $s-$open sets.

Section one:

We start this section by recalling the following from [2].

Let X be a topological space then a subset A of a space X is called $s-$open iff $A \subseteq \overline{A^0}$ and A is called $s-$closed iff A^c is $s-$open. It is clear that A is $s-$closed set iff $\overset{0}{\overline{A}} \subseteq A$.

Example 1.1

It is clear that every open set is $s-$open but the converse is not true :

Let $X = \{a,b,c\}$, $\tau = \{\{a\}, X, \phi\}$. The $s-$open sets are $\{a\}$, $\{a,b\}$, $\{a,c\}$, ϕ, X

Proposition 1.2[5]

Let X be a topological space then G is open set in X iff $\overline{G \cap \overline{A}} = \overline{G \cap A}$ for each $A \subset X$.

Remark 1.3

The intersection of an $s-$open set and an open set is $s-$open.

Example 1.4

The intersection of two $s-$open sets is not $s-$open in general. In fact let $X = \{a,b,c\}$, $\tau = \{\{a\},\{b\},\{a,b\},\phi, X\}$. Then each of $\{a,c\}$, $\{b,c\}$ is an $s-$open set, where as $\{a,c\} \cap \{b,c\} = \{c\}$ is not $s-$open.

Proposition 1.5 [6]

For any subset A of a space X the following statements are equivalent :

(i) A is $s-$open set.

(ii) $\overline{A}=\overline{A^0}$

(iii) There exists an open set G such that $G\subseteq A\subseteq \overline{G}$.

Proposition 1.6 [6]

Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a collection of $s-$open sets in a topological space X, then $\bigcup_{\lambda\in\Lambda} A_\lambda$ is $s-$open.

Proposition 1.7 [1]

Let X be a topological space. Let $Y\subset X$ and A is $s-$open set in Y. Then there exists an $s-$open set B in X such that $A=B\bigcap Y$.

Proposition 1.8 [3]

For any subset A of a space X the following statements are equivalent:

(i) A is $s-$closed

(ii) $A^0=\overset{0}{\overline{A}}$

(iii) There exists a closed set F in X such that $F^0\subseteq A\subseteq F$.

Proposition 1.9

Let X be a topological space. If A is closed set in F and F is $s-$closed set in X, then A is $s-$closed set in X.

Proof.

Let A be a closed in F and F be a $s-$closed in X, then $\overline{A}^F=A$ and $\overset{0}{\overline{F}}\subseteq F$.

$\overset{0}{\overline{A}}=\overline{A\cap F}^0\subseteq(\overline{A}\cap\overline{F})^0=\overset{0}{\overline{A}}\cap\overset{0}{\overline{F}}\subseteq\overset{0}{\overline{A}}\cap F\subseteq\overline{A}\cap F=\overline{A}^F=A.$

Then A is $s-$closed set in X.■

Proposition 1.10 [9]

If X is a topological space then $\overline{A\cap B}^0=\overline{A}^0\cap\overline{B}^0$ for every open sets A,B in X.

Proposition 1.11.

If X be a topological space and let G,H be disjoint $s-$open sets in X. If $G\cup H$ is both open and closed in X, then G and H are $s-$closed sets in X.

Proof.

Since G and H are disjoint $s-$open sets in X such that $G\cup H$ is open and closed set in X, then $G\subseteq\overline{G^0}$, $H\subseteq\overline{H^0}$ and $\overline{(G\cup H)^0}=G\cup H$.

But $\overset{0}{\overline{G}}=\overline{(G\cup H)\cap G}^0$

$$\subseteq \overline{(G\cup H)\cap \overline{\overline{G^0}}}^{\,0}$$
$$= \overline{(G\cup H)\cap G^0}^{\,0} \quad \text{by Proposition 1.2.}$$
$$= \overline{(G\cup H)}^{\,0} \cap \overline{G^0}^{\,0} \quad \text{by Proposition 1.10.}$$
$$= (G\cup H)\cap \overline{G^0}^{\,0}$$
$$= (G\cap \overline{G^0}^{\,0})\cup (H\cap \overline{G^0}^{\,0})$$
$$\subseteq G\cup (H\cap \overline{G^0}^{\,0})$$
$$\subseteq G\cup (H\cap \overline{H^{c0}}^{\,0}$$

Thus $\overline{G}^{\,0} \subseteq G\cup (H\cap \overline{H^0}^{\,c})$

$$\subseteq G\cup (H\cap H^c) = G .$$

Then G is $s-$closed in X and by the same way we can prove that H is $s-$closed in X .■

Definition 1.12 [7]

Let X be a topological space. If $B\subseteq X$, then the semi closure of B is defined by the intersection of all $s-$closed sets in X containing B and is denoted by $\overline{B}^s$.

Now, a point $x\in X$, is called an $s-$limit point of $A\subseteq X$ if each $s-$open set containing x, contains a point of A distinct from x. And the set of all $s-$limit points of A is called the $s-$derived set of A and is denoted by A'^s .

The following results hold:

Proposition 1.13.[7]

Let X be a topological space and $A\subseteq X$ then $\overline{A}^s = A\cup A'^s$.

Proposition 1.14.[4]

Let X be a topological space and $A\subseteq B\subseteq X$, then

(a) A is $s-$closed set in X iff $A=\overline{A}^s$.

(b) $\overline{A}^s \subseteq \overline{A}$.

(c) $\overline{A}^s \subseteq \overline{B}^s$.

Proposition 1.15.

Let X be a topological space , A, O are subsets of X with O $s-$open. If $x\in O$ and $O\cap A=\phi$ then $x\notin \overline{A}^s$.

Proof.

Suppose $x \in \overline{A}^s$. The either $x \in A$ or $x \in A^{'s}$. If $x \in A$ then $O \cap A \neq \phi$ which contradicts the assumption. And if $x \in A^{'s}$ and $x \notin A$ then $(U \cap A)/\{x\} = \phi$ for every $s-$open set in X containing x and hence $U \cap A \neq \phi$ which is contradiction since O is $s-$open set containing x and $O \cap A = \phi$ and hence $x \notin \overline{A}^s$.■

We recall the following from [8].

The family $\{A_\lambda\}_{\lambda\in\Lambda}$ of subsets of a space X is said to be locally finite if for each $x \in X$ there exists a neighbourhood N_x of x such that the set $\{\lambda \in \Lambda : N_x \cap A_\lambda \neq \phi\}$ is finite. If $\{A_\lambda\}_{\lambda\in\Lambda}$ is locally finite family of subsets of a space X and there exist a family $\{B_\lambda\}_{\lambda\in\Lambda}$, $B_\lambda \subseteq A_\lambda$ for each λ, then $\{B_\lambda\}_{\lambda\in\Lambda}$ is locally finite. So that if $\{A_\lambda\}_{\lambda\in\Lambda}$ is locally finite family of subsets of a space X, then $\{\overline{A_\lambda}\}_{\lambda\in\Lambda}$ is a locally finite family of subsets of X.

Theorem 1.16.

Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a locally finite $s-$closed family of a topological space X, then $\bigcup_{\lambda\in\Lambda}\overline{F_\lambda}^s = \overline{\bigcup_{\lambda\in\Lambda}F_\lambda}^s$.

Proof.

Since $F_\lambda \subseteq \bigcup_{\lambda\in\Lambda}F_\lambda$ then $\overline{F_\lambda}^s \subseteq \overline{\bigcup_{\lambda\in\Lambda}F_\lambda}^s$ by Proposition 1.14, and hence $\bigcup_{\lambda\in\Lambda}\overline{F_\lambda}^s \subseteq \overline{\bigcup_{\lambda\in\Lambda}F_\lambda}^s$. Now we want to prove that $\bigcup_{\lambda\in\Lambda}\overline{F_\lambda}^s \supseteq \overline{\bigcup_{\lambda\in\Lambda}F_\lambda}^s$. Let $x \in \overline{\bigcup_{\lambda\in\Lambda}F_\lambda}^s$ such that $x \notin \bigcup_{\lambda\in\Lambda}\overline{F_\lambda}^s$, then $x \notin \overline{F_\lambda}^s$ for every $\lambda \in \Lambda$. Since $\{F_\lambda\}_{\lambda\in\Lambda}$ is locally finite then there exists an open set V_x containing x such that $F_\lambda \cap V_x \neq \phi$ for only a finite number of $\lambda = \lambda_1,...,\lambda_n$. Since $x \notin \overline{F_\lambda}^s$ for every $\lambda \in \Lambda$, then $x \notin F_\lambda$ and $x \notin F_\lambda^{'s}$ for every $\lambda \in \Lambda$ by Proposition 1.13. Thus there exists an $s-$open set U_x which contains x such that $F_\lambda \cap U_x = \phi$ for every $\lambda = \lambda_1,...,\lambda_n$.

$x \in U_x \cap V_x = O$ is $s-$open and since $F_\lambda \cap U_x = \phi$ for every $\lambda = \lambda_1,...,\lambda_n$, $O \subseteq U_\lambda$, then $O \cap F_{\lambda_i} = \phi$ for every $i = 1,2,...,n$. Since $F_\lambda \cap V_x = \phi$ for $\lambda \neq \lambda_1,...,\lambda_n$ then $O \cap F_\lambda = \phi$ for every $\lambda \neq \lambda_1,...,\lambda_n$, hence $O \cap F_{\lambda_i} = \phi$ for every $\lambda \in \Lambda$. Now we have $O \bigcap (\bigcup_{\lambda\in\Lambda}F_\lambda) = \phi$, so that since $x \in O$, then $x \notin \overline{\bigcup_{\lambda\in\Lambda}F_\lambda}^s$ by Proposition 1.15 which contradicts. Thus $x \in \bigcup_{\lambda\in\Lambda}\overline{F_\lambda}^s$. So that $\bigcup_{\lambda\in\Lambda}\overline{F_\lambda}^s \supseteq \overline{\bigcup_{\lambda\in\Lambda}F_\lambda}^s$, then $\bigcup_{\lambda\in\Lambda}\overline{F_\lambda}^s = \overline{\bigcup_{\lambda\in\Lambda}F_\lambda}^s$.■

Corollary 1.17.

In a topological space, the union of members of a locally finite $s-$closed sets is $s-$closed.

Proof.

Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a family of locally finite $s-$closed sets. Then $\overline{\bigcup_{\lambda\in\Lambda} F_\lambda}^{s} = \bigcup_{\lambda\in\Lambda} \overline{F_\lambda}^{s} = \bigcup_{\lambda\in\Lambda} F_\lambda$ by Theorem 1.16 and hence $\bigcup_{\lambda\in\Lambda} F_\lambda$ is $s-$closed set by Proposition 1.14.

Theorem 1.18.

Let $\{E_\lambda\}_{\lambda\in\Lambda}$ be a family of subsets of a space X and let $\{B_\gamma\}_{\gamma\in\Gamma}$ be a locally finite $s-$closed covering of X such that foe each $\gamma\in\Gamma$, the set $\{\lambda\in\Lambda : B_\gamma \cap E_\lambda \neq \phi\}$ is finite. Then there exists a locally finite family $\{U_\lambda\}_{\lambda\in\Lambda}$ of $s-$open sets of X such that $E_\lambda \subseteq U_\lambda$ for each $\lambda\in\Lambda$.

Proof.

For each λ, let $U_\lambda = X / \bigcup\{B_\gamma : B_\gamma \cap E_\lambda = \phi\}$. Clearly $E_\lambda \subseteq U_\lambda$ and since $\{B_\gamma\}_{\gamma\in\Gamma}$ is locally finite, it follow that U_λ is $s-$open by Corollary 1,17. Let x be a point of X, there exists a neighbourhood N of x and a finite subset K of Γ such that $N \cap B_\lambda = \phi$ if $\gamma \notin K$. Hence $N \subseteq \bigcup_{\gamma\in K} B_\gamma$. Now $B_\gamma \cap U_\lambda \neq \phi$ iff $B_\gamma \cap E_\lambda \neq \phi$. For each $\gamma\in K$, the set $\{\lambda\in\Lambda : B_\gamma \cap E_\lambda \neq \phi\}$ is finite. Hence the set $\{\lambda\in\Lambda : N \cap U_\lambda \neq \phi\}$ is finite. ■

Section two.

Recall that the space X is called paracompact iff each open cover of X has a locally finite open refinement (see[8]).

This suggests the following

Definition 2.1

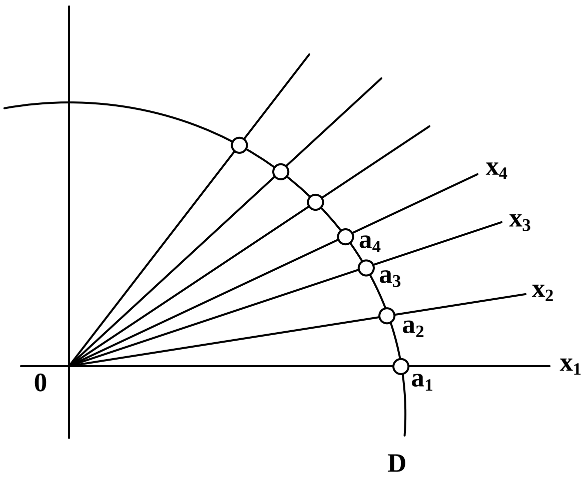
A space X is called $s-$paracompact space iff each open covering of X has a locally finite $s-$open refinement.

Example 2.2.

It is clear that every paracompact space is $s-$paracompact. However, the following example shows that the converse is false.

Suppose $X_m = \{(x,y)\in R^2, y = mx, m\in Z^+, x>0, y>0\}$ and let $X = (\bigcup_{m=1}^{\infty} X_m) \cup \{0\}$. Let a_m be the point in the intersection of the line $y = mx$ with the circumference of the unit open disc D with center 0, $a_m \notin D$. Denote the topology of X_m by τ_m, take a case for a point $x \in X_m, x \neq a_m$ to be the family of open intervals containing x but not a_m, and the base for a_m is X_m.

Let τ be the topology on X generated by $\bigcup_{m=1}^{\infty} \tau_m$ and the base at 0 the family D. Now we want to show that (X,τ) is not paracompact space for the open cover $\{X_m : m = 1,2,\ldots\} \cup \{D\}$ is an open cover having no locally finite open refinement, because every open refinement must contain D as a member and D intersects X in infinite number of points. Now to prove (X,τ) is s-paracompact. Let $\{G_\lambda\}$ be an open cover of X. If one member $G_\lambda = X$ then $\{X\}$ is a locally finite refinement of s-open sets, otherwise at least one $G_\lambda \ni D$ call it $G_{\lambda_0} \ni D$. Moreover for each m, at least one G_λ say $G_{\lambda_m} \ni X_m$ because the only open set containing a_m is X_m. There is no loss of generality if we suppose that G_λ is open interval when $\lambda \neq \lambda_0, \lambda_1, \ldots, \lambda_m, \ldots$ Then each $X_m / \{a_m\}$ is a collection of open intervals. So that $D \cup \{[a_1,\infty),[a_2,\infty),\ldots,[a_m,\infty),\ldots\}$ is an s-open refinement since each $[a_i,\infty)$ is s-open set for each i and since this s-open refinement are disjoint then it is locally finite s-open refinement. Then X is s-paracompact. It is clear that (X,τ) is T_0 space, but not T_1, not T_2 not regular and not normal. ■



We recall the following from [10].

A space X is called almost paracompact iff for every open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X there is a locally finite family $\{V_\lambda\}_{\lambda \in \Lambda}$ of open subsets of X, which $V_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$, and the family of the closures of members of $\{V_\lambda\}_{\lambda \in \Lambda}$ forms a covering of X.

Proposition 2.3.

If X is s-paracompact space, then it is almost paracompact.

Proof.

Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be an open covering of X, then it has a locally finite $s-$open refinement $\{W_\lambda\}_{\lambda\in\Lambda}$. Thus by Proposition 1.5, there exists an open set V_λ such that $V_\lambda \subseteq W_\lambda \subseteq \overline{V}_\lambda$.Therefore $\{V_\lambda\}_{\lambda\in\Lambda}$ is a locally finite open family such that $V_\lambda \subseteq G_\lambda$ for each $\lambda\in\Lambda$, and $\bigcup_{\lambda\in\Lambda}\overline{V}_\lambda = X$. Then X is almost paracompact space. ■

Proposition. 2.4[10]

If X is almost paracompact space regular space then is paracompact space.

Proposition. 2.5.

If X is $s-$paracompact space regular space then is paracompact space.

Proposition 2.6.

Let X be an $s-$paracompact space, let A be a subset of X and let B be a closed set of X which is disjoint from A. If every $x\in B$ there exist disjoint open sets U_x and V_x such that $A\subset U_x$, $x\in V_x$, then there exist an open set U and an $s-$open set V such that $A\subset U, B\subset V$ and $U\cap V=\phi$.

Proof.

The open covering of $s-$paracompact space X which consists of X/B together with the sets V_x for x in B has a locally finite $s-$open refinement $\{W_\gamma\}_{\gamma\in\Gamma}$. Let $\Gamma_1=\{\gamma\in\Lambda : W_\gamma \subset V_x : \mathit{for\ \ some}\ \ x\in B\}$. If $\gamma\in\Gamma_1$ then $U_x \cap W_\gamma = \phi$ for some x so that $A\cap\overline{W}_\gamma=\phi$. Now let $U = X/\bigcup_{\gamma\in\Gamma_1}\overline{W}_\gamma$ and $V=\bigcup_{\gamma\in\Gamma_1}W_\gamma$. Then $A\subset U, B\subset V$, U and V are disjoint. Clearly U is open set, and V is $s-$open set. Since $\{\overline{W}_\gamma\}_{\gamma\in\Gamma}$ is locally finite family so that $\bigcup_{\gamma\in\Gamma_1}\overline{W}_\gamma$ is closed set. ■

Theorem 2.7.

If X is $s-$paracompact Hausdorff space then for each x in X and a closed set B such that $x\notin B$, there exists disjoint $s-$open sets U,V such that $x\in U$, $B\subseteq V$.

Proof.

Let $x\in X$ and B be a closed set in X such that $x\notin B$. Then for every $y\in B$, there exist disjoint open sets U_x, V_x such that $x\in U_x, y\in V_x$. It follows from Proposition 2.6 that there exist disjoint $s-$open sets U and V such that $x\in U$, $B\subseteq V$.■

Proposition 2.8 [8].

If X is paracompact regular space then X is normal.

Corollary 2.9.

If X is $s-$ paracompact regular space then X is normal.

Theorem 2.10

If each finite open covering of a space X has a locally finite $s-$closed refinement, then for every disjoint closed sets A and B, there exist disjoint $s-$open sets U,V such that $A \subseteq U$, $B \subseteq V$.

Proof.

Let X be a space each finite open covering of which has a locally finite $s-$closed refinement and let A and B be disjoint closed sets of X. The open covering $\{X/A, X/B\}$ of X has a locally finite $s-$closed refinement Ω. Let E be the union of the members of Ω disjoint from A and let F be the union of the members of Ω disjoint from B. Then E and F are $s-$closed sets and $E \cup F = X$. Thus if $U = X/E$ and $V = X/F$, then U,V are disjoint $s-$open sets such that $A \subseteq U$, $B \subseteq V$. ■

Theorem 2.11

Let X be a topological space. If each open covering of X has a locally finite $s-$closed refinement, then X is $s-$paracompact space and for every disjoint closed sets A and B there exists disjoint $s-$open sets U,V such that $A \subseteq U$, $B \subseteq V$.

Proof.

Let Φ be an open covering of X and let $\{F_\lambda\}_{\lambda \in \Lambda}$ be a locally finite $s-$closed refinement of Φ. Since $\{F_\lambda\}_{\lambda \in \Lambda}$ is locally finite, each point x of X has a neighbourhood W_x such that $\{\lambda \in \Lambda : W_x \cap F_\lambda \neq \phi\}$ is finite. If $\{E_\gamma\}_{\gamma \in \Gamma}$ is a locally finite $s-$closed refinement of the open covering $\{W_x\}_{x \in X}$ of X, then for each γ in Γ the set $\{\lambda \in \Lambda : E_\gamma \cap F_\lambda \neq \phi\}$ is finite. It follows from Theorem 1.18. that there exists a locally finite family $\{V_\lambda\}_{\lambda \in \Lambda}$ of $s-$open sets such that $F_\lambda \subset V_\lambda$ for each λ. For each λ in Λ, let U_λ be a member of Φ such that $F_\lambda \subset U_\lambda$. Then $\{V_\lambda \cap U_\lambda\}_{\lambda \in \Lambda}$ is a locally finite $s-$open refinement of Φ. Thus X is $s-$paracompact and satisfy the last condition of theorem by using Theorem 2.10. ■

References .

[1] Adea Khaleefa Al-Obaidi "On some Maps and spaces" M.Sc. Thesis, University of Baghdad, College of Education, 1989.

[2] Nehmat Khder Ahmed "On some Application of Special subsets of Topological Spaces" M. Sc. Thesis, University of Salahaddin, College of Science, 1990.

[3] S,G. Crossley and S.K. Hildebrand " On Semi-Closure" Texas J. Sci.,22,(1971).

[4] P.Das "Not one some application on semi-open set" Progress of Mathematics ,7,(1973).
[5] J.Dugundji "Topology" Allyn and Bacon, Boston,1966.

[6] N. Levine "Semi-open sets and semi continuouity in Topological Space " Amer . Math. Monthly ,70,(1963).

[7] Ali Hasan Naser " On Separation Properties " M.Sc. Thesis University of Baghdad, College of Education, 1989.

[8] A.p.pears "On Dimension Theory of General Spaces" Cambridge University Press, 1975.
[9] W.J. Pervin " Foundation of General Topology " Acadimic Press, New York,1964.
[10] M.K Singal and S.P. Arya " On $\Pi-$Paracompact Spaces " Mathematische Annalen . 181.(1989)