Modules with Finite Submodule-Lengths

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Abstract

In this paper, the concepts of submodule with finite submodule-length, and module with finite submodule-lengths are introduced. These concepts are generalizations of the concepts of ideal with finite ideal-length and ring with finite ideal-length.

A submodule N of an R-module M is said to have finite submodule length if comp(I) is finite and $(M / I M)_{P'}$, is an Artinian and Noetherian $(R / I)_{P'}$-module, $\forall$ $P' = P / I$, $I = (N:M)$ and $P \in$ comp(I). An R-module M has finite submodule-lengths if each submodule of M has finite submodule-length. The basic results about these concepts, and some relationships between modules with finite submodule-lengths and other classes of modules are given.

المستخلص:

في هذا البحث قدمنا ,المفهومان مقاس جزئي ذي طول- مقاسي جزئي منتهي ومقاس ذي اطوال- مقاسية جزئية منتهية . هذان المفهومان هما تعميمان للمفهومين مثالي ذي طول –مثالي منتهي وحلقة ذات اطوال –مثالية منتهية .

يقال عن مقاس جزئي N من مقاس M على Rيمتلك طول- مقاسي جزئية منتهي اذا كان comp(I) مجموعة منتهية و$(M / I M)_{P}$ مقاس ارتيني و نوثيري على الحلقة -$(R / I)_{P'}$. $\forall$ $P' = P / I$, $I = (N:M)$ and $P \in$ comp(I). مقاس M على الحلقة R يمتلك اطوال – مقاسية جزئية اذاكان كل مقاس جزئي منه يمتلك طول مقاسي جزئي منتهي . النتائج الاساسية حول هذ ين

المفهومين وبعض العلاقات بين المقاسات ذات الأطوال – المقاسية الجزئية المنتهية و اصناف اخرى من اللمقاسات قد اعطيت .

Introduction

Zariski and Samuel in [11] introduced the concept of ideal-length for a decomposable ideal I , where a proper ideal I of a commutative ring R is said to be decomposable if it has a primary decomposition; that is, I can be written as $I = \bigcap_{i=1}^{n} Q_i$ with Q_i is P_i-primary, $\forall$ i =1, 2, …, n.

Zariski and Samuel put the following condition (*), where:

(*) : $M = \{P_1, P_2, \ldots, P_n\}$ is the set of all minimal prime ideals of I.

Zariski and Samuel in [11, P.233] gave the following definition: let R be a ring with identity and let I be a decomposable ideal, which satisfies (*). The length of the R-module $R_{S(I)}/IR_{S(I)}$ is called the length of I and is denoted by $\lambda(I)$, where $S(I) = R - \bigcup_{i=1}^{n} \{P_i : P_i \in M\}$.

J. Beachy and W. Weakley in [2], used the terminology ideal-length suggested by Zariski and Samuel in [11], they extended the notion ideal-length to an arbitrary ideal. In fact, they introduced the notion of finite ideal-length, where, an ideal I of a ring R is said to have finite ideal-length if the ring $R_{S(I)} / I R_{S(I)}$ is an Artinian ring, where $S(I) = R - \cup \{P : P \in comp(I)\}$ and comp(I) represents the set of all minimal prime ideals of I. In this case the length of a composition series of $R_{S(I)} / I R_{S(I)}$ which is denoted by $\lambda(I)$ is called the ideal-length of I.

Moreover, Beachy and Weakley introduced in [2] another concept, they said a ring R have finite ideal-lengths if each ideal of R has finite ideal-length.

Our aim in this paper is to extend these notions to submodules and modules.

In §.1 of this paper, the concepts of submodules with finite submodule-length and, modules with finite submodule-lengths are introduced. We call a submodule N of an R-module M, a submodule with finite submodule-length if comp($N \underset{R}{:} M$) is finite and $(M/(N:M)M)_{P'}$ is a Noetherian and Artinian $(R/(N \underset{R}{:} M))_{P'}$-module, $\forall$ $P' = P/(N \underset{R}{:} M)$,where $P \in$ comp(N:M). We also call the module M, a module of finite submodule-lengths if each submodule of M has finite submodule-length.

We study the basic properties of such submodules and modules; we give characterization for submodules with finite submodule-lengths. We also give descriptions for such submodules in the class of finitely generated modules or multiplication modules.

In §.2, we look for any relation between modules with finite submodule-lengths and Noetherian, Artinian and Laskerian modules.

Finally, we remark that R in this paper stands for a commutative ring with 1, and all modules are unitary

§.0 Preliminaries

In this section, we give some basic definitions and results which are needed later.

Definition 0.1 : **[8]**

For any ideal I of a ring R,the component of I (briefly comp (I)) is the set of all prime ideals of R which are minimal over I. We say that R has finite component (briefly FC) if comp (I) is finite for every ideal I of R.

Definition 0.2 : **[2]**

Let I be an ideal of R, and let S(I) denotes the complement of $\cup\{P : P \in \text{comp}(I)\}$ in R. Then I is said to have afinite ideal-length if the ring $R_{S(I)} / I R_{S(I)}$ is an Artinian ring.

In this case, the length of a composition series for $R_{S(I)} / I R_{S(I)}$ is denoted by $\lambda(I)$ and is called the ideal-length of I. If each ideal of R has finite ideal-length, then the ring R is said to have finite ideal-length.

Proposition 0.3 : **[2, prop.1.2]**

Let I be an ideal of R. Then I has finite ideal-length iff comp(I) is finite and the ring $R_P / I R_P$ is Artinian for each $P \in$ comp(I).

Theorem 0.4 **: [2, prop. 1.3]**

Let R be a ring. Then the following statements are equivalent :

1. R has finite ideal-lengths.
2. R has FC, and for each ideal I of R and each $P \in$ comp(I), the ring $R_P / I R_P$ ($\cong (R/I)_P$) is Artinian.
3. R has FC, and for each prime ideal P of R, the set of P-primary ideals satisfies a.c.c.

§.1 Modules with Finite Submodule-Lengths; Basic Properties

In this section, the basic properties about modules with finite submodule-lengths are given, some results are generalizations of known results about rings with finite ideal-lengths which are given in [2].

Definition 1.1 :

Let N be a submodule of an R-module M. Then N is said to have finite submodule-length if comp(I) is a finite set and $(M / I M)_{P'}$ is an Artinian and Noetherian $(R/I)_{P'}$-module, $\forall P' = P / I$, where $I = (N \underset{R}{:} M)$, $P \in$ comp(I).

If each submodule of M has finite submodule-length, then M is said to have finite submodule-lengths.

Definition 1.2 :

Let M be an R-module. M is said to have finite component (briefly M has FC), if for each submodule N of M, $(N \underset{R}{:} M)$, has FC.

Remarks and Examples 1.3 :

1. For any ring R, R has finite ideal-lengths iff R as R-module has finite submodule-lengths.

2. Let M be the Z-module Z_{p^∞}, where p is any prime number. M has FC since for each submodule N of M, $(N \underset{Z}{:} M) = (0)$ and hence $comp(N \underset{Z}{:} M) = \{(0)\}$. Moreover, for any $P' = P / (N \underset{Z}{:} M) = P/(0)$, $(M / (N \underset{Z}{:} M)M)_{P'} = M_{P'} \cong (Z_{p^\infty})_{P'}$, and $(Z / (N \underset{Z}{:} M))_{P'} = Z_{P'}$. Hence $(Z_{p^\infty})_{P'}$ is an Artinian $Z_{P'}$-module. Then by [10, lemma 1.5], $(Z_{p^\infty})_{P'}$ satisfies a.c.c. on semiprime submodule and hence satisfies a.c.c. on prime submodules. On the other hand $Z_{p'} \cong Q$ (the field of rational numbers). Then by [10, Lemma 1.1], $(Z_{p^\infty})_{P'}$ is a Noetherian $Z_{P'}$-module. Thus the Z-module Z_{p^∞} has finite submodule-lengths.

3. Let R be an integral domain. Let K be the total quotient field of R. Then K as R-module has finite submodule-lengths. In particular, the Z-module Q of rational numbers has finite submodule-lengths.

Before giving a characterization of modules with finite submodule-lengths, we give the following lemma:

Lemma 1.4 :

Let N be a submodule of an R-module M. Then $(N \underset{R}{:} M) = ((N \underset{R}{:} M) M \underset{R}{:} M)$.

Proof. It is straightforward, so is omitted

Theorem 1.5 :

Let M be an R-module. Then M has finite submodule-lengths iff M has FC and for each submodule N of M, $(M / N)_{P'}$ is an Artinian and Noetherian $(R/(N \underset{R}{:} M))_{P'}$- module, $\forall\, P' = P / (N \underset{R}{:} M)$, $P \in comp(N \underset{R}{:} M)$.

Proof. Assume M has finite submodule-lengths. Then the result follows by definition 1.2 and [9, lemma 9.12, p.173].

Conversely, let N be any submodule of M. Then by applying the hypothesis on $N' = (N:M)M$ and using lemma 1.4, the required result is obtained.

In the class of finitely generated modules, the following theorem is a characterization of submodules with finite submodule-lengths, but first we need the following lemma:

Lemma 1.6 :

Let N be a submodule of an R-module M, and let $P \in \text{comp}(N \underset{R}{:} M)$. Then $(M / N)_{P'}$ is a faithful $(R / (N:M))_{P'}$-module, where $P' = P / (N:M)$.

Proof. It is easy, so is omitted.

Theorem 1.7 :

Let M be a finitely generated R-module, and let N be a submodule of M. Then N has finite submodule-length iff $(N \underset{R}{:} M)$ has finite ideal-length.

Proof If N has a finite submodule-length, then $(N \underset{R}{:} M)$ has FC and $(M / N)_{P'}$ is an Artinian and Noetherian $(R / (N \underset{R}{:} M))_{P'}$-module, $\forall\ P' = P / (N \underset{R}{:} M)$, $P \in \text{comp}(N \underset{R}{:} M)$. Since M is a finitely generated R-module, so $(M / N)_{P'}$ is a finitely generated $\overline{R} = (R / (N \underset{R}{:} M))_{P'}$-module, $\forall\ P' = P / (N:M)$, $P \in \text{comp}(N:M)$. Hence $\overline{R} / \text{ann}_{\overline{R}}(M / N)_{P'}$ is an Artinian ring by [7, Th. 2, p.180]. However by lemma 1.6, $\text{ann}_{\overline{R}}\ (M/N)_{P'} = (\overline{0})$. Thus $\overline{R}$ is an Artinian ring, so the result is obtained.

The proof of the converse is similar.

The following corollaries follow directly by Th. 1.7:

Corollary 1.8 **:**

Let M be a finitely generated R-module. Then M has finite submodule-lengths iff for each submodule N of M, $(N \underset{R}{:} M)$ has a finite ideal-length.

Corollary 1.9 **:**

Let M be a finitely generated R-module. If R has finite ideal-lengths, then M has finite submodule-lengths.

Remark 1.10 **:**

The condition M is a finitely generated R-module cannot be dropped from theorem 1.7 and corollaries 1.8 and 1.9, as the following example shows:

Example **:**

Let M be the Z-module $Z \oplus Z \oplus \cdots$. For each submodule N of M, $(N \underset{Z}{:} M)$ has a finite ideal length, since Z has finite ideal-lengths by [2]. However if $N = 4Z \oplus 4Z \oplus \cdots$, then it is easy to see that N has no finite submodule-length.

Recall that an R-module M is called a multiplication R-module if for every submodule N of M, there exists an ideal I of R such that N = I M, [1].

Theorem 1.11 **:**

Let M be a multiplication R-module and let N be a submodule of M. Then N has finite submodule-length iff $(N \underset{R}{:} M)$ has finite ideal-length.

Proof. If N has finite submodule-length, and let $I = (N \underset{R}{:} M)$, then I has FC and $(M / N)_{P'}$ is an Artinian and Noetherian $(R / I)_{P'}$-module, $\forall$ $P' = P / I$, $P \in \text{comp}(I)$. Since M is a multiplication R-module, $(M / N)_{P'}$ is a multiplication $(R / I)_{P'}$-module, $\forall$ $P' = P / I$, $P \in \text{comp}(I)$ and by lemma 1.6, $(M / N)_{P'}$ is a faithful $(R / I)_{P'}$-module, $\forall$ $P' = P / I$, $P \in \text{comp}(I)$.Hence $(R / I)_{P'}$ is an Artinian ring, $\forall$ $P' = P / I$, $P \in \text{comp}(I)$. Thus I has finite ideal-length.

The proof of the converse is similar.

Corollary 1.12 **:**

Let M be a multiplication R-module. Then M has finite submodule-lengths iff for each submodule N of M, (N:M) has finite ideal-length.

Remark 1.13 **:**

The condition M being a multiplication R-module cannot be dropped from theorem 1.11 and corollary 1.12 as is seen by the example following remark 1.10.

For our next result, we prove the following lemma:

Lemma 1.14 **:**

Let M be a multiplication R-module. If M has FC or R has FC, then M is finitely generated.

Proof. If M has FC or R has FC, then comp($(0) \underset{R}{:} M$) = comp($ann_R M$) is a finite set. Hence by [3, remark. following corollary. 2.11], M has a finite number of minimal submodules. Then by [3, Th. 3.7], M is finitely generated.

Proposition 1.15 **:**

Let M be a faithful multiplication R-module. Then M has finite submodule-lengths iff R has finite ideal-lengths.

Proof. It follows by lemma 1.14, corollary1.12, and [3, theorem 3.1].

To give the next result, we need the following lemmas:

Lemma 1.16 **:**

Let P be a prime ideal of R, and let J, K be two P-primary ideals of R containing an ideal I of R such that $(J / I)_P \subseteq (K / I)_P$. Then $J \subseteq K$.

Proof. It is straightforward, so it is omitted.

Lemma 1.17 :

Let R be a ring. Then R has finite ideal-lengths iff R has FC and every primary ideal of R has finite ideal-length.

Proof. The if part is clear.

The proof of the converse follows by theorem 1.4 and lemma 1.16.

Recall that a proper submodule N of an R-module M is called primary if whenever $r \in$ R, $x \in$ M, $r\,x \in$ N, implies either $x \in$N or $r^n \in$ (N:M), for some $n \in Z_+$ [4, P. 39].

Proposition 1.18 :

Let M be a faithful multiplication R-module. Then M has FC and every primary submodule of M has finite submodule-lengths iff R has finite ideal-lengths.

Proof. Assume M has FC and every primary submodule of M has finite submodule-length. By lemma 1.14, M is finitely generated. Thus M is a finitely generated faithful multiplication R- module and so for any primary ideal I of R, N = I M is a primary submodule of M and I = (N:M). Since N has finite submodule-length, then by Th. 1.11, I has finite ideal-length. Moreover, for any ideal J of R, W = J M is a submodule of M which has FC and hence (W:M) = J has FC. Thus by Lemma 1.17, R has finite ideal-lengths.

The proof of the converse follows by Prop.1.15 and def. 1.1.

Corollary 1.19 :

Let M be a faithful multiplication R-module. Then the following statements are equivalent :

1. M has finite submodule-lengths.

2. M has FC and every primary submodule of M has finite submodule-length.

3. R has finite ideal-lengths.

Proof. (1) $\Rightarrow$ (2) : It is obvious.

(2) $\Rightarrow$ (3): It follows by proposition 1.18.

(3) $\Rightarrow$ (1): It follows by proposition 1.15.

Proposition 1.20 **:**

Let M be an R-module which has finite submodule-lengths. Then any factor module of M has the same property

Proof. It follows easily, so it is omitted.

The following result explains the behavior of modules with finite submodule – lengths under localization.

Theorem 1.21 **:**

Let M be a finitely generated R-module and let S be a multiplicatively closed subset of R. If M has finite submodule-lengths, then M_S as R_S-module has finite submodule-lengths.

Proof. Let N be a submodule of M_S, hence $N = W_S$ for some submodule W of M. Since M has finite submodule-lengths, $I = (W \underset{R}{:} M)$ has FC, and since M is a finitely generated, $I_S = (W_S \underset{R_S}{:} M_S)$ by [7, prop. 8, p.152]. However it is easy to see that, $comp(I_S) \subseteq \{P_S : P \in comp(I)\}$, hence $comp(I_S)$ is finite. Thus M_S has FC. Moreover, by using [7, prop. 10, p. 156], [7, prop. 19, p.165], [7, prop. 20, p. 166] and $(M / I\, M)_{P'}$ is an Artinian and Noetherian $(R / I)_{P'}$-module, $\forall\, P' = P / I$, $P \in comp(I)$, we have $(M_S / I_S M_S)_{P'_S}$ is an Artinian and Noetherian $(R_S / I_S)_{P'_S}$ -module, $\forall\, P'_S = P_S / I_S$, $P_S \in comp(I_S)$. Thus M_S as R_S-module has finite submodule-lengths.

§.2 Modules with Finite Submodule-Lengths and Related Concepts

In this section, we give some relationships between modules with finite submodule-lengths and other known classes of modules, for examples: Noetherian modules, Artinian modules and Laskerian modules.

First we have:

Proposition 2.1 :

Let M be a Noetherian R-module. Then M has finite submodule-lengths.

Proof. Since M is a Noetherian R-module, $\overline{R}$ = R / ann M is a Noetherian ring and hence $\overline{R}$ has finite ideal-lengths by [2]. Let N be any submodule of M, then I = (N:M) $\supseteq$ ann M, so $\overline{I}$ = I / ann M has finite ideal-length and hence $\overline{I}$ has FC. However it is easy to check that, for any P $\in$ comp(I), $\overline{P}$ = P / ann M $\in$ comp($\overline{I}$) and conversely , hence I = (N:M) has FC. Thus M has FC. Now, since $\overline{R}$ has finite ideal-lengths, $(\overline{R}/\overline{I})_{\overline{P}/\overline{I}}$ which is isomorphice to $(R / I)_{P/I}$ is an Artinian ring, $\forall$ P $\in$ comp(I) see [7, th. 2, p. 16] and th. 0.4. On the other hand, M is a Noetherian R-module, so $(M / I M)_{P/I}$ is a finitely generated over $(R / I)_{P/I}$, $\forall$ P $\in$ comp(I). Thus $(M / I M)_{P/I}$ is an Artinian and Noetherian $(R / I)_{P/I}$-module, $\forall$ P $\in$ comp(I). Thus M has finite submodule-lengths.

The converse of prop. 2.1 is false by the following example:

Let R be the ring of all matrices of the form $\begin{pmatrix} k & 0 \\ x & k \end{pmatrix}$, where $k \in$ Z, $x \in Z_{2^\infty}$, with usual addition and multiplication of matrices. R is not Noetherian, but R has finite ideal-lengths. (see [2]).

Proposition 2.3 :

Every finitely generated Artinian R-module M has finite submodule-lengths.

Proof. Since M is a finitely generated R-module, $\overline{R}$ = R / ann_RM is an Artinian ring, and so it is Noetherian. Then by a similar argument of proof of proposition 2.1, M has finite submodule-lengths.

Recall that if M be an R-module which contains prime submodule, then the height of P (denoted by ht P), is the greatest non-negative integer n such that there exists a chain of prime submodules of M, $P_0 \subset P_1 \subset \ldots \subset P_n = P$, and ht P = ∞ if no such *n* exists. The dimension of M is defined by:

D(M) = sup {ht P : P is a prime submodule of M}.[5]

Proposition 2.4 :

Let M be a Noetherian R-module with D(M) = 0. Then M is Artinian.

Proof. Since M is Noetherian, and D(M) = 0, every prime submodule N of M is maximal and hence M / N is simple module, so N is virtually maximal. Thus by [6, Th. 3.5], M is Artinian.

The converse of prop. 2.4 is not true as the following example shows.

Example :

The Q-module Q ⊕ Q , where Q is the set of rational numbers is Artinian, D(M) = 1 ≠ 0.

Recall that an R-module M is called Laskerian if every submodule of M can be written as a finite intersection of primary submodules.

Aring R is Laskerian iff the R-module R is Laskerian.

The following lemmas are needed for our next results.

Lemma 2.5 :

Every Laskerian module has FC.

Proof. It follows easily.

Lemma 2.6 :

Let M be a faithful multiplication R-module. Then M is a Laskerian module iff R is a Laskerian ring.

Proof. Assume M is Laskerian. Then by lemma 2.5, M has FC. Thus M is a multiplication R-module with FC, so that M is finitely generated by lemma 1.14, and this implies that R is a Laskerian ring.

The proof of the converse is similar.

Proposition 2.7 :

Let M be a faithful multiplication R-module. Then the following statements are equivalent:

1. M is Noetherian.

2. M is a Laskerian module and every primary submodule of M has finite submodule-length.

3. M is a Laskerian module with finite submodule-lengths.

4. R is Laskerian with finite ideal-lengths.

5. R is Noetherian.

Proof. (1) $\Rightarrow$ (2). It follows by [7, th. 14, P.192] and prop 2.1.

(2) $\Rightarrow$ (3). It follows by lemma 2.5 and cor. 1.19.

(3) $\Rightarrow$ (4). It follows by cor. 1.19 and lemma 2.6.

(4) $\Rightarrow$ (5). It follows by.[2 th.1.8].

(5) $\Rightarrow$ (1). It follows by [3, §.3, p.765].

Theorem 2.8 :

Let M be a Laskerian R-module. Then M has finite submodule-lengths iff every primary submodule of M has finite submodule-length.

Proof. If every primary submodule has finite submodule lengths. Let N be a submodule of M. Since M is Laskerian, N is a decomposable submodule and it has a minimal primary decomposition, so assume that N = $\bigcap_{i=1}^{k} N_i$ be a minimal primary decomposition of N. Then N_i is P_i-primary submodule in M. $1 \le i \le k$, such that $P_i = \sqrt{(N_i : M)}$, P_1, P_2, …, P_n are distinct and for all j =1, 2, …, n,

$N_j \not\supseteq \bigcap_{\substack{i=1 \\ i\neq j}}^{k} N_i$. It follows that $I = (N:M) = \bigcap_{i=1}^{k}(N_i : M)$ with $(N_i:M)$ is a P_i-primary ideal of R, for i = 1, 2, …, k. Therefore I is decomposable, so there are two cases:

(i) The above decomposition of I is a minimal primary decomposition. Then by [9, prop. 4.24, p.72], for any $P \in spec(R)$, $P \in comp(I)$ iff P is a minimal member of $\{P_1, P_2, \ldots, P_n\}$. Without loss of generality, assume that $P_1 \in comp(I)$. Then $(N:M)_{P_1} = (N_1:M)_{P_1} \cap (N_2:M)_{P_1} \cap \ldots \cap (N_k:M)_{P_1}$. Then one can check that $(N:M)_{P_1} = (N_1:M)_{P_1}$. Moreover, $R / (N:M))_{P_1/(N:M)} \cong R_{P_1} / (N:M)_{P_1}$ by [7, th. 10, p. 156] and hence $R_{P_1} / (N:M)_{P_1} \approx R_{P_1} / (N_1:M)_{P_1}$. Also, one can see that $(M / N)_{P_1/(N:M)} \cong M_{P_1} / (N_1)_{P_1}$, $(N)_{P_1} = (N_1)_{P_1}$, thus $(M / N)_{P_1/(N:M)} \cong M_{P_1} / (N_1)_{P_1} \cong (M / N_1)_{P_1/(N_1:M)}$. On the other hand, $(M / N_1)_{P_1/(N_1:M)}$ is an Artinian and Noetherian $(R / (N_1:M))_{P_1/(N_1:M)}$. Hence $(M / N)_{P_1/(N:M)}$ is an Artinian and Notherian $(R / (N:M))_{P_1/(N:M)}$. Furthermore M is a Laskerian so by lemma 2.4, M has F.C. Thus M has finite submodule-lengths.

(ii) If the decomposition of I = (N:M), where $I = \bigcap_{i=1}^{k}(N_i : M)$ is not a minimal primary decomposition. Assume that $(N_1:M) \supseteq \bigcap_{i=2}^{k}(N_i : M)$. Hence $I = \bigcap_{i=2}^{k}(N_i : M)$. Without loss of generality suppose that $I=\bigcap_{i=2}^{k}(N_i : M)$ is a minimal primary decomposition. Assume $P_2 \in comp(I)$, then by the same process of case (i), $(M / N)_{P_2/(N:M)}$ is an Artinian and Noetherian $(R / (N:M)_{P_2/(N:M)}$-module and M has FC. Thus M has finite submodule-lengths.

***<u>Note</u>*:**

We remark that the equivalence ((2) ⇔ (3)) of prop.2.7 can be obtained directly by th.2.8.

References

[1] A. Barnard, "Multiplication Modules", J. Algebra, 71(1981), 174-178.

[2] J. A. Beachy and W. D. Weakley, "Piecewise Noetherian Rings", Comm. In Algebra, 12(21), (1984), 2679-2706.

[3] Z. A. El-Bast and P. F. Smith, "Multiplication Modules", Comm. In Algebra, 16(4), (1988), 755-779.

[4] M. D. Larsen and P. J. McCarthy, "Multiplicative Theory of Ideals", Academic Press, London, New York, 1971.

[5] S. H. Man and P. F. Smith, "On Chains of Prime Submodules", Israel J. of Math., 127 (2002), 131-155.

[6] R. L. McCasland and P. F. Smith, "Prime Submodules of Noetherian Modules", Rocky Mountain J. of Math., 23(3), (1993), 1041-1061.

[7] D. G. Northcott, "Lessons on Rings, Modules and Multiplicities", Cambridge University Press, London and New York, 1968.

[8] J. Ohm and R. Pendleton, "Rings with Noetherian Spectrum", Duke Math. J., 35 (1968), 631-639.

[9] R. Y. Sharp, “Steps in Commutative Algebra”, Cambridge University Press, 1990.

[10] D. P. Yilmaz and P. F. Smith, “Chain Conditions in Modules with Krull Dimension”, Comm. In Algebra, 24 (13), (1996), 4123-4133.

[11] O. Zariski and P. Samuel, “Commutative Algebra”, Vol.I, Springer-Verlag, New York, Heidelberg, Berlin, 1960.