SOME PROPERTIES OF A NEW SUBCLASS OF MEROMORPHIC UNIAVLENT FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY RUSCHEWEYH DERIVATIVE II

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Abstract: We have introduced a new class $H(\alpha,\mu,\beta,\lambda)$ of univalent meromorphic functions defined by Ruscheweyh derivative in the punctured unit disk U*. We study several properties, like , coefficient estimates , region of univalency, Hadamard product (or convolution).
We also obtain some results connected to (n,s)- Neighborhoods on $H^{\sigma}(\alpha,\mu,\beta,\lambda)$ and integral operator.
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1.Introduction :

Let $\sum$ denote the class of functions f(z) of the form:

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n, \qquad (1)$$

which are analytic and meromorphic univalent in the punctured unit disk $U^*=\{z: z\in \mathbb{C} \text{ and } 0<|z|<1\} = U\setminus\{0\}$.

The class $\sum$ is closed under the Hadamard product (or convolution).

$$(f*g)(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n b_n z^n=(g*f)(z),$$

where $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n, g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_n z^n.$

Let M be a subclass of a class $\sum$ consisting of functions of the form :

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n, a_n \geq 0. \qquad (2)$$

Here , we introduce the class H$(\alpha,\mu,\beta,\lambda)$ consisting of functions $f \in M$ and satisfying :

$$\left|\frac{\alpha(z^2(D^\lambda f(z))'+zD^\lambda f(z))}{\mu z^2(D^\lambda f(z))'+\mu\alpha z D^\lambda f(z)}\right|<\beta , \qquad (3)$$

for $0\leq\alpha<1, 0<\beta\leq 1, 0\leq\mu\leq 1$ and $D^\lambda f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n D_n(\lambda)z^n$ (4)

(Ruscheweyh derivative of f of order λ [5] , [6]), where

$$D_n(\lambda)=\frac{(\lambda+1)(\lambda+2)\ldots(\lambda+n+1)}{(n+1)!}, \lambda>\text{-}1, z\in U^*. \qquad (5)$$

<u>2.Main Results :</u>

We need the following result in our work (coefficient estimates).

<u>Theorem 1</u> : A function f(z) defined by (2) belongs to the class $H(\alpha,\mu,\beta,\lambda)$ if and only if

$$\sum_{n=1}^{\infty}[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)a_n \leq \beta\mu(1-\alpha). \qquad (6)$$

The result is sharp.

<u>Proof :</u> Assume that the inequality (6) holds true and let $|z|=1$, then from (3), we have

$$\left|\alpha(z^2\left(D^\lambda f(z)\right)'+zD^\lambda f(z)\right|-\beta\left|\mu z^2\left(D^\lambda f(z)\right)'+\mu\alpha z D^\lambda f(z)\right|$$

$$=\left|\alpha\left(\sum_{n=1}^{\infty}(n+1)D_n(\lambda)a_n z^{n+1}\right)\right|-\beta\left|\mu(1-\alpha)-\sum_{n=1}^{\infty}(n\mu+\mu\alpha)D_n(\lambda)a_n z^{n+1}\right| \qquad (7)$$

$$\leq\sum_{n=1}^{\infty}[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)a_n-\beta\mu(1-\alpha)\leq 0.$$

Hence by the principle of maximum modulus, $f(z) \in H(\alpha, \mu, \beta, \lambda)$.
Conversely, suppose that $f(z)$ defined by (2) is in the class $H(\alpha, \mu, \beta, \lambda)$, then from (4) , we have

$$\left| \frac{\alpha(z^2 (D^\lambda f(z))' + zD^\lambda f(z)}{\mu z^2 (D^\lambda f(z))' + \mu\alpha z D^\lambda f(z)} \right|$$

$$= \left| \frac{\alpha\left(\sum_{n=1}^{\infty} (n+1) D_n(\lambda) a_n z^{n+1}\right)}{\mu(1-\alpha) - \sum_{n=1}^{\infty} (n\mu + \mu\alpha) D_n(\lambda) a_n z^{n+1}} \right| < \beta \ .$$

Since $|\mathrm{Re}(z)| \le |z|$ for all z, we have

$$\mathrm{Re}\left\{ \frac{\alpha\left(\sum_{n=1}^{\infty} (n+1) D_n(\lambda) a_n z^{n+1}\right)}{\mu(1-\alpha) - \sum_{n=1}^{\infty} (n\mu + \mu\alpha) D_n(\lambda) a_n z^{n+1}} \right\} < \beta \ .$$

Choose the value of z on the real axis so that $\dfrac{z(D^\lambda f(z))'}{D^\lambda f(z)}$ is real.
Upon clearing the denominator of (7) and letting $z \to 1$ through real values , we get

$$\sum_{n=1}^{\infty} \alpha(n+1) D_n(\lambda) a_n \le \beta\mu(1-\alpha) - \sum_{n=1}^{\infty} \beta\mu(n+\alpha) D_n(\lambda) a_n,$$

which implies the inequality (5). Sharpness of the result follows by setting :

$$f(z) = \frac{1}{z} + \frac{\beta\mu(1-\alpha)}{[\beta\mu(n+\alpha) + \alpha(n+1)] D_n(\lambda)} z^n, n \ge 1. \tag{8}$$

Corollary 1: Let $f(z) \in H(\alpha, \mu, \beta, \lambda)$. Then

$$a_n \le \frac{\beta\mu(1-\alpha)}{[\beta\mu(n+\alpha) + \alpha(n+1)] D_n(\lambda)},$$

where $0 \le \alpha$ <1, 0< $\beta \le 1$, $0 \le \mu \le 1$ and λ > -1 .
Next, we obtain the region of univalency, in particular, starlikeness and convexity for the class $H(\alpha, \mu, \beta, \lambda)$.

Theorem 2: Let $f(z) \in H(\alpha, \mu, \beta, \lambda)$. Then $f(z)$ is meromorphic univalent starlike of order θ $(0 \le \theta < 1)$ in $|z| < r_1 = r_1(\alpha, \mu, \beta, \lambda, \theta)$, where

$$r_1(\alpha, \mu, \beta, \lambda, \theta) = \inf_n \left\{ \frac{(1-\theta)[\beta\mu(n+\alpha) + \alpha(n+1)]D_n(\lambda)}{(2+n-\theta)\beta\mu(1-\alpha)} \right\}^{\frac{1}{n+1}} \tag{9}$$

The result (9) is sharp.

Proof : Let $f(z) \in H(\alpha, \mu, \beta, \lambda)$. Then by Theorem 1

$$\sum_{n=1}^{\infty} \frac{[\beta\mu(n+\alpha) + \alpha(n+1)]D_n(\lambda)}{\beta\mu(1-\alpha)} a_n \le 1. \tag{10}$$

It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \le 1-\theta \text{ for } |z| < r_1(\alpha, \mu, \beta, \lambda, \theta), \tag{11}$$

where $r_1(\alpha, \mu, \beta, \lambda, \theta)$, is given by (9).

Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$. Then $zf'(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} n a_n z^n$.

Therefore

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| = \left| \frac{\sum_{n=1}^{\infty} (n+1) a_n z^{n+1}}{1 + \sum_{n=1}^{\infty} a_n z^{n+1}} \right|$$

$$\le \frac{\sum_{n=1}^{\infty} (n+1) a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} a_n |z|^{n+1}} \le 1-\theta,$$

provided that

$$\sum_{n=1}^{\infty} \frac{(2+n-\theta)}{1-\theta} a_n |z|^{n+1} \le 1. \tag{12}$$

Now making use of (10). We observe that (12) holds true if

$$|z|^{n+1} \le \frac{(1-\theta)[\beta\mu(n+\alpha) + \alpha(n+1)]D_n(\lambda)}{(2+n-\theta)\beta\mu(1-\alpha)}.$$

Hence $|z| \le \left\{ \frac{(1-\theta)[\beta\mu(n+\alpha) + \alpha(n+1)]D_n(\lambda)}{(2+n-\theta)\beta\mu(1-\alpha)} \right\}^{\frac{1}{n+1}}$ (13)

setting $|z| = r_1(\alpha,\mu,\beta,\lambda)$ in (13), we get the radii of starlikeness , which completes the proof of Theorem2. The result is sharp for the function is given by

$$f_n(z) = \frac{1}{z} + \frac{\beta\mu(1-\alpha)}{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)} z^n .$$

Theorem 3: Let f(z) $\in H(\alpha,\mu,\beta,\lambda)$.Then f(z)is meromorphic univalent convex of order θ $(0 \le \theta < 1)$ in $|z| < r_2 = r_2(\alpha,\mu,\beta,\lambda,\theta)$, where

$$r_2(\alpha,\mu,\beta,\lambda,\theta) = \inf_n \left\{ \frac{(1-\theta)[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{n(2+n-\theta)\beta\mu(1-\alpha)} \right\}^{\frac{1}{n+1}} . \qquad (14)$$

Proof of Theorem 3 is similar to that of Theorem 2 and hence details are omitted .

Theorem 4: Let the function f(z) defined by (2) and the function $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n (b_n \ge 0, n \in /N)$ be in the class $H(\alpha,\mu,\beta,\lambda)$. Then the function w(z) defined by $w(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$ is in the class $H(\alpha,\mu,\sigma,\lambda)$, where $0 \le \alpha < 1, 0 < \beta \le 1, 0 < \sigma \le 1$, $0 \le \mu \le 1$, $n \in /N$ and $n \ge 1$ and σ is given by:

$$\sigma \le \frac{\beta^2 \mu\alpha(n+1)(1-\alpha)}{[\beta\mu(n+\alpha)+\alpha(n+1)]^2 D_n(\lambda) - \beta^2\mu^2(n+\alpha)(1-\alpha)} .$$

Proof : We must find the largest σ such that

$$\sum_{n=1}^{\infty} \frac{[\sigma\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{\sigma\mu(1-\alpha)} a_n b_n \le 1.$$

Since f(z)and g(z) are in $H(\alpha,\mu,\beta,\lambda)$, then

$$\sum_{n=1}^{\infty} \frac{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{\beta\mu(1-\alpha)} a_n \le 1,$$

and

$$\sum_{n=1}^{\infty} \frac{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{\beta\mu(1-\alpha)} b_n \le 1.$$

By Cauchy – Schwarz inequality , we get

$$\sum_{n=1}^{\infty} \frac{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{\beta\mu(1-\alpha)} \sqrt{a_n b_n} \le 1.$$

We want only to show that

$$\frac{[\sigma\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{\sigma\mu(1-\alpha)}a_nb_n \le \frac{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{\beta\mu(1-\alpha)}\sqrt{a_nb_n}.$$

This equivalently to

$$\sqrt{a_nb_n} \le \frac{\sigma[\beta\mu(n+\alpha)+\alpha(n+1)]}{\beta[\sigma\mu(n+\alpha)+\alpha(n+1)]}.$$ Then

$$\sigma \le \frac{\beta^2\mu\alpha(n+1)(n-\alpha)}{[\beta\mu(n+\alpha)+\alpha(n+1)]^2 D_n(\lambda)-\beta^2\mu^2(n+\alpha)(1-\alpha)}.$$

3. (n,δ)- Neighborhoods on $H^{\sigma}(\alpha,\mu,\beta,\lambda)$

The next , we determine the inclusion relation involving (n,δ)-neighborhoods. Following the earlier works on neighborhoods of analytic functions by Goodman [2] and Ruscheweyh [4], but for meromorphic function studied by Liu and Srivastava [3] and Atshan [1], we define the (n,δ)- neighborhoods of a function f(z)$\in M$ by

$$N_{n,\delta}(f)=\left\{g\in M : g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_nz^n \, and \sum_{n=1}^{\infty}n|a_n-b_n|\le\delta, 0\le\delta\right\}. \qquad (15)$$

Definition 1: A function g$\in$M is said to be in the class $H^{\sigma}(\alpha,\mu,\beta,\lambda)$ if there exists a function $f\in H(\alpha,\mu,\beta,\lambda)$ such that

$$\left|\frac{g(z)}{f(z)}-1\right|<1-\sigma,(z\in U,0\le\sigma<1). \qquad (16)$$

Theorem 5: Let $f(z)\in H(\alpha,\mu,\beta,\lambda)$ and

$$\sigma=1-\frac{\delta(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)}{(\beta\mu(1+\alpha)+2\alpha(\alpha+1)(\alpha+2)-2\beta\mu(1-\alpha)}. \qquad (17)$$

Then $N_{n,\delta}(f)\subset H^{\sigma}(\alpha,\mu,\beta,\lambda)$.

Proof: Let $g\in N_{n,\delta}(f)$.Then we have from (15) that

$$\sum_{n=1}^{\infty}n|a_n-b_n|\le\delta,$$

which implies the coefficient inequality $\sum_{n=1}^{\infty}|a_n-b_n|\le\delta,(n\in /N)$.

Also since $f\in H(\alpha,\mu,\beta,\lambda)$,we have from (6)

$$\sum_{n=1}^{\infty} a_n \le \frac{2\beta\mu(1-\alpha)}{(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)},$$

so that

$$\left|\frac{g(z)}{f(z)}-1\right| = \left|\frac{\sum_{n=1}^{\infty}(a_n-b_n)z^n}{\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n}\right| \le \frac{\sum_{n=1}^{\infty}|a_n-b_n|}{1-\sum_{n=1}^{\infty}a_n}$$

$$\le \frac{\delta(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+2)}{(\beta\mu(1+\alpha)+2\alpha)(\lambda+1)(\lambda+1)-2\beta\mu(1-\alpha)} = 1-\sigma$$

Thus, by definition 1, $g \in H^{\sigma}(\alpha,\mu,\beta,\lambda)$ for σ is given by (17). This complete the proof.

4. Integral Operator :

Next , we consider integral transform of functions in the class $H(\alpha,\mu,\beta,\lambda)$.

Theorem 6: Let the function f(z) is given by (2) be in the class $H(\alpha,\mu,\beta,\lambda)$. Then the integral operator

$$F(z) = c\int_0^1 u^c f(uz)du, \quad (0<u\le 1,\ 0<c<\infty) \qquad (18)$$

is in the class $H(\alpha,\mu,\beta,\lambda)$.

Proof : Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ in the class $H(\alpha,\mu,\beta,\lambda)$.

Then

$$F(z) = c\int_0^1 u^c f(uz)du$$

$$= c\int_0^1 \left(\frac{u^{c-1}}{z} + \sum_{n=1}^{\infty} a_n u^{n+c} z^n\right)du$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{n+c+1} a_n z^n.$$

It is sufficient to show that $\sum_{n=1}^{\infty} \frac{c[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{(c+n+1)\beta\mu(1-\alpha)} a_n \le 1.$ (19)

Since $f \in H(\alpha,\mu,\beta,\lambda)$, we have

$$\sum_{n=1}^{\infty} \frac{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{\beta\mu(1-\alpha)} a_n \le 1.$$

Note that (19) is satisfied if

$$\frac{c[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{(c+n+1)\beta\mu(1-\alpha)} \le \frac{[\beta\mu(n+\alpha)+\alpha(n+1)]D_n(\lambda)}{\beta\mu(1-\alpha)}.$$

Since $\frac{c}{c+n+1} < 1$ for all $n \in /N$. Hence , we obtain the request result.

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