On ParaLindelöf and semiparaLindelöf Spaces

by
Ihsan Jabbar Kadhim Al-fatlawee
University of Al-Qadisiya
College of Computer Science and Mathematics
Department of Mathematics

E-mail: Ihsan_j78@yahoo.com.

Abstract. *W*e define an $m-$paraLindelöf, countable paraLindelöf, $m-$semipara-Lindelöf, countable semiparaLindelöf topological and study some properties of these concepts and give the relation between these concepts. And we give the relationship between the paraLindelöf space and regular (normal) space.

Keywords: $m-$ paraLindelöf, countable paraLindelöf, semiparaLindelöf, and $a-$ paraLindelöf.

1. Introduction .The concept of paracompactness is due to Dieudonne [7] . The concept of para- Lindelöf is due to Fleissner [4]. A collection of subsets of X is locally finite (resp. locally countable) [3,6,7]if every $x \in X$ has a neighborhood meeting finitely many (resp. countable many) elements of the collection. A collection has the $\sigma-$property [7]if it is the union of countabley many collection with the property. A cover (or covering) of a space (X,τ) [3,6,7] is a collection of subset of X whose union is all of X . An open cover of X is a cover consisting of open sets , and other adjective applying to subsets apply similarly to cover . If Π and $\Phi$are covers of X , we say Φ refines Π[3,6,7]if each number of Φ is contained in some member of Π . Then we say Φ refines (or refinement of) Π . A subset of a topological space (X,τ) is an F_σ (G_δ)[3,6,7]if it is a countable union (intersection) of closed (open) sets . A topological space is said to be a $P-$space [1] if every G_δ is open. A topological space (X,τ) is said to be **(countable)** ***compact*** space [3,6,7]if each **(countable)** open cover of X has a finite open subcover, and is said to be $m-$compact [5] if each open cover of X with cardinality$\leq m$ has a finite open subcover. .A topological space (X,τ) is said to be**(countable)** ***paracompact*** space [6,7] if each **(countable)** open cover of X has a locally finite open refinement, and is said to be $m-$paracompact[5] if each open cover with cardinality$\leq m$ has a locally finite open refinement . A topological space is said to be $a-$paracompact[1] if every open cover has a locally finite refinement (not necessarily open or closed). A topological space (X,τ) is said to be**(countable)** ***Lindelof*** space [3,6,7]if each **(countable)** open cover of X has a countable open subcover, and is said to be $m-$compact [5] if each open cover of X with cardinality$\leq m$ has a countable open subcover. The function $f:(X,\tau)\to(Y,\xi)$is called $(\tau-\xi)-$closed if the image of each

$\tau-$closed set is $\xi-$closed set. And is called $(\tau-\xi)-$***continuous*** if the inverse image of each $\xi-$open set is $\tau-$open set [3,6,7].

2- Main results

We shall state below some new concepts such as $m-$ paraLindelöf (where m is an infinite cardinal number), countable paraLindelöf, semiparaLindelöf and $a-$para-Lindelöf spaces. Also we give some properties of these spaces and the relation among them.

Definition 2.1.[4] A topological space is said to be ($m-$) paraLindelöf if every open cover of the space has a locally countable open refinement (with cardinality $\leq m$).

Definition 2.2. A topological space is said to be countable paraLindelöf if every countable open cover of the space has a locally countable open refinement .

Definition2.3[3,7] A topological space (X,τ) is said to be semiparacompact , if each open cover of X has a $\sigma-$locally finite open refinement

Definition 2.4.A topological space (X,τ) is said to be semiparaLindelöf, if each open cover of X has a $\sigma-$locally countable open refinement .

Definition 2.5.A topological space is said to be $a-$paraLindelöf if every open cover has a locally countable refinement (not necessarily open or closed).

Cleary that every ($m-$,countable) compact, ($m-$countable) Lindelöf, and ($m-$countable) paracompact space is ($m-$countable) paraLindelöf.

Theorem 2.6.A topological space is paraLindelöf if and only if it is countable paraLindelöf and semiparaLindelöf.

Proof. Let $\Phi=\{U_\lambda:\lambda\in\Lambda\}$ be an open covering of X. By hypothesis, Φhas a $\sigma-$locally countable open refinement, Ω say. Then $\Omega=\bigcup_{n=1}^{\infty}\Omega_n$, where each Ω_n is locally countable ,say $\Omega_n=\{V_\gamma:\gamma\in\Gamma\}$ and let $V_\delta=\bigcup\{V_\gamma:\gamma\in\Gamma\}$. Since Ω covers X, therefore $\{V_\delta:\delta\in N\}$ is a countable open covering of X. Since X is countable paraLindelöf, then the collection $\{V_\delta:\delta\in N\}$ has a locally countable open refinement $\{W_\delta:\delta\in N\}$ such that $W_\delta\subseteq V_\delta$ for each $\delta\in N$. The collection

$$\Xi=\{V_\gamma\cap W_\delta:\gamma\in\Gamma,\delta\in N\}$$

form a locally countable open refinement of Φ. Thus X is paraLindelöf. The converse is obvious.■

Now, every paracompact is semiparacompact [3,6,7]Since every locally countable collection is $\sigma-$locally countable, then we can conclude that every semiparacompact is semiparaLindelöf. Consequently, every paracompact space is semiparaLindelöf.

Theorem 2.7. Every semiparaLindelöf space is $a-$ paraLindelöf.

Proof. Let $\Phi = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X. By hypothesis Φ has $\sigma-$locally countable open refinement $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, where each where each Ω_n is locally countable ,say $\Omega_n = \{V_{n\gamma} : \gamma \in \Gamma\}$ and let $W_n = \bigcup\{V_{n\gamma} : \gamma \in \Gamma\}$. Then $\{W_n : n \in N\}$ covers X. Define $A_n = W_n - \bigcup_{i<n} W_i$. Then $\{A_n : n \in N\}$ is locally countable refinement of $\{W_n : n \in N\}$. Now consider $\{A_n \cap V_{n\gamma} : n \in N, \gamma \in \Gamma\}$. This is a locally countable refinement of Ω and hence of Φ.■

Corollary 2.8. Every paraLindelöf space is $a-$ paraLindelöf.

Theorem 2.9. [7].A regular topological space is paracompact if and only if it is $a-$paracompact.

Since every paracompact space is paraLindelöf, then we have the following :

Corollary 2.10.A regular topological space is paraLindelöf if it is $a-$paracompact.

Theorem 2.11. Every Hausdorff paraLindelöf $P-$space is regular.

Proof. Suppose that X is Hausdorff and paraLindelöf $P-$space. Let A be a closed set, and $x \in X - A$. Since X is Hausdorff, then for each $y \in A$, we can find two disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. The collection

$$\Pi = \{V_y : y \in A\} \cup \{X - A\}$$

form an open cover of X. By paraLindelöfness of X the collection Π admit locally countable open refinement

$$\Omega = \{W_\gamma : \gamma \in \Gamma\}$$

Set

$$V = \bigcup_{\gamma \in \Gamma} \{W_\gamma : W_\gamma \cap A \neq \phi\}$$

Then V is an open set containing A. Now, since Ω is locally countable, then x admit an open neighborhood N which meet countable manyW_{γ_n}, $n = 1,2,\ldots$.
If $W_{\gamma_n} \cap A \neq \phi$, $n = 1,2,\ldots$, then $W_{\gamma_n} \subseteq X - A$ is impossible. Thus there exists V_{y_n} such that $W_{\gamma_n} \subseteq V_{y_n}$. Set

$$U = N \cap \left(\bigcap_{n=1}^{\infty} U_{y_n} \right).$$

Since X is $P-$space ,then the $G_\delta-$set $\bigcap_{n=1}^{\infty} U_{y_n}$ is open. Hence U is an open set and $x \in U$. Finally, we have $U \cap V = \phi$, which implies that X is regular. ■

Theorem 2.12. Every Hausdorff paraLindelöf $P-$space is normal.

Proof. Suppose that X is Hausdorff and paraLindelöf $P-$space. Let A and B are disjoint closed subsets of X. Since X is Hausdorff, then for each $x \in A$ and $y \in B$, we can find two disjoint open sets U_x and V_x such that $x \in U_x$ and $y \in V_x$. The collection

$$\Pi = \{U_x : x \in A\} \bigcup \{X - A\}$$

form an open cover of X. By paraLindelöfness of X the collection Π has locally countable open refinement

$$\Omega = \{W_\gamma : \gamma \in \Gamma\}$$

Set

$$U = \bigcup_{\gamma \in \Gamma} \{W_\gamma : W_\gamma \cap A \neq \phi\}$$

Then U is an open set containing A. For each $y \in B$ we can find an open neighborhood N_y which meet countable many W_γ ,say $W_{\gamma_1(y)}(y), W_{\gamma_2(y)},....$(the value of n also depending on y)

If $W_{\gamma_n} \cap A \neq \phi,\ n = 1,2,...$, then $W_{\gamma_n} \subseteq X - A$ is impossible. Thus there exists V_{y_n} such that $W_{\gamma_n(y)} \subseteq V_{y_n}$. Set

$$G_y = N_y \bigcap \left(\bigcap_{n=1}^{\infty} V_{x_n} \right).$$

Since X is $P-$space ,then the $G_\delta-$set $\bigcap_{n=1}^{\infty} V_{x_n}$ is open. Hence G_y is an open set which contains y but does not meet U. Let $V = \bigcup_{y \in B} G_y$,then V is an open set containing B and disjoint from U. Therefore X is normal. ■

Theorem 2.13. Let (X, τ) be a regular space and $x \in X$ having a fundamental system of open neighborhoods $\aleph(x)$ with property that $X - N$ is $m-$paraLindelöf for each $N \in \aleph(x)$. Then the topological space (X, τ) is $m-$paraLindelöf.

Proof. Let $\Phi = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X with cardinality$\leq m$. The some member of Φ contains x, say $U_{\lambda(x)}$. Since X is regular space, therefore there exists an $N \in \aleph(x)$ such that $x \in N \subseteq Cl(N) \subseteq U_{\lambda(x)}$. Then $\Omega = \{(X - N) \bigcap U_\lambda : \lambda \in \Lambda\}$ is an open cover of $X - N$ with cardinality$\leq m$. By hypothesis Ω has a locally countable open refinement $\Psi = \{V_\gamma : \gamma \in \Gamma\}$. Set

$$\Xi = \{U_{\lambda(x)}\} \bigcup \{(X - N) \cap V_\gamma : \gamma \in \Gamma\}.$$

Then Ξ is a locally countable open refinement of Φ. Hence X is $m-$paraLindelöf. ■

Corollary 2.14. Let (X,τ) be a regular space and $x \in X$ having a fundamental system of open neighborhoods $\aleph(x)$ with property that $X - N$ is paraLindelöf for each $N \in \aleph(x)$. Then the topological space (X,τ) is paraLindelöf.

Corollary 2.15. Let (X,τ) be a regular space and $x \in X$ having a fundamental system of open neighborhoods $\aleph(x)$ with property that $X - N$ is countable paraLindelöf for each $N \in \aleph(x)$. Then the topological space (X,τ) is countable paraLindelöf.

In [1] P.T. Daniel Thanapalan state and proof the following theorem

Theorem 2.16. Let $f : X \to Y$ be a continuous closed surjection with the point inverse being Lindelöf subsets of X, where X and Y are $P-$spaces. Then if Y is paraLindelöf, so is X.

Theorem 2.17. Let $f : X \to Y$ be a continuous closed surjection with the point inverse being $m-$Lindelöf subsets of X. Then if Y is $m-$paraLindelöf, so is X.

Proof. Let $\Phi = \{U_\lambda : \lambda \in \Lambda\}$, $|\Lambda| \le m$ be an open covering of X. And let Γ be the family of all countable subsets γ of Λ then $|\Gamma| \le m$. Since $f^{-1}(y)$ is $m-$Lindelöf for every y of Y, there exists a countable subset γ of Λ such that $f^{-1}(y) \subseteq \bigcup_{\lambda\in\gamma} U_\lambda$. Let

$$V_\gamma = Y - f(X - \bigcup_{\lambda\in\gamma} U_\lambda),$$

then V_γ is open by the closedness of f and $y \in V_\gamma$ and $f^{-1}(V_\gamma) \subseteq \bigcup_{\lambda\in\gamma} U_\lambda$. Therefore

$$\Im = \{V_\gamma : \gamma \in \Gamma\}$$

is an open covering of Y with cardinality $\le m$. If Y is $m-$paraLindelöf, then there exists a locally countable open refinement $\{W_\delta : \delta \in \Delta\}$ of $\Im$. Since, for each δ there exists a $\gamma_\delta \in \Gamma$ such that $f^{-1}(W_\delta) \subseteq f^{-1}(V_{\gamma_\delta}) \subseteq \bigcup_{\lambda\in\gamma_\delta} U_\lambda$, and

$$\Pi = \left\{ f^{-1}(W_\delta) \cap U_\gamma : \delta \in \Delta, \lambda \in \gamma_\delta \right\}$$

is locally countable open refinement of Φ. Thus we get the theorem. ■

Corollary 2.18. Let $f : X \to Y$ be a continuous closed surjection with the point inverse being Lindelöf subsets of X. Then if Y is paraLindelöf, so is X.

Corollary 2.19. Let $f : X \to Y$ be a continuous closed surjection with the point inverse being countable Lindelöf subsets of X. Then if Y is countable paraLindelöf, so is X.

Theorem 2.20. Let $f: X \to Y$ be a continuous closed surjection with the point inverse being $m-$Lindelöf subsets of X .Then if Y is $m-$semiparaLindelöf, so is X .

Proof. Let $\Phi = \{U_\lambda : \lambda \in \Lambda\}$, $|\Lambda| \le m$be an open covering of X . And let Γ be the family of all countable subsets γ of Λ then $|\Gamma| \le m$. Since $f^{-1}(y)$ is $m-$Lindelöf for every y of Y, there exists a countable subset γ of Λ such that $f^{-1}(y) \subseteq \bigcup_{\lambda \in \gamma} U_\lambda$. Let

$$V_\gamma = Y - f(X - \bigcup_{\lambda \in \gamma} U_\lambda),$$

then V_γ is open by the closedness of f and $y \in V_\gamma$ and $f^{-1}(V_\gamma) \subseteq \bigcup_{\lambda \in \gamma} U_\lambda$. Therefore

$$\Im = \{V_\gamma : \gamma \in \Gamma\}$$

is an open covering of Y with cardinality $\le m$. If Y is $m-$semiparaLindelöf, then there exists a refinement $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ where every Ω_n is locally countable. Set $\Omega_n = \{W_{n\delta} : \delta \in \Delta\}$. Thus $\Omega = \bigcup_{n=1}^{\infty} \{W_{n\delta} : \delta \in \Delta\}$. Set $\Pi = \bigcup_{n=1}^{\infty} \Pi_n$, where

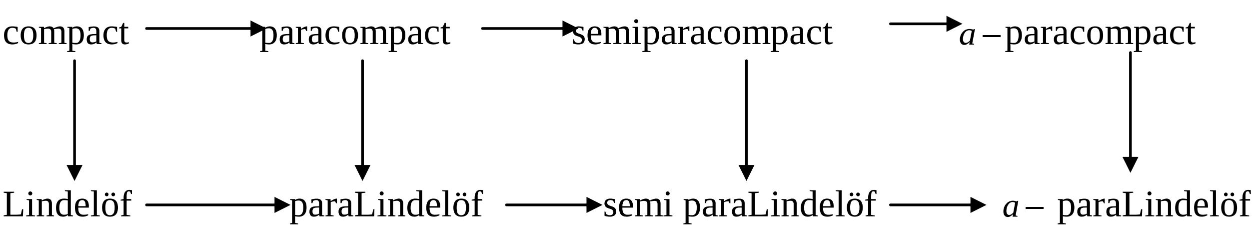
$$\Pi_n = \{f^{-1}(W_{n\delta}) \cap U_\gamma : \delta \in \Delta, \lambda \in \gamma_\delta\} .$$

Then Π is $\sigma-$locally countable open refinement of Φ.Thus we get the theorem. ■

Corollary 2.21. Let $f: X \to Y$ be a continuous closed surjection with the point inverse being Lindelöf subsets of X . Then if Y is semiparaLindelöf, so is X .

Corollary 2.22. Let $f: X \to Y$ be a continuous closed surjection with the point inverse being countable Lindelöf subsets of X . Then if Y is countable semiparaLindelöf, so is X .

It illustrates the relation among some of the spaces given in this paper:



References .

[1] P.T.Daniel Thanapalan," On Mappings of (Locally) Almost and (Locally) Nearly ParaLindelöf Spaces" India J.pure appl. Math.,23(1992),No 12. 867-872.

[2] J. Dugundji, " Topology " , Allyn and Bacon Inc., Boston, Mass, 1978.

[3] R. Engelking " General Topology " polish. Scientific publishers. Warsawx 1989.
[4] W.G. Fleissner "Normal, Not Paracompact Spaces"American Math. Soc. 7(1982).No.1, 233-236.

[5] S. Hanai, " Inverse Images of Closed Mappings", proc. Jaban, Acad. 37(1961), 298-301.
[6] J.R. Munkres, " Topology ", Prentic Hall, upper saddle River 2000.

[7] S.Willard, "General Topology " ,Addison-Wesry Pub.Co.,Inc.(1970).