JORDAN *-DERIVATIONS ON PRIME AND SEMIPRIME *-RINGS

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مشتقات-* جوردن في الحلقات-* الاولية والشبه الاولية

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Abstract

Let R be a 2-torsion free *-ring, and d: R→R be a Jordan *-derivation. In this paper we prove the following results: (1) If R is a non-commutative prime *-ring, and d(h) h + h d(h) $\in$ Z(R) for all $h \in$ H(R), then d(h) =0 for all $h \in$ H(R).(2) If R be a non-commutative prime *-ring, and d([x,y])= [x,y] for all x, $y \in$ R, then R is normal *-ring.(3) If R is a semiprime *-ring, then there is no d satisfies d(xy+yx)=xy+yx for all x, y R, where H(R)={x; $x \in$ R s.t x*=x }.

ألمستخلص

لتكن R حلقة-* طليقة الالتواء من النمط 2,و لتكن d: R→R دالة مشتقة-* جوردان في هذا البحث سنبرهن النتائج التالية:(1) اذا كان R حلقة-* اولية (غير ابدالية) وان d(h) h + h d(h) في Z(R) لكل h في H(R), فان d(h)=0 لكل h في H(R).(2) اذا كان R حلقة-* اولية (غير ابدالية) وان d([x,y])= [x,y] لكل x,y في R, فان R تكون حلقة-* سوية .(3)اذا كان R حلقة-* شبه اولية,فانه لا توجد دالة d تحقق d(xy+yx)=xy+yx لكل x,y في R. عندما H(R)={x; $x \in$ R s.t x*=x }.

1. Introduction

Throughout, R will represent an associative ring with center Z(R). A ring R is n-torsion free, if nx = 0, $x \in$ R implies x = 0, where n is a positive integer. Recall that R is prime if aRb = (0) implies a = 0 or b = 0, and semiprime if aRa = (0) implies a =0. A mapping *: R→R is called an involution if (x+y)*=x*+y*(additive), (xy)* = y* x* and (x)** = x for all x, $y \in$ R. A ring equipped with an involution is called *-ring [1]. An element x in a *-ring R is said to be hermitian if x* = x and skew-hermitian if x* = -x. The sets of all hermitian and skew-hermitian elements of R will be denoted by H(R) and S(R), respectively. If R is 2-torsion free then every $x \in$ R can be uniquely represented in the form $2x = h + k$ where $h \in$ H(R) and $k \in$ S(R). An element $x \in$ R is called normal element if $xx^* = x^*x$, and if all the elements of R are normal then R is called a normal ring (see [2]). As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$ for all $x,y,z \in$R. An additive mapping d: R$\rightarrow$R is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x,y\in$R, and is called a Jordan derivation in case $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in$ R. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [3] asserts that every Jordan derivation on a prime ring of characteristic different from 2 is a derivation. Cusack [4] generalized Herstein's theorem to 2-torsion free semiprime ring . An additive mapping d: R$\rightarrow$R is called a *-derivation if $d(xy) = d(x)y^* + xd(y)$ holds for all pairs $x,y\in$R and is called a Jordan *-derivation in case $d(x^2) = d(x)x^* + xd(x)$ is fulfilled for all $x \in$ R, the concepts of *-derivation and Jordan*-derivation were first mentioned in [5] for more details see also ([6] and [7]]). Every *-derivation is a Jordan *-derivation but the converse in general not true, for example let R be a 2-torsion free semiprime *-ring and let $a \in$ R such that $[a,x]\neq 0$, for some $x \in$ R, define a map d: R$\rightarrow$R as follows, $d(x)=ax^*-xa$ for all $x\in$ R, then d is a Jordan *-derivation but not a *-derivation.

2. The Main Results

In the present note, we explore more about Jordan *-derivations on prime and semiprime *-rings. We will provide some properties for Jordan *-derivations on semiprime *-ring. Also we will study a normalization of a non-commutative prime *-ring. We begin with the following known results.

Theorem 2.1. [5]. Let R be a non-commutative prime *-ring of characteristic different from 2, then R is normal ring if and only if there exists a nonzero commuting Jordan *-derivation.

Lemma 2.2. [8]. Let R be a prime*-ring such that a H(R) $b= 0$, where either $a \in$ H(R) or $b \in$ H(R).Then either a=0 or b=0.

Lemma 2.3. [5]. Let R be a 2-torsion free non-commutative prime *-ring, and let d: R→R be a Jordan *-derivation, then $d(c)=0$ for all $c \in Z(R)\cap H(R)$.

In the following theorem we proved that, a Jordan *-derivation d on a non-commutative prime *-ring of characteristic different from 2, which satisfies $d(h)\, h + h\, d(h) \in Z(R)$ for all $h \in H(R)$, is finish on H(R).

Theorem 2.4. Let R be a non-commutative prime *-ring of characteristic different from 2, and d: R→R be a Jordan *-derivation which satisfies $d(h)\, h + h\, d(h) \in Z(R)$ for all $h \in H(R)$, then $d(h) =0$ for all $h \in H(R)$.

To prove above theorem we need the following lemmas

Lemma 2.5. Let R be a 2-torsion free non-commutative prime *-ring, and d: R→R be a Jordan *-derivation which satisfies $d(h)\, h + h\, d(h) \in Z(R)$ for all $h \in H(R)$, then $d(h^2) =0$ for all $h \in H(R)$.

Proof: We have

$$d(h)h+hd(h)= d(h^2) \in Z(R) \text{ for all } h \in H(R), \quad (1)$$

Replace h by h^2 in (1) we get

$$d(h^2)h^2+h^2\, d(h^2)=2h^2\, d(h^2) \in Z(R) \text{ for all } h \in H(R), \quad (2)$$

Therefore,

$$[2h^2\, d(h^2),y]=0 \text{ for all } h \in H(R), y \in R, \quad (3)$$

From the relation (1), and since R is a 2-torsion free we get

$$[h^2,y]\, d(h^2)=0 \text{ for all } h \in H(R), y \in R, \quad (4)$$

Putting yz for y in (4) we obtain

$$[h^2,y]\, z\, d(h^2)=0 \text{ for all } h \in H(R), y, z \in R, \quad (5)$$

By primness of a *-ring R, we get either $d(h^2)=0$ for all $h \in H(R)$ or $h^2 \in Z(R)$ for all $h \in H(R)$, If $h^2 \in Z(R)$, then $h^2 \in Z(R)\cap H(R)$. Therefore by Lemma 2.3 we get $d(h^2) =0$ for all $h \in H(R)$.

Proof of Theorem2.4: By using Lemma2.5

$$d(h^2) =0 \quad \text{for all } h \in H(R). \quad (6)$$

Linearization the relation (6) we get

$$d(hk+kh)=d(h)k+hd(k)+d(k)h+kd(h)=0 \quad \text{for all } h, k \in H(R). \qquad (7)$$

If we replace k by $(hk+kh) \in H(R)$ in (7), and since is 2-torsion free we obtain

$$d(hkh)=d(h)k\,h+hd(k)h+hkd(h)=0 \quad \text{for all } h, k \in H(R), \qquad (8)$$

Putting h for k and $(h_1\, k\, h_1)$ for h in (8), we get

$$(h_1\, kh_1)\, d(h)\, (h_1\, kh_1)=0 \quad \text{for all } h,h_1, k \in H(R). \qquad (9)$$

Now replace k by $(h_1k\, h_1)$ in (8) we obtain

$$d(h)\,(h_1k\, h_1)\, h+ h\,(h_1k\, h_1)\, d(h)=0 \quad \text{for all } h,h_1, k \in H(R). \qquad (10)$$

Left multiplication the relation (10) by $(h_1\, k\, h_1)$, and using (9) we get

$$(h_1\, k\, h_1)\, h\, (h_1\, k\, h_1)\, d(h)=0 \quad \text{for all } h,h_1, k \in H(R). \qquad (11)$$

Linearization the relation (11) on h we get

$$(h_1\, k\, h_1)\, h\, (h_1\, k\, h_1)\, d(l) + (h_1\, k\, h_1)\, l\, (h_1\, k\, h_1)\, d(h)=0 \quad \text{for all } h,h_1, l, k \in H(R). \quad (12)$$

Replace l by (bab) in (12), we get

$$(h_1\, k\, h_1)\,(bab)\,(h_1\, k\, h_1)\, d(h)=0 \quad \text{for all } h,h_1,a,b,k \in H(R), \qquad (13)$$

Left and right multiplication (13) by b we get

$$b(h_1\, k\, h_1)\, bab\, (h_1\, k\, h_1)\, d(h)\, b=0 \quad \text{for all } h,h_1,a,b,k \in H(R), \qquad (14)$$

Setting $b=h=h_1$, then by using (6) we get

$$(h^2\, k\, h^2)\;\; a\,(h^2\, k\, h^2)\, d(h)=0 \quad \text{for all } h,h_1,a,b,k \in H(R), \qquad (15)$$

Since $(h^2\, k\, \mathrm{h}^2) \in H(R)$, then by using Lemma2.2, we get, either $(h^2\, k\, h^2)=0$, or $(h^2\, k\, h^2)$ $d(h)=0$ for all $h, k \in H(R)$. Therefore

$$(h^2\, k\, h^2)\, d(h)=0 \quad \text{for all } h,k \in H(R). \qquad (16)$$

Then also by using Lemma 2.2, we obtain

$$h^2\, d(h)=0 \quad \text{for all } k \in H(R), \qquad (17)$$

Linearization the above relation we get

$$(h\, k+kh)\, d(k)+(\, h\, k+kh)d(h)+h^2\, d(k)+k^2\, d(h)=0 \quad \text{for all } h,k \in H(R). \qquad (18)$$

Replace k by $-\, k$ in the above relation and comparing the relation so obtained with the relation (18) we get

$$(h\,k+kh)d(h)+h^2\,d(k)=0 \qquad \text{for all } h,k \in H(R).$$

Putting ($h\,k+k\,h$) for k in the above relation we obtain

$$2\,h\,k\,h\,d(h)+h^2\,k\,d(h)=0 \qquad \text{for all } h,k \in H(R). \qquad (19)$$

Right multiplication the above relation by h and using (6) we get

$$h^2\,k\,d(h)\,h=0 \qquad \text{for all } h,k \in H(R).$$

By using Lemma2.2, we get if $h^2=0$, then from relation (20), we get $h\,k\,d(h)\,h=0$ for all $k \in H(R)$, therefore we obtain $d(h)\,h=0$ for all $h \in H(R)$, let $h=h+h_1\,k\,h_1$, then we get

$$d(h)\,h_1\,k\,h_1=0 \quad \text{for all } h,h_1,k \in H(R).$$

Then from above relation and Lemma2.2 we get $d(h)=0$ for all $h \in H(R)$. The proof of Theorem 2.4, is complete. □

In the following proposition we will give a condition on a Theorem2.4 to get R is a normal *-ring.

Proposition 2.6. Let R be a non-commutative prime *-ring of characteristic different from 2, and d: R→R be a Jordan *-derivation which satisfies $d(h)\,h+h\,d(h) \in Z(R)$ for all $h \in H(R)$, and $[d(s),h] \in Z(R)$ for all $h \in H(R)$, and $s \in S(R)$, then R is normal *-ring.

Proof: we have, $[d(s),h] \in Z(R)$ for all $h \in H(R)$, and $s \in S(R)$, Since $h^2 \in H(R)$, for all $h \in H(R)$, $[d(s), h^2] \in Z(R)$, for all $s \in S(R)$, and $h \in H(R)$. By assumption $[d(h),s] \in Z(R)$ for all $h \in H(R)$, $s \in S(R)$, then we get

$$2h[d(s), h] \in Z(R), \text{ for all } s \in S(R), \text{ and } h \in H(R).$$

Hence,

$$2[d(s), h[d(s), h]]=0 \quad \text{for all } s \in S(R), \text{ and } h \in H(R).$$

Since $[d(s),h] \in Z(R)$, and R is a 2-torsion free, then from above relation we get

$$[d(s),h]^2=0 \text{ for all } s \in S(R), \text{ and } h \in H(R).$$

By the semiprimness of R, we get

$$[d(s),h]=0 \text{ for all } s \in S(R), \text{ and } h \in H(R).$$

To prove $[d(x),x]=0$, Since R be a 2-torsion free we only show, $4[d(x),x]=0$ for all $x \in R$, we have for all $x \in R$ then ($2x=s+h$ for $s \in S(R)$, and $h \in H(R)$), therefore

$$4[d(x),x]=[d(2x),2x]=[d(s+h), s+h] \quad \text{for } s \in S(R), \text{ and } h \in H(R).$$

Hence,

$$4[d(x),x]=[d(s),s]+[d(s),h]+[d(h), h]+[d(h), s]$$

From above relation and Theorem2.4, and characteristic of R not equal 2, we get

$$[d(x),x] =0 \quad \text{for all } x \in R.$$

Then from Theorem2.1, we get R is normal *-ring. □

Daif and Bell[9] established that a semiprime ring R must be commutative if it admits a derivation d such that $d([x,y])=[x,y]$ for all $x, y \in R$. In the following theorem we will prove if R be a 2-torsion free non-commutative prime *-ring, and d: R→R be a Jordan *-derivation which satisfies $d([x,y])= [x,y]$ for all $x, y \in R$, then R is normal *-ring, but under some conditions on a *-ring R.

Theorem 2.7. Let R be a 2-torsion free non-commutative prime *-ring, and let d: R→R be a Jordan *-derivation which satisfies $d([x,y])= [x,y]$ for all $x, y \in R$, then R is normal *-ring.

Proof: we have

$$d([x,y])=[x,y] \quad \text{for all } x, y \in R, \tag{20}$$

Since $[x^2,y]=[x,y]x+x[x,y]$ then from (20) we get

$$\begin{aligned} d([x,y]x+x[x,y]) &= [x,y]\, x^*+[x,y]d(x)+d(x)[x,y]^*+x[x,y] \\ &= [x,y]x+x[x,y] \quad \text{for all } x, y \in R, \end{aligned} \tag{21}$$

Replace x by $[h,s] \in H(R)$, where $h \in H(R)$, and $s \in S(R)$, in (21) and using (20)we obtain, $x[x,y]^*+[x,y]x=0$ for all $y \in R$, Replace y by xy, we get, $x[x,y]^*x+x[x,y]x=0$ for all $y \in R$, hence we get

$$[[x,y],x]\, x =0 \quad \text{for all } y \in R, \tag{22}$$

Define an additive mapping, f_x: R→R by $f_x(y)=[x,y]$, then f_x is inner derivation and from (22) we get

$$f_x^2\,(y)\, x =0 \quad \text{for all } y \in R. \tag{23}$$

Therefore, one can show from relation (23) that

$$x\, f_x^2\,(y)=0 \quad \text{for all } y \in R. \tag{24}$$

Putting yw from y in (23) we get, $f_{x^2}(y)wx+2f_x(y)f_x(w)x=0$ for all $y, w \in R$, left multiplication by x and using (24), therefore since R is a 2-torsion free we obtain

$$xf_x(y)f_x(w)x=0 \quad \text{for all } y, w \in R, \tag{25}$$

Putting yv for y in (25) we get, $xf_x(y)vf_x(w)x+ xyf_x(v)f_x(w)x=0$ for all $y,w,v \in R$, replace v by xv, and using (25) we get, $xf_x(y)x\ v\ f_x(w)x =0$ for all $y,w,v \in R$, Setting $y=w$, and putting vx for v, we obtain, $xf_x(y)x\ v\ x\ f_x(y)x =0$ for all $y,v \in R$, By primness of a *-ring R, we get, $x\ f_x(y)\ x =0$ for all $y \in R$, Putting yw from y we get, $x\ f_x(y)\ w\ x+ x\ yf_x(w)\ x=0$ for all $y, w \in R$, since $x\ f_x(y)\ w\ x-xf_x(y)xw-yxf_x(w)x+ x\ yf_x(w)\ x=0$ for all $y, w \in R$, therefore $f_x(y)\ f_x(w)\ x=xf_x(y)f_x(w)$ for all $y, w \in R$. Then from relation (25) we get, $x^2f_x(y)f_x(w)=0$ for all $y, w \in R$, replace w by rx^2y we obtain, $x^2f_x(y)rx^2\ f_x(y)=0$ for all $y, w \in R$, Since R is a *-prime ring we get, $x^2f_x(y) = 0$ for all $y\in R$, Putting wy from y in the above relation we get

$$x^2\ w\ f_x(y)=0 \quad \text{for all } y, w \in R, \tag{26}$$

Putting yw for w in the relation (26) we get

$$x^2\ y\ w\ f_x(y)=0 \quad \text{for all } y, w \in R, \tag{27}$$

Left multiplication the relation (27) by y we get

$$y\ \ x^2\ w\ f_x(y)=0 \quad \text{for all } y, w \in R, \tag{28}$$

Comparing the relations (27) and (28) we obtain

$$[x^2,y]\ w\ [x,y]=0 \quad \text{for all } y, w \in R, \tag{29}$$

Replace w by wx in (29) we get

$$[x^2,y]\ w\ x\ [x,y]=0 \quad \text{for all } y, w \in R, \tag{30}$$

Right multiplication the relation (29) by x we get

$$[x^2,y]\ w\ [x,y]\ x=0 \quad \text{for all } y, w \in R, \tag{31}$$

Comparing the relations (30) and (31) we obtain

$$[x^2,y]\ w\ [x^2,y]=0 \quad \text{for all } y, w \in R, \tag{32}$$

By primness of a *-ring R, $x^2 \in Z(R)$, and hence $x^2 \in Z(R)\cap H(R)$. Therefore by Lemma 2.3 we get $d(x^2) =0$, then we obtain $0=d(x^2)=2\ x^2$, therefore $x^2=0$, from relations (21) one can obtain

$$x\ [x,k]=0 \text{ for all } k \in H(R),$$

Therefore, $x\ k\ x=0$ for all for all $k \in H(R)$, then by using Lemma 2.2 we get$[s,h]=0$

for all $h \in$ H(R), s $\in$ S(R), hence we obtain R is a normal *-ring. □

M. Hongan In [10] proved that, if R is a 2-torsion free ring with an identity element. Then there is no a derivation d: R→R such that d($xy+yx$)=$xy+yx$ for all $x, y \in$ R. In the following Proposition we will give a result similar to the result of M. Hongan [10], but in case Jordan*-derivation.

Proposition 2.8. Let R be a 2-torsion free semiprime *-ring, then there is no Jordan*-derivation d: R→R which satisfies d($xy+yx$)=$xy+yx$ for all $x, y \in$ R.

To prove above proposition we the following lemma

Lemma 2.9. Let R be a semiprime *-ring, if there exist an element $h \in$H(R) which satisfied $h\,x\,h$=0 for all $x \in$H(R), then h=0.

Proof: We have, $h\,x\,h$=0 for all $x \in$H(R), Since ($y+y$*) $\in$H(R), for all $y \in$R, hence $h\,y\,h$=- $h\,y^*\,h$ for all $y \in$ R. Also since (yhy*) $\in$H(R), therefore $h\,y\,h\,y^*\,h$=- $h\,y\,h\,y\,h$=0 for all $y \in$ R, linearization we get, $h\,y\,h\,z\,h$+ $h\,zh\,y\,h$=0 for all $z,y \in$ R, left multiplication by $y\,h$ we get, $h\,y\,h\,z\,h\,y\,h$=0 for all $z,y \in$ R. By the semiprimness of R, we get h=0.

Proof of Proposition2.8: If d is a non-zero Jordan *-derivation, then we have

$$d(xy+yx)=d(x)y^*+xd(y)+d(y)x^*+yd(x)=xy+yx \quad \text{for all } x, y \in R. \quad (33)$$

Setting $y=ab+ba$, $x=cd+dc$ where $a,b,c,d \in$ H(R), then from (33) we get

$$xy+yx=0 \quad \text{for all } y=ab+ba,\ x=cd+dc \text{ where } a,b,c,d \in H(R), \quad (34)$$

Now setting $x= (cd+dc)^2$, $y=(ab+ba)$ in (33), we get

$$x^2y+yx^2=0 \quad \text{for all } y=ab+ba,\ x=cd+dc \text{ where } a,b,c,d \in H(R), \quad (35)$$

Left multiplication the relation (34) by x we get

$$x^2y+xyx=0 \quad \text{for all } y=ab+ba,\ x=cd+dc \text{ where } a,b,c,d \in H(R), \quad (36)$$

Right multiplication the relation (34) by x we get

$$y\,x^2+xyx=0 \quad \text{for all } y=ab+ba,\ x=cd+dc \text{ where } a,b,c,d \in H(R), \quad (37)$$

According to (35), (36) and (37) we get

$$x(\,ab+ba\,)\,x=0 \quad \text{for all } a,b \in H(R), \quad (38)$$

Replace a by $ab+ba$ in (49) we obtain

$$x\,a\,b\,a\,x=0 \quad \text{for all } a,b \in H(R),$$

Lift and right multiplying by a, we get

$$a\,x\,a\,b\,a\,x\,a=0 \quad \text{for all } a,b \in H(R),$$

By Lemma2.9 we get

$$a\,(cd+dc)\,a=0 \quad \text{for all } a,d,c \in H(R),$$

Replace c by $cd+dc$ we get H(R) =0, therefore $x=-x^*$ for all $x \in$ R, hence R=0, which is contradiction. Then assume d=0, therefore $xy+yx=0$ for all $x,y \in$ R, then also we get contradiction. Therefore d(x)=0 for all $x \in$ R. □

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