Global Estimates for Monotone Approximation

[1] *Malik Saad Al-Muhja*

Abstract. *In 1995 Kopotun [3], introduced a paper approximation of* k *-monotone function . In this paper , we show that if* f *and* g *are* k *-monotone functions on* $[a,b]$, *such that* $f^{(\ell_j-1)}(x_j)=g^{(\ell_j-1)}(x_j)$ *for all* $0\le j\le N$ *, then a direct theorem for the rate of* k *-monotone approximation in* L_p *-spaces of the* k *th usual modulus of smoothness .*

مبرهنات عامة في التقريب الرتيب

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المستخلص : في عام 1995م قدم الرياضي الكندي كبوتون [3] بحث في التقريب للدالة الرتيبة k . وفي هذا البحث بينا أذا كانت f,g دوال رتيبة من الدرجة k معرفة على الفترة $[a,b]$ بحيث آن مشتقات هذه الدوال تكون متساوية عند نقاط الاندراج فان المبرهنة المباشرة تكون التقريب الرتيب k في الفضاءات L_p بدلالة مقياس النعومة ω_k . وان النتيجة لهذه المبرهنة تكون التقريب الرتيب k ولكن بدلالة مقياس النعومة ω_k^{ϕ} .

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Global Estimates for Monotone Approximation

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Abstract. *In 1995 Kopotun [3], introduced a paper approximation of k-monotone function . In this paper , we show that if f and g are k-monotone functions on $[a,b]$, such that $f^{(\ell_j-1)}(x_j)=g^{(\ell_j-1)}(x_j)$ for all $0 \le j \le N$, then a direct theorem for the rate of k-monotone approximation in L_p-spaces of the k th usual modulus of smoothness .*

1. Introduction.

In [5] Sammer , departure from these previous works is that you will prove simultaneous direct estimates for the rate of polynomial approximation in terms of the Ditzian-Totik modulus of smoothness . An other variant of her work in [5] is to consider the constrained and unconstrained problem of coapproximation and approximation of k-monotone and other functions in $L_P[-1,1]$, $0<p<1$.

Let $L_p[a,b]$, $0<p<\infty$, be the set of all measurable functions on $[a,b]$ such that $\|f\|_{L_p[a,b]}<\infty$, where

$$\|f\|_{L_P[a,b]} := \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{P}} .$$

Let us recall some definitions of moduli of smoothness used throughout this paper . The k *th symmetric difference* of f is given by:

$$\Delta_h^k(f,x,[a,b]) := \begin{cases} \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} f\left(x-\frac{kh}{2}+ih\right) & x\pm\frac{kh}{2}\in[a,b] \\ 0 & o.w. \end{cases} , o<h<1 .$$

The k *th usual modulus of smoothness* of $f\in L_p[a,b]$ is defined by :

$$\omega_k(f,\delta,[a,b])_p := \sup_{0\le h\le\delta}\left\|\Delta_h^k(f,\cdot,[a,b])\right\|_{L_p[a,b]} .$$

The Ditzian-Totik modulus of smoothness whichis defined for such an f, as follows

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$$\omega_k^{\phi}(f,\delta,[a,b])_p := \sup_{0\le h\le\delta}\left\|\Delta^k_{h\phi(.)}(f,\cdot,[a,b])\right\|_{L_p[a,b]} .$$

It will be omitted for the sake of simplicity ,

$$\omega_k(f,[a,b])_P = \omega_k(f,\delta,[a,b])_P$$

For $f \in L_p[a,b]$, let

$$E_n(f)_P = \inf_{p_n\in\Pi_n}\|f - p_n\|_P ,$$

denote the *degree of unconstrained approximation*, where Π_n, the set of all polynomials of degree $\le n$, and n is natural , i.e., $n \in \mathrm{N}$.

2. *Notations and Definitions.*

Let $\theta_N = \theta_N[a,b] = \{x_i\}_{i=0}^N = \{a = x_\circ \le ... \le x_{N-1} \le x_N = b\}$ be a partition of $[a,b]$ into $\le N$, subintervals . We denote $\|\theta_N\| = \|\theta_N[a,b]\| = \max_{0\le i\le N-1}\{x_{i+1} - x_i\}$, the length of the largest interval in that partition (the norm of the partition), and denote the length of the smallest interval by

$$\langle\theta_N\rangle = \langle\theta_N[a,b]\rangle = \min_{0\le i\le N-1}\{x_{i+1} - x_i\} .$$

We call a partition $\theta_N[a,b]$, *almost uniform* if

$$\|\theta_N[a,b]\| \le 3\langle\theta_N[a,b]\rangle ,[4] .$$

It was shown in [4], that any partition $\theta_N[a,b]$ of $[a,b]$ can be made almost uniform by deleting some of the partition points : For any partition $\theta_N[a,b]$, there exists a superpartition $\tilde{\theta}_N[a,b]$ (i.e., partition $\tilde{\theta}$ is obtained from θ, by deleting some of the points of θ)
which is almost uniform and such that

$$\|\theta\| \le \langle\tilde{\theta}\rangle \le \|\tilde{\theta}\| \le 3\|\theta\| .$$

In this result we obtain a relationship between two functions by using a partition θ_N .

3. The Main Result and Auxiliary Lemma.

Let us introduce the following auxiliary Lemma .

***Lemma A.* [2]**
Let $k \ge 2$, *and an interval* $I \subset [a,b]$ *be such that* $b - a \le \mathrm{A}.dist(I,\{a,b\})$, *for some* $A \in R$, *and* $\{t_1,...,t_{k-1}\}$ *be a set of any* $k-1$, *points in* I . *If* f,g *in* $\Delta^k[a,b]\cap L_P[a,b]$ *are such that* $f^{(\ell_j-1)}(t_j) = g^{(\ell_j-1)}(t_j)$, *for all* $0 \le j \le k$, *(where* $t_\circ = a$, $t_k = b$, *and* $\ell_j = \ell_j(\{t_i\}_{i=0}^k)$ *), then*

$$\|f - g\|_P \le C\min\{\omega_k(f,[a,b])_P, \omega_k(g,[a,b])_P\},$$

where the constant C, *depends only on* k *and* A .

Theorem I.
Suppose that $N \ge k \ge 2$, *and let* $\theta_N = \theta_N[a,b] = \{a = x_\circ \le ... \le x_{N-1} \le x_N = b\}$ *be a*

partition of $[a,b]$ *into* $\leq N$ *, subintervals such that* $\|\theta_N\| < \dfrac{b-a}{3(k-1)}$ *. Also , let* f, g *in* $\Delta^k[a,b] \cap L_P[a,b]$ *be such that* $f^{(\ell_j - 1)}(x_j) = g^{(\ell_j - 1)}(x_j),\ 0 \leq j \leq N$ *. Then ,*

$$\|f-g\|_P \leq C \min\{\omega_k(f, \|\theta_N\|, [a,b])_P, \omega_k(g, \|\theta_N\|, [a,b])_P\},$$

where the constant C *, depends only on* k *.*

Corollary II.

Suppose that $N \geq k \geq 2$*, and let* $\theta_N = \theta_N[a,b] = \{a = x_\circ \leq \ldots \leq x_{N-1} \leq x_N = b\}$ *be a partition of* $[a,b]$ *into* $\leq N$ *, subintervals such that* $\|\theta_N\| < \dfrac{b-a}{3(k-1)}$ *. Also , let* f, g *in* $\Delta^k[a,b] \cap L_P[a,b]$ *be such that* $f^{(\ell_j - 1)}(x_j) = g^{(\ell_j - 1)}(x_j),\ 0 \leq j \leq N$ *. Then ,*

$$\|f-g\|_P \leq C \min\{\omega_k^\phi(f, N^{-1}, [a,b])_P, \omega_k^\phi(g, N^{-1}, [a,b])_P\},$$

where the constant C *, depends only on* k *.*

PROOF OF THEOREM I.

Without loss of generality , assume that

$$\omega_k(f, \|\theta_N\|, [a,b])_P \leq \omega_k(g, \|\theta_N\|, [a,b])_P \quad .$$

An almost uniform partition of $[a,b]$, $\tilde{\theta}$ is a superpartition of $\theta_N[a,b]$, such that

$$\|\theta_N\| \leq \langle\tilde{\theta}\rangle \leq \|\tilde{\theta}\| \leq 3\|\theta_N\| \quad .$$

Since $\|\theta_N\| < \dfrac{b-a}{3(k-1)}$, this implies that $\|\tilde{\theta}\| < \dfrac{b-a}{(k-1)}$, and therefore $\tilde{\theta}$, consists of at least k intervals . Now , it is sufficient to prove (theorem I), for the partition $\tilde{\theta}$, instead of θ_N .

Equivalently , we can assume that the original partition θ_N is almost uniform . Hence , we finish the proof of the theorem assuming that $\|\theta_N\| \leq 3\langle\theta_N\rangle$, and that θ_N , consist of at least k intervals .

Let i , $0 \leq i \leq N-1$ be fixed , and denote $\alpha(i) = \max\{0, i-k+1\}$, and $J_i = [x_{\alpha(i)}, x_{\alpha(i)+k}]$. Since $\theta_N[a,b]$, consists of at least k intervals , then $[x_i, x_{i+1}] \subset J_i \subset [a,b]$. Taking into account that $|J_i| \approx \|\theta_N\|$, we can now apply theorem A, with $[a,b] = J_i$, $t_j = x_{\alpha(i)+j}$, $0 \leq j \leq k$, and $I = [t_1,\ t_{k-1}]$, to conclude that

$$\begin{aligned}\|f-g\|_{L_P[x_i, x_{i+1}]} &\leq \|f-g\|_{L_P(J_i)} \\ &\leq C\omega_k(f, J_i)_P \\ &\leq C\omega_k(f, \|\theta_N\|, [a,b])_P,\end{aligned}$$

where C, depends only on k, because we can choose a constant A, in the statement of theorem A, to be $3k$, since

$$\frac{|J_i|}{dist(I, \{x_{\alpha(i)}, x_{\alpha(i)+k}\})} \leq \frac{k\|\theta_N\|}{\langle\theta_N\rangle} \leq 3k \ . \ [4]$$

We get

$$\left|J_i\right| \le 3k\, dist\left(I, \left\{x_{\alpha(i)}, x_{\alpha(i)+k}\right\}\right) .$$

Since *there exists* i , $0 \le i \le N-1$, such that $\|f-g\|_{L_P[a,b]} = \|f-g\|_{L_P[x_i, x_{i+1}]}$,

then

$$\|f-g\|_{L_P[a,b]} \le C\omega_k\left(f, \|\theta_N\|, [a,b]\right)_P .$$

□

PROOF OF COROLLARY II.

Let $[a,b] \subseteq [-1,1]$ be $\theta_N[a,b] = \{x_i\}_{i=0}^N$, a partition and

$$C_2^{-1}\|\theta_N\| \le \langle\theta_N\rangle \le C_2\|\theta_N\| . \qquad (1)$$

Since ,

$$\|f-g\|_P \le C \min\left\{\omega_k\left(f, \|\theta_N\|, [a,b]\right)_P, \omega_k\left(g, \|\theta_N\|, [a,b]\right)_P\right\} .$$

Then from (1), and Lemma 2.2.5 in [5], if we assume $\langle\theta_N\rangle = \dfrac{c}{N}$, $c > 0$, we get

$$\|f-g\|_P \le C \min\left\{\omega_k^\phi\left(f, N^{-1}, [a,b]\right)_P, \omega_k^\phi\left(g, N^{-1}, [a,b]\right)_P\right\} .$$

□

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