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Existence of the Global Attractor in the Asymptotically Smooth Random Dynamical Systems

Zainab Hayder Hasan[a, *], Ihsan Jabbar Kadhim[b]

[a]University of Kufa, College of Education for Girls, Department of Mathematics, , Najaf, Iraq.
[b]University of Al-Qadisiyah, College of Science, Department of Mathematics, Al Diwaniyah, Iraq.
.Email: zainabh.alali@uokufa.edu.iq, ihsan.kadhim@qu.edu.iq

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ABSTRACT

The main objective of this article is to prove that convergence smoothness and asymptotic convergence are interchangeable, as well as to provide some sufficient conditions that ensure the random dynamical system is asymptotically compact. The system of infinite symmetric stochastic dynamics is described using the Kuratowski measure of non-compactness. Additionally, several results are presented at the end of this paper that provide useful criteria for the convergence smoothness and compressibility of stochastic dynamics. Furthermore, we discuss the asymptotic smoothness of random dynamics and explain some key properties of these systems by proving some equivalent statements of the concept. And since the global random attractor is the most practical idea when considering systems with infinite dimensions, it was also discussed in this research. The pointwise decay condition was used more appropriately than the (bounded) decay in some cases.

MSC..

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1. Introduction

The idea of asymptotically smooth systems was presented by Hale, LaSalle,and Slemrod [3] in 1972, along with broad existence proofs of maximal compact invariant sets. Asymptotic smoothness was first proposed by J. Hale in [2] proved that all continuous transformations across vector spaces of finite dimensions are asymptotically smooth, but this isn't the case for transformations between metric spaces or infinite dimensional Banach spaces.

∗Corresponding author: Zainab Hayder Hasan

Email addresses: *zainabh.alali@uokufa.edu.iq*

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The fact that a compact set that attracts locally points also attracts locally compact sets is a crucial characteristic of an asymptotically smooth map from a dynamical. For asymptotic smooth transformations in infinite dimensional spaces are provided by J. Hale [3]. To ensure investigate the long-term dynamics of infinite-dimensional systems, certain asymptotic compactness criteria are also required. These features can be expressed in a variety of ways, depending on the relevant model's structure (see, for example, [12]). Many researchers have been examining asymptotic compactness and global attractors in the last few years to ensure better understand the long-term behavior of dynamical systems, both stochastic and deterministic, see (for example), [4-11]. Furthermore, research has been done on the many kinds of dissipative RDS [12,13] and how they relate to global attractors. By examining asymptotic smooth RDSs in infinite dimensional spaces and proving the existence of a global attractor based on asymptotic smoothness and point dissipatveness, we will investigate the behavior of stochastic dynamical systems in this article. We'll talk about concepts pertaining to the article's subject in Section 2. A few fundamental characteristics of asymptotically smooth RDSs are demonstrated in Section 3. In Section 4, the preconditions for the existence of global attractors are enumerated.

2. Basic concepts

We have devoted this section to mentioning some fundamental ideas pertaining to the article's subject.

Notation 2.1

we use the notation $\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega) \rightrightarrows K(\omega)$ as $t \longrightarrow \infty$ in the case when:

$$\lim_{t\longrightarrow+\infty} d_X(\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)|K(\omega)) = 0$$

where $d_X(A|B) = \sup_{x\in A} \operatorname{dist}_X(x,B)$ holds.

Definition 2.1 [14]

A **metric dynamical system** (MDS) is an invariant action $\theta: \mathbb{R} \times \Omega \longrightarrow \Omega$ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$.

Definition 2.2 [14]

A **random dynamical system** (RDS) is a pair (θ,φ) including an MDS θ and a cocycle φ over θ of continuous mappings of X.

Definition 2.3 [14]

If (X,d) be a metric space.

1. The multivalued function $\omega \longmapsto D(\omega) \neq \emptyset$ is said to be random set if for every $x \in X$ the function

$$\omega \longmapsto \operatorname{dist}_X(x,D(\omega))$$

is measurable. If $D(\omega)$ is closed for every $\omega \in \Omega$, then $D(\omega)$ is said to be **closed random set**.

2. A random set $D(\omega)$ is called bounded if $D(\omega)$ containes in a random ball.

3. A variable $\varepsilon: \Omega \longrightarrow \mathbb{R}$ is said to be **tempered random variable** (TRV) if

$$\lim_{t\longrightarrow +\infty} \frac{1}{|t|} \log|\varepsilon(\theta_t \omega)| = 0.$$

Definition 2.4 [14]

A random set $D(\omega)$ in an RDS (θ, φ) is said to be **forward invariant** (**backward invariant**) then $\varphi(t, \omega)S(\omega) \subseteq S(\theta_t \omega)$ $(S(\theta_t \omega) \subseteq \varphi(t, \omega)S(\omega))$ respectively) for every $t > 0$ and $\omega \in \Omega$.

Definition 2.5 [14]

The family $\mathcal{U}$ of closed random sets is said to be a **universe** of sets when it is closed under inclusions.

Definition 2.6 [14]

Let (θ, φ) be an RDS and $\mathcal{U}$ be a universal . A closed random set $\{B(\omega)\}$ is said to be an **absorbing random set** for the RDS (θ, φ) in $\mathcal{U}$ if $B(\omega)$ attracts every member in $\mathcal{U}$, i.e. if for any $U \in \mathcal{U}$ and for any ω there exists $t_0(\omega)$ such that

$$\varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega) \subset B(\omega) \text{ for all } t \geq t_0(\omega) \text{ and } \omega \in \Omega.$$

Definition 2.7 [13,14]

An RDS (θ, φ) is said to be **dissipative** in a universe $\mathcal{U}$, if there is an absorbing set in $\mathcal{U}$ contains in a random ball.

Definition 2.8 [13, 14]

An RDS (θ, φ) on a Banach space X is said to be

1. **compact** if it has an absorbing compact random set;
2. **conditionally compact** if any bounded invariant random set $D(\omega)$ there exist $t_D > 0$ and a compact random set $K(\omega)$ in $\overline{D(\omega)}$, wih $\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset K(\omega)$ for all $t \geq t_D$;
3. **asymptotically compact** if the following condition: for any bounded random set $B(\omega)$ such that the $\gamma_B^\tau(\omega) = \bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega)$ is bounded for some $\tau \geq 0$, the sequence $\{\varphi(t_n, \theta_{-t_n}\omega)x_n\}$ is relatively compact, where $x_n \in B(\theta_{-t_n}\omega)$ and $t_n \longrightarrow \infty$.

Definition 2.9 [13,14]

For any random set $M(\omega)$ we define the omega-limit set by the random set:

$$\omega \mapsto \Gamma_M(\omega) := \cap \overline{\gamma_M^t(\omega)} = \cap \overline{\cup\, \varphi(\tau, \theta_{-\tau}\omega)M(\theta_{-\tau}\omega)},\ \tau \geq t > 0.$$

Proposition 2.1 [14]

For any RDS (θ,φ), $x \in \Gamma_M(\omega)$ if and only if there exist divergent sequences $t_n \to +\infty$ and $y_n \in M(\theta_{-t_n}\omega)$ such that:

$$x = \lim_{n\to+\infty} \varphi\big(t_n, \theta_{-t_n}\omega\big) y_n .$$

3. Asymptotically smooth RDSs

The focus of this section is the investigation of the long-term behavior of smooth RDS. We will give the definition of an asymptotically smooth RDS and then demonstrate some of the main characteristics of these systems. Also, some equivalent statements for the concept of an asymptotically smooth RDS are given.

Definition 3.1

An RDS (θ,φ) is said to be **asymptotically smooth** if for each forward invariant bounded random set $D(\omega)$, there is a compact random set $K(\omega)$ in $\overline{D(\omega)}$ such that:

$$\lim_{t\longrightarrow+\infty} \sup_{x\in D(\theta_{-t}\omega)} \inf_{y\in K(\omega)} \|\varphi(t,\theta_{-t}\omega)x - y\| = 0 . \qquad (1)$$

Some lemmas for studying smooth systems are listed below.

Lemma 3.1

If $B_0(\omega)$ is an absorbing bounded random set for an RDS (θ,φ) in the universe $\mathcal{U}$, then the random set:

$$B_*(\omega) = \bigcup_{t\geq t_*} \varphi(t,\theta_{-t}\omega) B_0(\theta_{-t}\omega),$$

admits the following properties for some $t_* \geq 0$:

1. $B_*(\omega)$ is forward invariant;
2. $B_*(\omega)$ is bounded;
3. $B_*(\omega)$ is an absorbing in $\mathcal{U}$.

Proof

1. To show that $\varphi(\tau,\omega)B_*(\omega) \subseteq B_*(\theta_\tau\omega)$ for every $\tau > 0$ and $\omega \in \Omega$.

$$\varphi(\tau,\omega)B_*(\omega) = \bigcup_{t\geq t_*} \varphi(\tau,\omega)\varphi(t,\theta_{-t}\omega)B_0(\theta_{-t}\omega)$$

$$= \bigcup_{t\geq t_*} \varphi(\tau,\theta_{-\tau}\omega)\varphi(t,\theta_{-\tau}\theta_{-\tau}\omega)B_0(\theta_{-\tau}\theta_{-t}\omega)$$

$$= \bigcup_{t\geq t_*} \varphi(\tau+t,\theta_{-(\tau+t)}\omega)B_0(\theta_{-(\tau+t)}\omega) = B_*(\theta_\tau\omega).$$

2. Since $B_0(\omega)$ is bounded, then so is $B_0(\theta_{-t}\omega)$. Since $\varphi(t,\theta_{-t}\omega): X \longrightarrow X$ is linear and hmeomorphism, then $B_*(\omega) = \bigcup_{t\geq t_*} \varphi(t,\theta_{-t}\omega)B_0(\theta_{-t}\omega)$ is bounded.

3. Suppose that $B_0(\omega)$ is an absorbing random set in $\mathcal{U}$, then if for any $U \in \mathcal{U}$ and for any ω there exists $\tau_0(\omega)$ such that $\varphi(\tau, \theta_{-\tau}\omega)U(\theta_{-\tau}\omega) \subset B_0(\omega)$ for all $\tau \geq \tau_0(\omega)$ and $\omega \in \Omega$. Thus

$$\varphi(\tau, \theta_{-\tau}\theta_{-t}\omega)U(\theta_{-\tau}\theta_{-t}\omega) \subset B_0(\theta_{-t}\omega) \text{ for all } \tau \geq \tau_0(\omega) \text{ and } \omega \in \Omega.$$

So,

$$\varphi(t, \theta_{-t}\omega)\varphi(\tau, \theta_{-\tau}\theta_{-t}\omega)U(\theta_{-\tau}\theta_{-t}\omega) \subset \varphi(t, \theta_{-t}\omega)B_0(\theta_{-t}\omega), \text{for all } \tau \geq \tau_0(\omega) \text{ and } \omega \in \Omega.$$

This implies that

$$\varphi\big(\tau + t, \theta_{-(\tau+t)}\omega\big)U\big(\theta_{-(\tau+t)}\omega\big) \subset \cup_{t \geq t_*} \varphi(t, \theta_{-t}\omega)B_0(\theta_{-t}\omega) = B_*(\omega).$$

Thus $B_*(\omega)$ is an absorbing random set in $\mathcal{U}$.

Lemma 3.2

If (θ, φ) is a compact RDS, then it is conditionally compact.

Proof

Suppose that (θ, φ) is compact RDS. Let $D(\omega)$ be a bounded random set such that $\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset D(\omega)$ for $t > 0$. By hypothesis, there exists a compact set $K(\omega)$ with the property that:

$$\varphi(\tau, \theta_{-\tau}\omega)D(\theta_{-\tau}\omega) \subset K(\omega) \text{ for all } \tau \geq \tau_0(\omega) \text{ and } \omega \in \Omega.$$

Take $K(\omega) = \overline{D}(\omega)$, we get the result.

Lemma 3.3

A dissipative RDS (θ, φ) is compact if it is conditionally compact.

Proof. Suppose that (θ, φ) is dissipative and conditionally compact RDS. Let $D(\omega)$ be a bounded random set. By Lemma 3.1 there exists a bounded forward invariant absorbing set, say $B(\omega)$. Set $K(\omega) = \overline{B}(\omega)$. Then $K(\omega)$ is the desired compact set, and the proof is completed.

Lemma 3.4

If $D(\omega)$ is bounded random set in an asymptotically compact RDS (θ, φ) such that $\gamma_D^{\tau}(\theta_{-t}\omega)$ is bounded random set for some $\tau \geq 0$. Then $\Gamma_D(\omega)$ is a nonvoid invariant compact random set with

$$\varphi\,(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \rightrightarrows \Gamma_D(\omega) \text{ as } t \longrightarrow \infty.$$

Proof

Suppose that (θ, φ) is asymptotically compact, then for any $t_n \to \infty$ and $x_n \in D(\theta_{-t_n}\omega)$ the sequence $\{\varphi(t_n, \theta_{-t_n}\omega)x_n\}$ is relatively compact:

$$\exists \{n_m\}, z \in X \ni \varphi\left(t_{n_m}, \theta_{-t_{n_m}}\omega\right)x_{n_m} \to z \quad as\ m \to \infty.$$

By Proposition 2.1,

$$\Gamma_D(\omega) = \left\{z \in X: z = \lim_{n\to\infty} \varphi(t_n, \theta_{-t_n}\omega)x_n \text{ for some } t_n \to +\infty,\ x_n \in D(\theta_{-t_n}\omega)\right\}. \quad (2)$$

So, $z \in \Gamma_D(\omega)$, and consequently $\Gamma_D(\omega) \neq \emptyset$. To show that $\Gamma_D(\omega)$ is compact. For every sequence $\{z_n\}$ in $\Gamma_D(\omega)$ there exist $t_n \to \infty$ and $x_n \in D(\theta_{-t_n}\omega)$ such that $\|\varphi(t_n, \theta_{-t_n}\omega)x_n - z_n\| < \frac{1}{n}$. Since (θ, φ) is asymptotically compact, there exist a subsequence $\{n_m\}$ and an $\hat{z} \in X$ with $\varphi(t_{nm}, \theta_{-t_{nm}}\omega)x_{nm} \to \hat{z} \in \Gamma_D(\omega)$ whenever $m \to \infty$. It follows that $z_{nm} \to \hat{z}$, so $\Gamma_D(\omega)$ is relatively compact. Similarly, if $z_n \to \hat{z}$ as $n \to \infty$ then $\bar{z} = \hat{z} \in \Gamma_D(\omega)$ i.e. $\Gamma_D(\omega)$ is closed. we now prove invariance of $\Gamma_D(\omega)$. If $z \in \Gamma_D(\omega)$ and $z = \lim_{n\to\infty} \varphi(t_n, \theta_{-t_n}\omega)x_n$, then

$$\varphi(t, \theta_{-t}\omega)z = \lim_{n\to\infty} \varphi(t + t_n, \theta_{-t_n}\omega)x_n.$$

According to Equation (2), the omega-limit set $\Gamma_D(\omega)$ is forward invariant. To show that $\Gamma_D(\omega)$ is backward invariant. Consider the sequence $\{\varphi(t_n - t, \theta_{-t_n+t}\omega)x_n\}$ for some fixed $t > 0$ and n such that $t_n > t$. This sequence is relatively compact because (θ, φ) is asymptotic compact. Consequently, there exist a sequence $\{n_m\}$ and $v \in \Gamma_D(\omega)$ with

$$y_m \equiv \varphi\left(t_{n_m} - t, \theta_{-t_{n_m}+t}\omega\right)x_{n_m}.$$

Also, $\varphi(t, \theta_{-t}\omega)y_m \longrightarrow z$. Hence $z = \varphi(t, \theta_{-t}\omega)v$ and so $\varphi(t, \theta_{-t}\omega)\Gamma_D(\theta_{-t}\omega) \supset \Gamma_D(\omega)$, i.e., $\Gamma_D(\omega)$ is backward invariant.

Assume that:

$$\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \rightrightarrows \Gamma_D(\omega),$$

is false. Then there exist $\delta > 0$, sequences $t_n \longrightarrow \infty$ and $x_n \in D(\omega)$ such that $\text{dist}_X\left(\varphi(t_n, \theta_{-t_n}\omega)x_n, \Gamma_D(\omega)\right) \geq \delta$ for all n. As above, $\{\varphi(t_n, \theta_{-t_n}\omega)x_n\}$ is relatively compact. Therefore,

$$\varphi\left(t_{n_m}, \theta_{-t_{n_m}}\omega\right)x_{n_m} \longrightarrow z \in \Gamma_D(\omega),$$

for some subsequence $\{n_m\}$. This opposes the inequality:

$$\text{dist}_X\left(\varphi(t_n, \theta_{-t_n}\omega)x_n, \Gamma_D(\omega)\right) \geq \delta.$$

Some sufficient requirements for the asymptotic compactness of RDS are given in the following lemma.

Theorem 3.1

An RDS (θ,φ) is asymptotically smooth if and only if it is asymptotically compact.

Proof

Let (θ,φ) be asymptotically compact and $B(\omega)$ be an invariant bounded random set. By Lemma 3.6, $\Gamma_B(\omega)$is a compact set which attracts $B(\omega)$. Thus, the condition in Definition 3.1 holds. Conversely, let (θ,φ) be asymptotically smooth and $B(\omega)$ be a bounded set such that the tail $\gamma_B^\tau(\omega)=\bigcup_{t\geq\tau}\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)$is bounded for some $\tau\geq 0$. Since $B_*(\omega)\equiv\gamma_B^\tau(\omega)$ is forward invariant, by Definition 3.1 $\varphi(t,\theta_{-t}\omega)B_*(\theta_{-t}\omega)$ converges uniformly to a compact set $K(\omega)$. Thus $\varphi(t_n,\theta_{-t_n}\omega)x_n\longrightarrow K(\omega)$ for any sequences $x_n\in B(\theta_{-t_n}\omega)$ and $t_n\longrightarrow\infty$. Hence $\{\varphi(t_n,\theta_{-t_n}\omega)x_n\}$ is relatively compact.

Lemma 3.5

An RDS (θ,φ) is asymptotically compact if any conditions are hold:

1. Each bounded random set $B(\omega)$ in X, there is a compact random set $K(\omega)$ with $\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\rightrightarrows K(\theta_{-t}\omega)$ as $t\longrightarrow\infty$.
2. Each bounded random set $B(\omega)$ there is a compact set $K_B(\omega)$ with $\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\rightrightarrows K_B(\theta_{-t}\omega)$ as $t\longrightarrow\infty$.
3. There exists a decomposition $\varphi=\varphi_1+\varphi_2$, where (θ,φ_1) is uniformly compact for large t i.e. every bounded random set $B(\omega)\ \exists\ t_0=t_0(B)$ such that the set

$$\gamma^1(B;t_0):=\bigcup_{\tau\geq t_0}\varphi_1(\tau,\theta_{-\tau}\omega)B(\theta_{-\tau}\omega),$$

is compact in X and (θ,φ_2) is uniformly stable, meaning that

$$r_B(t,\omega)=\sup\{\|\varphi_2(t,\theta_{-t}\omega)x\|_X : x\in B(\theta_{-t}\omega)\}\longrightarrow 0\,, \qquad (3)$$

as $t\longrightarrow\infty$.

Proof

(1) follows from definition of asymptotically compact RDS.

(3) implies (2) with $K_B(\omega)=\overline{\gamma^1(B;t_0)}$. Statement (2) applied to a bounded sequence $B(\omega)=\{x_n(\omega)\}$ yields the convergence of $\varphi\left(t_n,\theta_{-t_n}\omega\right)x_n$ to a compact random set as $t_n\longrightarrow\infty$.

Definition 3.2[1,2]

The Kuratowski's $\alpha-$ measure of non-compact random set. The formula defines:

$$\alpha\big(B(\omega)\big)=\inf\left\{\begin{matrix}\text{d}(\omega)\text{: B}(\omega)\text{has afinit cover}\\ \text{by open random sets}\\ \text{of diameter }<\text{d}(\omega)\end{matrix}\right\}$$

on bounded random sets of X.

Proposition 3.1[1]

For any bounded random sets $A(\omega)$ and $B(\omega)$ in a Banach space X we have:

$$\alpha\big(A(\omega)+B(\omega)\big) \leq \alpha\big(A(\omega)\big)+\alpha\big(B(\omega)\big) \tag{4}$$

Proof

Take TRV $\varepsilon > 0$, let:

$$\mathcal{O}_i\big(A(\omega)\big) = \{x \in X: \inf_{y\in A(\omega)}\|x-y\| < \varepsilon(\omega)\}$$

and

$$\mathcal{O}_i\big(B(\omega)\big) = \{z \in X: \inf_{c\in B(\omega)}\|z-c\| < \varepsilon(\omega)\}$$

be coverings of A and B with diameters less than $\alpha\big(A(\omega)\big)+\varepsilon$ and $\alpha\big(B(\omega)\big)+\varepsilon$. Then $\{\mathcal{O}_i\big(A(\omega)\big)+\mathcal{O}_i\big(B(\omega)\big)\}$ is a covering for $A(\omega)+B(\omega)$ it is clear that

$$diam\{\mathcal{O}_i\big(A(\omega)\big)+\mathcal{O}_i\big(B(\omega)\big)\} \leq diam\{\mathcal{O}_i\big(A(\omega)\big)\}+diam\{\mathcal{O}_i\big(B(\omega)\big)\} \leq \alpha\big(A(\omega)\big)+\alpha\big(B(\omega)\big)+2\varepsilon$$

Proposition 3.2 [3]

If $U_1 \supset U_2 \supset U_3 \ldots$ is nonvoid closed sets in a complete metric space X. If $\alpha(U_n) \longrightarrow 0$ as $n \longrightarrow \infty$, then $\bigcap_{n\geq 1} U_n \neq \emptyset$ and compact.

Proposition 3.3 [3]

If $\alpha(U_t) \longrightarrow 0$ as $t \longrightarrow \infty$, for some decreasing collection $\{U_t\}$ of nonvoid closed sets, then $\bigcap_{t\geq 0} U_t$ is nonvoid and compact.

Proposition 3.4

An RDS (φ,θ) is asymptotically smooth if and only if

$$\lim_{t\longrightarrow\infty} \alpha\big(\varphi(t,\theta_{-t}\omega)B(\omega)\big) = 0,$$

for each forward invariant bounded random set $B(\omega)$.

Proof

Suppose that $B(\omega)$ is a forward invariant bounded random set of (θ,φ). If (θ,φ) is an asymptotically smooth RDS, then by Definition 3.1 there is a compact set $K_B(\omega)$ so that $\varphi(t,\theta_{-t}\omega)B(\omega) \rightrightarrows K_B(\omega)$ at $t \longrightarrow \infty$. Since $K_B(\omega)$ is compact, Afterward, for $\varepsilon > 0$ there is a finite set $\{x_k: k=1,\ldots,N_\varepsilon\}$ in $K_B(\omega)$ such that $K_B(\omega) \subset \bigcup_{k=1}^{N_\varepsilon} \mathcal{B}_k(\omega)$ where:

$$\mathcal{B}_k(\omega) = \{x \in X: \inf_{x_k\in k_B(\omega)}\|x_k - x\| < \varepsilon(\omega)\}.$$

Since $\varphi(t,\theta_{-t}\omega)B(\omega) \subset \bigcup_{k=1}^{N_\varepsilon} \mathcal{B}_k(\omega)$ for all $t \geq t_\varepsilon$ thus $\alpha\big(\varphi(t,\theta_{-t}\omega)B(\omega)\big) < 2\varepsilon$ for all $t \geq t_\varepsilon$ this implies that $\alpha\big(\varphi(t,\theta_{-t}\omega)B(\omega)\big) \to 0$ $as\ t \to \infty$. Then we can apply the result of Proposition 3.3 to a family of the set $U_t = \overline{\varphi(t,\theta_{-t}\omega)B(\omega)}$ and conclude that $\Gamma_{B(\omega)} = \bigcap_{t>0} \overline{\varphi(t,\theta_{-t}\omega)B(\omega)}$ is anon empty compact set, hence it is enough to indication that $\varphi(t,\theta_{-t}\omega)B(\omega) \rightrightarrows \Gamma_{B(\omega)}$ if this is false, then there exist $\delta > 0$ and sequence $t_n \to \infty$ and $x_n \in B(\omega)$ such that

$$\left\|\varphi(t_n,\theta_{-t}\omega)x_n - \Gamma_{B(\omega)}\right\| \geq \delta.$$

For all n one can see that for any $t > 0$ there exist $t_\varepsilon(\omega) > 0$ there exists N_t such that:

$$\{\varphi(t_n,\theta_{-t}\omega)x_n : n = 1,2,\dots\} \subset \{\varphi(t_n,\theta_{-t}\omega)x_n : n = 1,2,\dots,N_t\} \cup \overline{\varphi(t,\theta_{-t}\omega)B(\omega)}.$$

Thus:

$$\alpha\{\varphi(t_n\,,\theta_{-t}\omega)x_n : n = 1,2,\dots\} \leq \alpha\{\varphi(t,\theta_{-t}\omega)B(\omega)\},$$

which implies that

$$\alpha(\varphi(t_n\,,\theta_{-t}\omega)x_n) = 0.$$

Hence $\varphi(t_n\,,\theta_{-t}\omega)x_n$ is relatively compact, there for $\varphi\big(t_{n_m}\,,\theta_{-t}\omega\big)x_{n_m} \to z \in \Gamma_{B(\omega)}$ for some sub sequence $\{n_m\}$ this contradiction the relation $\left\|\varphi(t_n,\theta_{-t}\omega)x_n - \Gamma_{B(\omega)}\right\| \geq \delta$.

Proposition 3.5

An RDS (θ,φ) in Banach space X. Suppose that for each $t > 0$ there is a decomposition $\varphi(t,\theta_{-t}\omega) = \varphi_1(t,\theta_{-t}\omega) + \varphi(t,\theta_{-t}\omega)$ where $\varphi_2(t,\theta_{-t}\omega) : X \longrightarrow X$ is a function filling Equation (3) and $\varphi_1(t,\theta_{-t}\omega)$ is compact in means $\forall\, t > 0$ the random set $\varphi_1(t,\theta_{-t}\omega)\, B(\omega)$ is a relatively compact set in X each $t > 0$ sufficiently large and each forward invariant bounded random set $B(\omega)$ in X. Hence an RDS $(\theta,\varphi)$is a asymptotically smooth.

Proof

For any forward invariant bounded random set $B(\omega)$,

$$\varphi(t,\theta_{-t}\omega)\, B(\omega) \subset \varphi(t,\theta_{-t}\omega)^{(1)}B(\omega) + \varphi(t,\theta_{-t}\omega)^{(2)}B(\omega).$$

Therefore, Proposition 3.1 yields

$$\alpha\big(\varphi(t,\theta_{-t}\omega)\, B(\omega)\big) \leq \alpha\left(\varphi(t,\theta_{-t}\omega)^{(1)}B(\omega)\right) + \alpha\left(\varphi(t,\theta_{-t}\omega)^{(2)}B(\omega)\right)$$

$$\leq \alpha\left(\varphi(t,\theta_{-t}\omega)^{(2)}B(\omega)\right)$$

$$\leq diam\big\{\varphi(t,\theta_{-t}\omega)^{(2)}B(\omega)\big\} \leq 2\sup_{y\in B(\omega)}\left\|\varphi(t,\theta_{-t}\omega)^{(2)}y\right\|,$$

for all t large enough. Thus, by Equation (3) $\alpha\big(\varphi(t,\theta_{-t}\omega)\, B(\omega)\big) \longrightarrow 0$

as $t \longrightarrow \infty$. Hence by Proposition 3.4, (θ, φ) is asymptotically smooth.

Proposition 3.6

An RDS (θ, φ) is asymptotically smooth if $\exists\ t > 0$ such that $\varphi(t, \theta_{-t}\omega) : X \longrightarrow X$ is an α −contraction [12].

Proof

Note that

$$\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset \varphi\big(nt_*, \theta_{-nt_*}\omega\big)D(\theta_{-t_*}\omega),$$

for every forward invariant random set $D(\omega)$, where n is the integer part of t/t_*.

This section is concluded with a number of claims that provide practical standards for the asymptotic smoothness and compactness of RDSs.

Proposition 3.7

Let $(\theta, \varphi\,)$ be a RDS on Banach space X. Suppose that each bounded forward invariant random set $B(\omega)$ in X and any TRV $\varepsilon(\omega) > 0$

$$\lim_{m\to\infty} \inf \lim_{n\to\infty} \inf\|\varphi\,(t\,, \theta_{-t}\omega)y_n - \varphi\,(t\,, \theta_{-t}\omega)y_m\| \le \varepsilon(\omega)\,, \qquad (5)$$

for every sequence $\{y_n\} \subset B(\omega)$, then $(\theta, \varphi\,)$ is an asymptotically smooth RD.

Proof

According to Proposition 3.4 it is enough to show that:

$$\lim_{t\to\infty} \alpha\big(\varphi\,(t\,, \theta_{-t}\omega)B(\theta_{-t}\omega)\big) = 0,$$

where $\alpha\big(B(\omega)\big)$ is Kuratowski's α − measure of non compactness. Since:

$$\varphi\,(t_1\,, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset \varphi\,(t_2\,, \theta_{-t}\omega)B(\theta_{-t}\omega)\ for\ t_1 > t_2,$$

the function:

$$\alpha(t, \omega) \equiv \varphi\,(t\,, \varphi_{-t}\omega)B(\theta_{-t}\omega),$$

is non- increasing. So, it is enough to show that for any TRV $\varepsilon(\omega)$ there is $T > 0$, $\alpha\big(\varphi\,(t\,, \theta_{-t}\omega)B(\theta_{-t}\omega)\big) \le \varepsilon(\omega)$, if this is not true, then there is $\varepsilon_o(\omega) > 0$ such that

$$\alpha\big(\varphi\,(t\,, \theta_{-t}\omega)B(\theta_{-t}\omega)\big) \ge 5\varepsilon_o(\omega) \forall\ T > 0,$$

for $\varepsilon_o(\omega)$ pick T_0 so that Equation (5) fulfills. The inequality

$$\alpha\big(\varphi\,(t\,, \theta_{-t}\omega)B(\omega)\big) \ge 5\varepsilon_o(\omega)$$

implies that there is a sequence $\{y_n\}_{n=1}^{\infty}$ with

$$\left\|\varphi\left(t_0, \theta_{-t_0}\omega\right)y_n - \varphi\left(t_0, \theta_{-t_0}\omega\right)y_m\right\| \geq 2\varepsilon_0(\omega)\,, \qquad (6)$$

for all $n+m$ such that $n = 1,2,\ldots$ In the event that such a sequence is not found, the following construction can be applied: pick a random variable $y_1 \in B(\omega)$ and $y_2 \in B(\omega)$ such that:

$$\left\|\varphi\left(t_o, \theta_{-t_o}\omega\right)y_3 - \varphi\left(t_o, \theta_{-t_o}\omega\right)y_i\right\| \geq 2\varepsilon_0(\omega),$$

for $i = 1,2,\ldots$ and so on. If this process is stopped, we get a TRV $2\varepsilon_0(\omega)$ − net for $\varphi\left(t_0, \theta_{-t}\omega\right)B(\omega)$. This means that $\alpha\left(\varphi\left(t_o, \theta_{-t_o}\omega\right)B(\omega)\right) \geq 4\varepsilon_o(\omega)$ and contradicts the inequality $\alpha\left(\varphi\left(t_0, \theta_{-t_0}\omega\right)B(\omega)\right) \geq 5\varepsilon_0(\omega)$, thus Equation (6) holds true this contradiction in Equation (5).

Proposition 3.8

An RDS (θ,φ) on a reflexive Banach space X is asymptotically smooth if any forward invariant bounded random set $B(\omega)$ in an RDS (θ,φ) and for any TRV $\varepsilon(\omega)$, $\exists\, T > 0$ and compact operator K s.t.:

$$\|(I-K)\varphi\left(t, \theta_{-t}\omega\right)y\| \leq \varepsilon(\omega) \quad \forall y \in B(\omega). \qquad (7)$$

Proof

By Equation (7) we have that:

$$\begin{aligned}\|\varphi\left(t, \theta_{-t}\omega\right)y_1 - \varphi\left(t, \theta_{-t}\omega\right)y_2\| &\\ \leq \|(I-K)\varphi\left(t, \theta_{-t}\omega\right)y_1\| + \|(I-K)\varphi\left(t, \theta_{-t}\omega\right)y_2\| + \|K(\varphi\left(t, \theta_{-t}\omega\right)y_1 - \varphi\left(t, \theta_{-t}\omega\right)y_2)\| &\\ \leq 2\epsilon + \|K(\varphi\left(t, \theta_{-t}\omega\right)y_1 - \varphi\left(t, \theta_{-t}\omega\right)y_2)\|, \quad \forall y_1, y_2 \in B(\omega)&\end{aligned}$$

which implies that:

$$\lim_{m\to\infty}\inf \lim_{n\to\infty}\inf\|k(\varphi\left(t, \theta_{-t}\omega\right)y_n - \varphi\left(t, \theta_{-t}\omega\right)y_m)\| = 0$$

for every $\{y_n\} \subset B(\omega)$.

Theorem 3.2

Let (θ,φ) be an RDS on a Hilbert space X. Suppose that $\varphi\left(t, \theta_{-t}\omega\right): X \longrightarrow X$ is weakly continuous $\forall\, t > 0$; i.e., $x_n \longrightarrow x$ weakly in X suggests that $\varphi\left(t, \theta_{-t}\omega\right)x_n \longrightarrow \varphi\left(t, \theta_{-t}\omega\right)x$ weakly. Then the RDS (θ,φ) is asymptotically smooth if any forward invariant bounded random set B and any $\varepsilon > 0$ there is $T \equiv T(\varepsilon, B)$ such that:

$$\lim_{n\to\infty}\sup\|\varphi\left(T, \theta_{-T}\omega\right)y_n\| \leq \|\varphi\left(T, \theta_{-T}\omega\right)y\| + \varepsilon, \qquad (8)$$

for every sequence $\{y_n\} \subset B(\theta_{-T}\omega)$ such that $y_n \longrightarrow y$ weakly.

Proof

By hypothesis we have

$$\lim_{n\to\infty}\sup\|\varphi\left(T, \theta_{-T}\omega\right)y_n - \varphi\left(T, \theta_{-T}\omega\right)y\| \leq \varepsilon,$$

then apply Proposition 3.7 we get the result.

Theorem 3.3

Let (θ,φ) be RDS on Banach space X. Suppose that any bounded forward invariant random set $B(\omega)$ in X $\exists\, T>0$, a continuous non decreasing function $g:\mathbb{R}^+\longrightarrow\mathbb{R}^+$, and a pseudoment $\varrho^T_{B(\omega)}$ on the random set $B(\omega)$ such that

1. $g(0)=0\,;g(s)<s\,,s>0.$
2. The pseudometric $\varrho^T_{B(\omega)}$ is recompact (with respect to the topology of X) in means any sequence $\{x_n\}\subset B(\omega)$ has a subsequence $\{x_{n_k}\}$ which is Cauchy with respect to $\varrho^T_{B(\omega)}$.
3. For each, the following estimate satisfies: $y_1\,,y_2\in B(\omega)$

$$\|\varphi\,(t\,,\theta_{-t}\omega)y_1-\ \varphi\,(t\,,\theta_{-t}\omega)y_2\|\le g(\|y_1-y_2\|)+\varrho^T_{B(\omega)}(y_1,y_2). \tag{9}$$

Then the RDS (θ,φ) is asymptotically smooth.

Proof

We use Properties 3.7. Let $B(\omega)$ be bounded forward invariant set in X with diameter $L(\omega)$. One can see that for any $\varepsilon(\omega)>0$ we can choose N such that $g^N\big(L(\omega)\big)\le\varepsilon(\omega)$, where g^N denoted the composition $g\circ\ldots\circ g$. Iterating Equation (9) we have that:

$$\left\|\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^N y_1-\big(\varphi\,(t\,,\theta\omega)\big)^N y_2\right\|\le g\left\|\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^{N-1}y_1-\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^{N-1}y_2\right\|$$

$$+\varrho^T_{B(\omega)}\left(\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^{N-1}y_1,\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^{N-1}y_2\right)$$

$$+g\left(g\left(\ldots g\big(g(L)\big)+\varrho^T_{B(\omega)}(y_1,y_2)\right)\right)+\varrho^T_{B(\omega)}(\varphi\,(t\,,\theta_{-t}\omega)y_1,\varphi\,(t\,,\theta_{-t}\omega)y_2)\ldots)$$

$$+\varrho^T_{B(\omega)}\left(\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^{N-1}y_1\,,\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^{N-1}y_2\right).$$

On the right-hand side of the link above are the expressions of the form and a continuous function of L.

$$\varrho^T_{B(\omega)}\big(\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^m y_1\,,\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^m y_2\big),m=1,\ldots,N-1.$$

Since the pseudo metric $\varrho^T_{B(\omega)}$ is precompact, any sequence $\{x_n\}\subset B(\omega)$ such that

$$\lim_{p,q\to\infty}\varrho^T_{B(\omega)}\left(\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^m\hat{x}_{n_p}\,,\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^m\hat{x}_{n_q}\right)=0\ \ \forall m=1,\ldots N-1.$$

This implies that

$$\lim_{k\to\infty}\inf\lim_{n\to\infty}\inf\left\|\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^N x_n-\big(\varphi\,(t\,,\theta_{-t}\omega)\big)^N x_k\right\|\le g^N(L)\le\varepsilon.$$

According to Proposition 3.7, this suggests that (θ,φ) is asymptotically smooth. Theorem 3.3 implies the following result.

Proposition 3.9

Let (θ,φ) be a RDS in a Banach space X. Suppose that any bounded forward invariant set B in X there exist functions $C_B(t) \geq 0$ and $K_B(t) \geq 0$ such that $\lim_{t\to\infty} K_B(t) = 0$, a time $t_0 = D_{t_0}(B)$, and a recompact pseudometric ϱ on X such that

$$\|\varphi(t,\omega)y_1 - \varphi(t,\omega)y_2\| \leq K_B(t) \cdot \|y_1 - y_2\| + K_B(t) \cdot \varrho(y_1, y_2), t \geq t_0, \tag{10}$$

each $y_1, y_2 \in B(\omega)$. Then (θ,φ) is an asymptotically smooth RDS.

Proof

We use Theorem 3.3 with $g(s) = K_B(T) \cdot s$, where T is chosen such that $K_B(T) < 1$.

4. Global attractors

The term "attractor" has many different meanings. For infinite-dimensional systems, the most feasible concept is a global random attractor [15]. In some cases, it is more convenient to utilize the condition of point dissipative rather than (bounded) dissipative.

Theorem 4.1

Let (θ,φ) be a dissipative asymptotically compact RDS on a Banach space X. Then (θ,φ) possesses a unique compact global attractor $\mathfrak{A}(\omega)$ such that:

$$\mathfrak{A}(\omega) = \Gamma_{B_0}(\omega) = \bigcap_{t>0} \overline{\bigcup_{\tau\geq t} \varphi(\tau,\theta_{-\tau}\omega)B_0(\theta_{-\tau}\omega)}, \tag{11}$$

for each bounded absorbing set $B_0(\omega)$ and

$$\lim_{t\to+\infty} \left(\sup_{x\in B_0(\theta_{-\tau}\omega)} \inf_{y\in\mathfrak{A}(\omega)} \|\varphi(t,\theta_{-t}\omega)x - y\| + \sup_{y\in\mathfrak{A}(\omega)} \inf_{x\in B_0(\theta_{-\tau}\omega)} \|y - \varphi(t,\theta_{-t}\omega)x\|\right). \tag{12}$$

Furthermore, if a connected absorbing bounded set exists, then $\mathfrak{A}(\omega)$ is connected.

Proof

Since (θ,φ) is dissipative, there is a bounded absorbing set $B_0(\omega)$. This suggests that each bounded set $D(\omega)$ the tail $\gamma_D^t(\omega)$ lies in $B_0(\omega)$ $\forall\, t \geq t_D$. Consequently, with the asymptotic compactness of (θ,φ), from Lemma 3.5 we conclude that $\Gamma_{B_0}(\omega)$ is a nonempty compact strictly invariant set such that Equation (11) satisfies. We thus obtain a global random attractor from the formula in Equation (12). To Equation (13), we must demonstrate that:

$$\sup_{y\in\mathfrak{A}(\omega)} \inf_{x\in B_0(\theta_{-\tau}\omega)}\|y-\varphi(t,\theta_{-t}\omega)x\| = 0.$$

This is derived from the fact that $\mathfrak{A}(\omega) \subseteq B_0(\omega)$, it suggests that:

$$\mathfrak{A}(\omega) = \varphi(t,\theta_{-t}\omega)\mathfrak{A}(\theta_{-t}\omega) \subset \varphi(t,\theta_{-t}\omega)B_0(\theta_{-t}\omega), \text{for all } t > 0.$$

We employ Equation (13) and the contradiction argument to demonstrate connection.

Let $B_0(\omega)$ be connected. Suppos that $\mathfrak{A}(\omega)$ is not connected, i.e., $\mathfrak{A}(\omega) = K(\omega) \cup K_*(\omega)$, where $K(\omega)$ and $K_*(\omega)$ are two compact sets that are nonempty and disjoint, so that $\text{dist}(K(\omega), K_*(\omega)) > 3\delta$. By Equation (13), we possess that:

$$\varphi(t,\theta_{-t}\omega)B_0(\theta_{-t}\omega) \subset \mathcal{O}_\delta(\mathfrak{A}(\omega)) \coloneqq \{x \in X: \text{dist}_X(x,\mathfrak{A}(\omega)) < \delta\}, \quad (14)$$

for all t large enough. Obviously $\varphi(t,\theta_{-t}\omega)B_0(\theta_{-t}\omega)$ is connected for each t. Thus, by Equation (14) we have that $\varphi(t,\theta_{-t}\omega)B_0(\theta_{-t}\omega) \subset \mathcal{O}_\delta\left(\widetilde{K}(\omega)\right)$, where $\widetilde{K}(\omega)$ is either $K(\omega)$ or $K_*(\omega)$, say $\widetilde{K}(\omega) = K(\omega)$. By Equation (13) again we possess that:

$$K_*(\omega) \subset \mathfrak{A}(\omega) \subset \mathcal{O}_\delta\big(\varphi(t,\theta_{-t}\omega)B_0(\theta_{-t}\omega)\big) \subset \mathcal{O}_{2\delta}\left(\widetilde{K}(\omega)\right),$$

every t large enough. This is impossible because $\text{dist}(K(\omega), K_*(\omega)) > 3\delta$.

Corollary 4.1

Let (θ,φ) be a dissipative asymptotically compact RDS on a Banach space X. Then the compact global attractor $\mathfrak{A}(\omega)$ is given by

$$\mathfrak{A}(\omega) = \bigcap_{n\geq N} \varphi(nT,\theta_{-nT}\omega)B_0(\theta_{-nT}\omega), \text{ for every } N \in \mathbb{Z}^+ \text{ and } T > 0, \quad (15)$$

for each bounded absorbing set $B_0(\omega)$.

Proof

Since $\mathfrak{A}(\omega) \subset B_0(\omega)$ and thus:

$$\mathfrak{A}(\omega) = \varphi(nT,\theta_{-nT}\omega)\mathfrak{A}(\theta_{-nT}\omega) \subset \varphi(nT,\theta_{-nT}\omega)B_0(\theta_{-nT}\omega).$$

for every $n \in \mathbb{Z}^+$ and $T > 0$.

Corollary 4.2

Let an RDS (θ,φ) be dissipative and $V(t,\omega) \coloneqq \varphi(t_*,\theta_{-t_*}\omega): X \longrightarrow X$ be an α −contraction for some $t > 0$. such that (θ,φ) possesses a compact global random attractor.

Proof

By Proposition 3.7.

Theorem 4.2

A compact global attractor is present in an RDS (θ,φ) on a Banach space X if:

1. (θ,φ) is point dissipative
2. Each bounded set B $\exists\, \tau > 0$ s.t the tail $\gamma_B^\tau(\omega) = \bigcup_{t\geq\tau} \varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega)$ is bounded;
3. (θ,φ) is asymptotically smooth.

Proof

It is sufficient to demonstrate that the system (θ,φ) is (bounded) dissipative under the aforementioned conditions because of Theorem 4.1. The following "locally compact" dissipative property is established first. Specifically, we demonstrate the existence of a limited forward invariant random set $B_*(\omega)$ with the following property: for each compact random set $K(\omega)$, there is $\varepsilon = \varepsilon_K \geq 0, t_K \geq 0$ such that if $x \in X$ with $\inf_{y\in K(\omega)}\|x - y\| < \varepsilon(\omega)$, then

$$\varphi(t,\theta_{-t}\omega)x \in B_*(\omega), \forall t \geq t_K\,. \tag{16}$$

Indeed, since (θ,φ) is point dissipative, $\exists$ a bounded set $B_0(\omega)$ such that for every $x_0 \in X$, there exists $t_{x_0} \geq 0$ with $\varphi(t,\theta_{-t}\omega)x_0 \in B_0(\omega)$ for all $t \geq t_{x_0}$.

We suppose that $B_0(\omega)$ is open. In this instance, by the continuity of $\varphi\left(t_{x_0},\theta_{-t_{x_0}}\omega\right): X \longrightarrow X$ there is $\varepsilon = \varepsilon_{x_0} > 0$ such that

$$\varphi\left(t_{x_0},\theta_{-t_{x_0}}\omega\right)x \in B_0(\omega), \text{ whenever } \|x - x_0\| < \varepsilon_{x_0}.$$

Let τ_0 be such that $B_*(\omega) \equiv \gamma_{B_0}^{\tau_0}(\omega)$ is bounded. In this case,

$$\varphi\left(t + t_{x_0},\theta_{-t_{x_0}}\omega\right)x \equiv \gamma_{B_0}^{\tau_0}(\omega) \text{ for all } \quad \tau \geq \tau_0.$$

If $K(\omega)$ is a compact random set, then we can find a finite set in such that a finite set $\{x_i\}$ in $K(\omega)$ can be found if $K(\omega)$ is a compact random set-in s.t:

$$K(\omega) \subset U(\omega) \equiv \bigcup_i \mathcal{O}_{\varepsilon_{x_i}}(x_i)\,.$$

Since $\mathcal{O}_{\varepsilon_{x_i}}(x_i) = \left\{x \in X: \|x - x_i\| < \varepsilon_{x_i}\right\}$. Hence

$$K(\omega) \subset \cup_i\left\{x \in X: \|x - x_i\| < \varepsilon_{x_i}(\omega)\right\},$$

so there is an i_0 such that

$$K(\omega) \subset \left\{x \in X: \left\|x - x_{i_0}\right\| < \varepsilon_{x_{i_0}}(\omega)\right\}.$$

It is clear that

$$\varphi(t,\theta_{-t}\omega)U(\theta_{-t}\omega) = \varphi(t,\theta_{-t}\omega)\cup_i \mathcal{O}_{\varepsilon_{x_i}}(x_i)$$

$$= \cup_i\{\varphi(t,\theta_{-t}\omega)x : \|x - x_i\| < \varepsilon_{x_i}\} \subset B_*(\omega),$$

for all $\tau \geq \tau_0 + \max_i t_{x_i}$. Since $U(\omega)$ is an open set that we can find $\varepsilon = \varepsilon_K > 0$ such that $\mathcal{O}_\varepsilon(K) \subset U(\omega)$. Hence there exist a TRV $\varepsilon(\omega) = \varepsilon_K(\omega)$ and $t_K \geq 0$ such that

$$\text{if } x \in X \text{ with } \inf_{y\in K(\omega)}\|x - y\| < \varepsilon(\omega), \text{ then } \varphi(t,\theta_{-t}\omega)x \in B_*(\omega), \forall t \geq t_K.$$

This establishes Equation (16). As we wrap up the proof, we observe that each bounded set $B(\omega)$, $\exists\ \tau = \tau_B$ s.t $\gamma_B^\tau(\omega)$ is bounded and forward invariant. Accordingly, by asymptotic smoothness, there is a compact set K such that

$$\forall\varepsilon > 0, \exists t_\varepsilon \geq 0 \text{ such that } \varphi(t,\theta_{-t}\omega)\gamma_B^\tau(\theta_{-t}\omega) \subset \mathcal{O}_\varepsilon(K), \text{for all } t \geq t_\epsilon.$$

Therefore, the intended conclusion is implied by the locally compact dissipative Proposition 3.6.

5. Conclusion

This article includes several results, among them the proof of the equivalence of asymptotically smooth and asymptotically compact RDSs. The trajectory of an absorbing bounded random set is forward invariant; bounded; and an absorbing. A compact RDS is conditionally compact. In addition, sufficient conditions have been found to ensure that the RDS is asymptotically smooth. It is evident that an RDS is dissipative and asymptotically compact if it is a compact global attractor [12, 13]. Therefore, if an RDS (θ, φ) is dissipative and asymptotically compact (or asymptotically smooth), Theorem 4.1 suggests that it possesses a compact global attractor.

This article concludes with a number of significant and novel findings. It is demonstrated that if an RDS is asymptotically compact, it is also asymptotically smooth. Furthermore, we demonstrate that the RDS becomes asymptotically compact if $\varphi(t,\theta_{-t}\omega): X \longrightarrow X$ is an α −contraction for any $t > 0$. We discovered that an asymptotically smooth RDS is a necessary requirement for a dynamic system. When the RDS is dissipative, it has a unique compact global attractor. Moreover, the omega limit set for a bounded absorbing random set is this compact global attractor. Furthermore, any point dissipative and asymptotically smooth RDS contains a compact global attractor.

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