An Equations Related to θ-Centralizers on Lie Ideals

Abdulrahman H. Majeed

Department of Mathematics
College of Science
Baghdad University, Iraq
e-mail: ahmajeed6@yahoo.com

Mushreq I. Meften

Department of Mathematics
College of Science
Baghdad University, Iraq
e-mail: mushreq.meften@yahoo.com

علاقات قريبة من تمركزات θ في مثاليات لي

عبد الرحمن حميد مجيد

قسم الرياضيات / كلية العلوم
جامعة بغداد

مشرق إبراهيم مفتن

قسم الرياضيات / كلية العلوم
جامعة بغداد

Abstract : The purpose of this paper is to prove the following result :

Let R be a 2-torsion free prime ring , U a square closed Lie ideal, and $T,\theta: R\rightarrow R$ are additive mappings, such that $3T(xyx) = T(x)\theta(yx) + \theta(x)T(y)\theta(x) + \theta(xy)T(x)$ and $\theta(x)T(xy+yx)\theta(x) = \theta(x)T(y)\theta(x^2) + \theta(x^2)T(y)\theta(x)$ holds for all pairs $x, y \in U$, if θ is a surjective endomorphism on U, and $T(u) \in U$, for all $u\in U$, then $T(xy)= T(x)\theta(y) = \theta(x)T(y)$ for all $x,y\in U$.

الخلاصة :- الهدف من البحث هو برهان النتيجة الآتية : لتكون R حلقة أولية طليقة الالتواء من الدرجة الثانية و U مثالي لي مغلق تربيعيا في R و $\theta,T: R\rightarrow R$ دالتان جمعيتان و T تحقق المعادلتين التاليتين لكل x,y في U

$$3T(xyx)=T(x)\theta(yx)+\theta(x)T(y)\theta(x)+\theta(xy)T(x)\}$$

و $\theta(x)T(xy+yx)\theta(x)=\theta(x)T(y)\theta(x^2)+\theta(x^2)T(y)\theta(x)$

اذا كان θ دالة تشاكل شامل ذاتي على U و $T(u)\in U$ لكل $u\in U$ فان $T(xy)=T(x)\theta(y)=\theta(x)T(y)$ لكل x,y في U.

Keywords: prime ring, semiprime ring, derivation, Jordan derivation, Jordan triple derivation, left (right) centralizer, left (right) Jordan centralizer, centralizer , left (right) θ-centralizer, left (right) Jordan θ-centralizer, θ-centralizer

Introduction

This note is motivated by the work of Vukman and Kosi-Ulbl [14]. Throughout this note, R will represent an associative ring with center Z(R). A ring R is n-torsion free, where n is an integer, in case $nx = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basic commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xz, y] = [x, y]z + x[z, y]$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D: R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$, and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. A derivation D is inner in case there exists $a \in R$ such that $D(x) = [a, x]$. Every derivation is a Jordan derivation. The inverse is in general not true. A classical result of Herstein ([7]) asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in ([2]). Cusack ([5]) has generalized Herstein's result to 2-torsion free semiprime rings (see also [3] for an alternative proof). An additive mapping $T: R \to R$ is called a left (right) centralizer in case $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all $x, y \in R$. We follow Zalar [17] and call T a centralizer in case T is both a left and a right centralizer. If $a \in R$ then $La(x) = ax$ is a left centralizer and $Ra(x) = xa$ is a right centralizer. An additive mapping $T: R \to R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$). Following ideas from [3], Zalar ([17]) has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Also, Vukman ([15]) proved that if $T: R \to R$ is an additive mapping such that $2T(x^2) = T(x)x + xT(x)$ holds for all $x \in R$, then T is a centralizer. Also, Vukman ([16]) proved that if R is a 2-torsion free semiprime ring and $T: R \to R$ is an additive mapping such that $T(xyx) = xT(y)x$ holds for all $x, y \in R$, then T is a centralizer. In ([14]) Vukman and Kosi-Ulbl proved that if R is a 2-torsion free semiprime ring and $T: R \to R$ is an additive mapping such that $2T(xyx) = T(x)yx + xyT(x)$ holds for all $x, y \in R$, then T is a centralizer. An additive mapping $D: R \to R$, where R is an arbitrary ring, is a Jordan triple derivation in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all $x, y \in R$. One can easily prove that any triple derivation is a Jordan triple derivation. Bresar ([4]) has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a triple derivation. In [8], We has introduced the notation of θ-centralizer and Jordan θ-centralizer, which is a generalization of the definition of centralizer and Jordan centralizer, and we proved, on a 2-torsion free semiprime ring, with some condition that every Jordan θ-centralizer is a θ-centralizer. In [9], [10], [11], [12] we generalized results on centralizer in rings to θ-centralizer in rings and lie ideals . In [6] Daif , Tammam and Muthana proved that if R be 2-tortion free semiprime ring and $T: R \to R$ an additive mapping such that $3T(xyx)=T(x)\theta(yx)+\theta(x)T(x)\theta(x) +\theta(xy)T(x)$ and $\theta(x)T(xy + yx)\theta(x) = \theta(x)T(y)\theta(x^2) + \theta(x^2)T(y)\theta(x)$ hold for all $x,y \in R$, where θ is a homomorphism from R onto R, then T is a θ- centralizer .In this paper we generalize the result in [6] on lie ideals .

The Main Result

We now give the main result of this paper.

Theorem 1

Let R be a 2-torsion free prime ring , U a square closed Lie ideal, and let $T,\theta: R \rightarrow R$ are additive mappings. Suppose that $3T(xyx) = T(x)\theta(yx) + \theta(x)T(y)\theta(x) + \theta(xy)T(x)$ and $\theta(x)T(xy+yx)\theta(x) = \theta(x)T(y)\theta(x^2) + \theta(x^2)T(y)\theta(x)$ holds for all pairs $x, y \in U$, where θ is a surjective endomorphism on U, and $T(u) \in U$, for all $u \in U$, then $T(xy)= T(x)\theta(y) = \theta(x)T(y)$ for all $x,y \in U$.

Lemma 1.[1]

If $U \not\subset Z$ is Lie ideal of a 2-torsion free prime ring R and a , b $\in$ R such that aUb = {0} , then a=0 or b=0 .

Lemma 2.[13]

Let R be a 2-tortion free prime ring , U be a square closed Lie ideal of R. Suppose that the relation axb + bxc = 0 holds for all $x \in U$ and some a , b , c $\in$ U. In this case (a + c)xb = 0 is satisfied for all $x \in U$.

Proof. of Theorem (1):

$$3T(xyx) = T(x)\theta(yx) + \theta(x)T(y)\theta(x) + \theta(xy)T(x) \quad \text{for all } x, y \in U \qquad (1)$$

And,

$$\theta(x)T(xy + yx)\theta(x) = \theta(x)T(y)\theta(x^2) + \theta(x^2)T(y)\theta(x) \text{ for all } x, y \in U \qquad (2)$$

(i) If $U \not\subset Z(R)$

After replacing x by x + z in (1), we obtain

$$3T(xyz+zyx) = T(x)\theta(yz) + T(z)\theta(yx) + \theta(x)T(y)\theta(z) + \theta(z)T(y)\theta(x) + \theta(zy)T(x) +\theta(xy)T(z), \quad \text{for all } x, y, z \in U \ (3)$$

Letting y = x and z = y in (3) gives

$$3T(x^2y+yx^2) = T(x)\theta(xy) + T(y)\theta(x^2) + \theta(x)T(x)\theta(y) + \theta(y)T(x)\theta(x) + \theta(yx)T(x) + \theta(x^2)T(y) \quad \text{for all } x , y \in U \ (4)$$

After replacing x by 3x and z by $2x^3$ in (3) and using (1), we obtain

$$9T(xyx^3+x^3yx) = 3T(x)\theta(yx^3) + 3T(x^3)\theta(yx) + 3\theta(x)T(y)\theta(x^3) + 3\theta(x^3) T(y) \theta(x) + 3\theta(x^3y)T(x) + 3\theta(xy)T(x^3)= 3T(x)\theta(yx^3)+ T(x)\theta(x^2yx) + \theta(x) T(x)\theta(xyx) + \theta(x^2)T(x)\theta(yx) + 3\theta(x)T(y)\theta(x^3) + 3\theta(x^3)T(y)\theta(x) +\theta(xy) T(x)\theta(x^2)+\theta(xyx)T(x)\theta(x)+\theta(xyx^2)T(x) + 3\theta(x^3y)T(x) \quad \text{for all } x, y \in U (5)$$

Replacing y by $3(x^2y + yx^2)$ in (1) and using (4), we obtain

$9T(xyx^3+ x^3yx) = 3T(x)\theta(x^2y+yx^2)\theta(x) + 3\theta(x)T(x^2y+yx^2)\theta(x) +3\theta(x)\, \theta(x^2y+yx^2)\, T(x) = 3T(x)\theta(x^2y+yx^2)\theta(x) + \theta(x)(T(x)\theta(xy) + T(y)\theta(x^2) + \theta(x)T(x)\theta(y) + \theta(y)T(x)\theta(x) + \theta(x^2)T(y) + \theta(yx)T(x))\theta(x) + 3\theta(x)\, \theta(x^2y+yx^2)\, T(x) = 3T(x)\theta(x^2yx) + 3T(x)\theta(yx^3) + \theta(x)T(x)\theta(xyx) + \theta(x)\, T(y)\, \theta(x^3) + \theta(x^2)T(x)\theta(yx) + \theta(xy)T(x)\theta(x^2) + \theta(x^3)T(y)\theta(x) + \theta(xyx)\, T(x)\theta(x) +3\theta(x^3y)T(x) + 3\theta(xyx^2)T(x)$ for all $x, y \in U$ (6)

Subtracting (6) from (5), we obtain

$T(x)\theta(x^2yx)+\theta(xyx^2)T(x)-\theta(x^3)T(y)\theta(x)-\theta(x)T(y)\theta(x^3)= 0$, for all $x, y \in U$ (7)

Replacing y by 6xyx in (4), we obtain

$9T(x^3yx+xyx^3) = 3T(x)\theta(x^2yx) + T(x)\theta(yx^3) + \theta(x)T(y)\theta(x^3) + \theta(xy)T(x)\theta(x^2) + 3\theta(x)T(x)\theta(xyx) + 3\theta(xyx)T(x)\theta(x) + \theta(x^2)T(x)\theta(yx) + \theta(x^3)T(y)\, \theta(x) + \theta(x^3y)T(x) + 3\theta(xyx^2)T(x)$ for all $x,y \in U$ (8)

On the other hand by replacing z by $6x^3$ in (3), we obtain

$9T(x^3yx+xyx^3) = 3T(x)\theta(yx^3) + T(x)\theta(x^2yx) + \theta(x)T(x)\theta(xyx) + \theta(x^2)T(x)\theta(yx) + 3\theta(x)T(y)\theta(x^3) + 3\theta(x^3)T(y)\theta(x) + \theta(xy)T(x)\theta(x^2) + \theta(xyx)T(x)\theta(x) + \theta(xyx^2)T(x) + 3\theta(x^3y)T(x)$ for all $x,y \in U$ (9)

Comparing (8) and (9), we arrive at

$T(x)\theta(yx^3) - T(x)\theta(x^2yx) + \theta(x)T(y)\theta(x^3) - \theta(x)T(x)\theta(xyx) - \theta(xyx^2)T(x) + \theta(x^3y)T(x) - \theta(xyx)T(x)\theta(x) + \theta(x^3)T(y)\theta(x) = 0$ for all $x, y \in U$ (10)

From (7) and (10), we obtain

$T(x)\theta(yx^3)-\theta(x)T(x)\theta(xyx)+\theta(x^3y)T(x)-\theta(xyx)T(x)\theta(x)=0$ for all $x, y \in U$ (11)

Replacing y by 2yx in the above relation gives

$T(x)\theta(yx^4)-\theta(x)T(x)\theta(xyx^2)+\theta(x^3yx)T(x)-\theta(xyx^2)T(x)\theta(x)=0$, for all $x,y\in U$ (12)

On the other hand right multiplication of (11) by θ(x) gives

$T(x)\theta(yx^4)-\theta(x)T(x)\theta(xyx^2)+\theta(x^3y)T(x)\theta(x)-\theta(xyx)T(x)\theta(x^2)= 0$, for all $x, y \in U$ (13)

Subtracting (13) from (12) gives

$\theta(x^3y)[T(x), \theta(x)] - \theta(xyx)[T(x), \theta(x)]\theta(x) = 0$ for all $x, y \in U$ (14)

Left multiplication of (14) by T(x) gives

$T(x)\theta(x^3y)[T(x),\theta(x)]-T(x)\theta(xyx)[T(x),\theta(x)]\theta(x)=0$ for all $x, y \in U$. (15)

Replacing θ(y) by 2T(x)θ(y) in (14) gives,

$\theta(x^3)T(x)\theta(y)[T(x),\theta(x)]-\theta(x)T(x)\theta(yx)[T(x),\theta(x)]\theta(x)=0$ for all $x, y \in U$ (16)

After subtracting (16) from (15), we arrive at

$[T(x),\theta(x^3)]\theta(y)[T(x),\theta(x)]-[T(x),\theta(x)]\theta(yx)[T(x),\theta(x)]\theta(x)=0$ for all $x,y \in U$ (17)

In the above relation let

$a = [T(x),\theta(x^3)]$, $b = [T(x), \theta(x)]$, $c = -\theta(x)[T(x), \theta(x)]\theta(x)$ and $z = \theta(y)$

From the above substitutions, we have

$$azb + bzc = 0.$$

We apply Lemma 2 to the above relation to obtain

$$\{[T(x),\theta(x^3)] - \theta(x)[T(x), \theta(x)]\theta(x)\}\theta(y)[T(x), \theta(x)] = 0, \text{ for all } x, y \in U,$$

this reduces to

$$\{[T(x), \theta(x)]\theta(x^2) + \theta(x^2)[T(x), \theta(x)]\}\theta(y)[T(x), \theta(x)] = 0, \text{ for all } x, y \in U \quad (18)$$

Right multiplication of the above relation by $\theta(x^2)$ gives

$$\{[T(x),\theta(x)]\theta(x^2)+\theta(x^2)[T(x),\theta(x)]\}\theta(y)[T(x),\theta(x)]\theta(x^2)=0 \quad \text{for all } x , y \in U \quad (19)$$

After replacing $\theta(y)$ by $2\theta(y)\theta(x^2)$ in (18), we get

$$\{[T(x),\theta(x)]\theta(x^2)+\theta(x^2)[T(x),\theta(x)]\}\theta(y)\theta(x^2)[T(x),\theta(x)]=0 \quad \text{for all } x, y\in U \quad (20)$$

Adding (19) to (20), we obtain

$$\{[T(x),\theta(x)]\theta(x^2)+\theta(x^2)[T(x),\theta(x)]\}\theta(y)\{[T(x),\theta(x)]\theta(x^2) + \theta(x^2)[T(x),\theta(x)]\} = 0 \text{ for all } x,y\in U$$

By the surjectivity of θ and Lemma 1, we get

$$[T(x),\theta(x)]\theta(x^2) + \theta(x^2)[T(x),\theta(x)] = 0 \quad \text{for all } x \in U \quad (21)$$

Replacing y by 2yx in (14) gives

$$\theta(x^3yx)[T(x), \theta(x)] - \theta(xyx^2)[T(x), \theta(x)]\theta(x) = 0 \text{ for all } x,y\in U \quad (22)$$

Replacing $\theta(y)$ by $2[T(x), \theta(x)]\theta(y)$ in the above relation gives

$$\theta(x^3)[T(x),\theta(x)]\theta(yx)[T(x),\theta(x)]- \theta(x)[T(x),\theta(x)]\theta(yx^2)[T(x),\theta(x)]\theta(x) = 0 \text{ for all } x,y \in U \quad (23)$$

In the above relation let

$a = \theta(x^3)[T(x),\theta(x)]$, $b = \theta(x)[T(x),\theta(x)]$, $c = - \theta(x^2)[T(x),\theta(x)]\theta(x)$ and $z = \theta(y)$

From the above substitutions, we have

$$azb + bzc = 0.$$

We apply Lemma 2 to the above relation to obtain

$$\{\theta(x^3)[T(x),\theta(x)]-\theta(x^2)[T(x),\theta(x)]\theta(x)\}\theta(yx)[T(x),\theta(x)]=0 \quad \text{for all } x, y\in U \quad (24)$$

Replacing y by $2yx^2$ in the above relation gives

$$\{\theta(x^3)[T(x),\theta(x)]-\theta(x^2)[T(x), \theta(x)]\theta(x)\}\theta(yx^3)[T(x),\theta(x)]=0 \quad \text{for all } x,y\in U \quad (25)$$

On the other hand replacing y by 2yx in relation (24) and right multiplying of this relation by $\theta(x)$ gives

$$\{\theta(x^3)[T(x),\theta(x)] - \theta(x^2)[T(x),\theta(x)]\theta(x)\}\theta(yx^2)[T(x),\theta(x)]\theta(x) = 0, \quad \text{for all } x, y \in U. \quad (26)$$

Subtracting (26) from (25) gives

$$\{\theta(x^3)[T(x),\theta(x)]-\theta(x^2)[T(x),\theta(x)]\theta(x)\}\theta(y)\{\theta(x^3)[T(x),\theta(x)]-\theta(x^2)[T(x),\theta(x)]\theta(x)\}=0$$

for all $x,y\in U$

By the surjectivity of θ and Lemma 1, we get

$$\theta(x^3)[T(x),\theta(x)] - \theta(x^2)[T(x),\theta(x)]\theta(x) = 0, \quad \text{for all } x \in U \tag{27}$$

Right multiplication of (21) by θ(x) gives

$$[T(x),\theta(x)]\theta(x^3) + \theta(x^2)[T(x),\theta(x)]\theta(x) = 0, \quad \text{for all } x \in U \tag{28}$$

According to (27) and (28), we have

$$[T(x),\theta(x)]\theta(x^3) + \theta(x^3)[T(x),\theta(x)] = 0, \quad \text{for all } x \in U \tag{29}$$

Left multiplication of (22) by [T(x), θ(x)] gives

$$[T(x),\theta(x)]\theta(x^3yx)[T(x),\theta(x)] - [T(x),\theta(x)]\theta(xyx^2)[T(x),\theta(x)]\theta(x) = 0, \text{ for all } x, y \in U \tag{30}$$

Adding relations (23) and (30) and using (29), we obtain

$$\{[T(x),\theta(x)]\theta(x)+\theta(x)[T(x),\theta(x)]\}\theta(yx^2)[T(x),\theta(x)]\theta(x)= 0 \text{ for all } x,y\in U \tag{31}$$

Using (27) we obtain from the above relation

$$\{[T(x),\theta(x)]\theta(x)+\theta(x)[T(x),\theta(x)]\}\theta(yx^3)[T(x),\theta(x)]=0 \quad \text{for all } x,y \in U \tag{32}$$

Left multiplication of (32) by $\theta(x^2)$ gives

$$\{\theta(x^2)[T(x),\theta(x)]\theta(x)+\theta(x^3)[T(x),\theta(x)]\}\theta(yx^3)[T(x),\theta(x)] = 0 \text{ for all } x, y \in U$$

According to (27) one can replace $\theta(x^2)[T(x),\theta(x)]\theta(x)$ by $\theta(x^3)[T(x),\theta(x)]$ in the above relation. Thus, we have

$$\theta(x^3)[T(x),\theta(x)]\theta(y)\theta(x^3)[T(x),\theta(x)] = 0, \quad \text{for all } x, y \in U$$

By the surjectivity of θ and Lemma 1, we get

$$\theta(x^3)[T(x),\theta(x)] = 0, \quad \text{for all } x \in U \tag{33}$$

Because of (29), we have

$$[T(x), \theta(x)]\theta(x^3) = 0, \quad \text{for all } x \in U \tag{34}$$

Replacing θ(y) by 2[T(x), θ(x)]θ(y) in (14) gives

$$\theta(x^3)[T(x),\theta(x)]\theta(y)[T(x),\theta(x)]-\theta(x)[T(x),\theta(x)]\theta(yx)[T(x),\theta(x)]\theta(x)=0, \text{ for all } x,y\in U \tag{35}$$

Using (33) the above relation reduces to

$$\theta(x)[T(x),\theta(x)]\theta(yx)[T(x),\theta(x)]\theta(x) = 0 \text{ for all } x,y\in U \tag{36}$$

Replacing y by 2xy in (36) gives

$$\theta(x)[T(x), \theta(x)]\theta(x)\theta(y)\theta(x)[T(x), \theta(x)]\theta(x) = 0 \text{ for all } x, y \in U$$

By the surjectivity of θ and Lemma 1, we get

$$\theta(x)[T(x), \theta(x)]\theta(x) = 0, \quad \text{for all } x \in U \tag{37}$$

Putting x + y for x in (37), we obtain

$$\theta(x)[T(x),\theta(x)]\theta(y) + \theta(x)[T(x), \theta(y)]\theta(x) + \theta(x)[T(y), \theta(x)]\theta(x) + \theta(y)[T(x), \theta(x)]\theta(x)$$
$$+ \theta(x)[T(x),\theta(y)]\theta(y) + \theta(x)[T(y),\theta(x)]\theta(y)+ \theta(y)[T(x),\theta(x)]\theta(y) + \theta(x)[T(y),(y)]\theta(x) +$$

$$\theta(y)[T(x), \theta(y)]\theta(x) + \theta(y)[T(y), \theta(x)]\theta(x) + \theta(x)[T(y), \theta(y)]\theta(y) + \theta(y)[T(x), \theta(y)]\theta(y)$$
$$+ \theta(y)[T(y), \theta(x)]\theta(y) + \theta(y)[T(y), \theta(y)]\theta(x) = 0 \text{ for all } x, y \in U \quad (38)$$

Putting −x for x in the above relation and combining the relation so obtained with (38), we obtain

$$\theta(x)[T(x),\theta(y)]\theta(y) + \theta(x)[T(y), \theta(x)]\theta(y) + \theta(y)[T(x), \theta(x)]\theta(y) + \theta(x)[T(y), \theta(y)]\theta(x)$$
$$+ \theta(y)[T(x), \theta(y)]\theta(x) + \theta(y)[T(y), \theta(x)]\theta(x) = 0, \text{ for all } x, y \in U \quad (39)$$

After comparing (38) and (39), we have

$$\theta(x)[T(x), \theta(x)]\theta(y) + \theta(x)[T(x), \theta(y)]\theta(x) + \theta(x)[T(y), \theta(x)]\theta(x) + \theta(y)[T(x), \theta(x)]\theta(x)$$
$$+ \theta(x)[T(y), \theta(y)]\theta(y) + \theta(y)[T(x), \theta(y)]\theta(y) + \theta(y)[T(y), \theta(x)]\theta(y) + \theta(y)[T(y),$$
$$\theta(y)]\theta(x) = 0, \text{ for all } x, y \in U \quad (40)$$

Replacing x by 2x in the above relation and subtracting the relation so obtained from the above relation multiplied by 8, we obtain

$$\theta(x)[T(y),\theta(y)]\theta(y) + \theta(y)[T(x),\theta(y)]\theta(y) + \theta(y)[T(y),\theta(x)]\theta(y) + \theta(y)[T(y),\theta(y)]\theta(x) = 0$$
$$\text{for all } x, y \in U \quad (41)$$

Comparing (40) and (41), we obtain

$$\theta(x)[T(x),\theta(x)]\theta(y)+\theta(x)[T(x),\theta(y)]\theta(x)+\theta(x)[T(y),\theta(x)]\theta(x) + \theta(y)[T(x) ,\theta(x)]\theta(x) = 0$$
$$\text{for all } x, y \in U \quad (42)$$

Right multiplication of (42) by $\theta(x^2)[T(x), \theta(x)]$ and using (33) gives

$$\theta(x)[T(x),\theta(x)]\theta(y)\theta(x^2)[T(x),\theta(x)] = 0 \quad \text{for all } x, y \in U \quad (43)$$

Left multiplication of (43) by θ(x) gives

$$\theta(x^2)[T(x), \theta(x)]\theta(y)\theta(x^2)[T(x), \theta(x)] = 0 \qquad \text{for all } x, y \in U$$

By the surjectivity of θ and Lemma 1, we get

$$\theta(x^2)[T(x), \theta(x)] = 0, \qquad \text{for all } x \in U \quad (44)$$

Because of (21), we also have

$$[T(x), \theta(x)]\theta(x^2) = 0, \qquad \text{for all } x \in U \quad (45)$$

Right multiplication of (42) by θ(x)[T(x), θ(x)] gives because of (44)

$$\theta(x)[T(x), \theta(x)]\theta(y)\theta(x)[T(x), \theta(x)] = 0, \quad \text{for all } x, y \in U$$

By the surjectivity of θ and Lemma 1, we get

$$\theta(x)[T(x), \theta(x)] = 0, \qquad \text{for all } x \in U \quad (46)$$

Left multiplication of (42) by [T(x), θ(x)]θ(x) and use of (45) gives

$$[T(x), \theta(x)]\theta(x)\theta(y)[T(x), \theta(x)]\theta(x) = 0 \qquad \text{for all } x, y \in U$$

By the surjectivity of θ and Lemma 1, we get

$$[T(x),\theta(x)]\theta(x) = 0 \qquad \text{for all } x \in U \quad (47)$$

Replacing x by x + y in (47) and then using (47) gives

$[T(x),\theta(x)]\theta(y) + [T(x),\theta(y)]\theta(x) + [T(x),\theta(y)]\theta(y) + [T(y),\theta(x)]\theta(x) + [T(y), \theta(x)]\theta(y) + [T(y),\theta(y)]\theta(x) = 0$ for all $x, y \in U$

Putting −x for x in the above relation and comparing the relation so obtained with the above relation gives:

$$[T(x),\theta(y)]\theta(x)+[T(y),\theta(x)]\theta(x)+[T(x),\theta(x)]\theta(y)= 0 \text{ for all } x,y\in U \quad (48)$$

Right multiplication of the above relation by $[T(x), \theta(x)]$ use of (46) gives

$$[T(x), \theta(x)]\theta(y)[T(x), \theta(x)] = 0 \qquad \text{for all } x, y \in U$$

By the surjectivity of θ and Lemma 1, we get

$$[T(x), \theta(x)] = 0, \qquad \text{for all } x \in U \quad (49)$$

Now, we will prove that

$$T(xy + yx) = T(y)\theta(x) + \theta(x)T(y) \text{ for all } x,y\in U \quad (50)$$

In order to prove the above relation, we need to prove the following relation

$$[A(x, y), \theta(x)] = 0 \qquad \text{for all } x,y\in U \quad (51)$$

where $A(x, y)$ stands for $T(xy + yx) - T(y)\theta(x) - \theta(x)T(y)$. With respect to this notation equation (2) can be rewritten as,

$$\theta(x)A(x, y)\theta(x) = 0 \text{ for all } x,y\in U \quad (52)$$

Replacing x by x + y in relation (49) gives

$$[T(x), \theta(y)] + [T(y), \theta(x)] = 0 \qquad \text{for all } x,y\in U \quad (53)$$

After replacing y by xy + yx in (53) and using (49), we obtain

$$\theta(x)[T(x),\theta(y)] + [T(x),\theta(y)]\theta(x) + [T(xy+yx),\theta(x)] = 0 \qquad \text{for all } x, y \in U$$

According to (53) we can replace in the above relation $[T(x) , \theta(y)]$ by $- [T(y) , \theta(x)]$. We then have

$$[T(xy+yx),\theta(x)] - \theta(x)[T(y),\theta(x)] - [T(y),\theta(x)]\theta(x) = 0 \text{ for all } x, y \in U$$

This can be written in the form

$$[T(xy+yx) - T(y)\theta(x) - \theta(x)T(y) , \theta(x)] = 0, \text{ for all } x, y \in U$$

The proof of relation (51) is therefore complete.
Replacing x by x + z in (52) and using (52) gives

$\theta(x)A(x,y)\theta(z) + \theta(x)A(z, y)\theta(x) + \theta(z)A(x, y)\theta(x) + \theta(z)A(z, y)\theta(x) +\theta(z)A(x, y)\theta(z) + \theta(x)A(z, y)\theta(z) = 0$ for all $x, y, z \in U$

After replacing x for −x in the above relation and adding the relation so obtained to the above relation, we arrive at:

$$\theta(x)A(x,y)\theta(z)+\theta(x)A(z,y)\theta(x) + \theta(z)A(x,y)\theta(x) = 0 \qquad \text{for all } x,y,z \in U$$

Right multiplication of the above relation by $A(x, y)\theta(x)$ and using (52) gives

$$\theta(x)A(x, y)\theta(z)A(x, y)\theta(x) = 0 \qquad \text{for all } x,y,z \in U \quad (54)$$

Using (54), the above relation can be written in the form

$$\theta(x)A(x, y)\theta(z)\theta(x)A(x, y) = 0 \quad \text{for all } x,y,z \in U \tag{55}$$

By the surjectivity of θ and Lemma 1, we get

$$\theta(x)A(x, y) = 0 \quad \text{for all } x,y\in U \tag{56}$$

From (51) and (56), we also get

$$A(x, y)\theta(x) = 0 \quad \text{for all } x,y\in U \tag{57}$$

Replacing x by x + z in (57) gives

$$A(x, y)\theta(z) + A(z, y)\theta(x) = 0 \quad \text{for all } x, y, z \in U$$

Right multiplication of the above relation by A(x, y) and using (56) gives

$$A(x, y)\theta(z)A(x, y) = 0 \quad \text{for all } x,y, z \in U$$

By the surjectivity of θ and Lemma 1, we get

$$A(x, y) = 0 \quad \text{for all } x,y \in U$$

The proof of (50) is therefore complete.

(ii) If $U \subset Z(R)$

Right multiplication of relation(2) by r A(x, y), r∈R and by primness of R, we get

$$\theta(x)A(x, y) = 0 \text{ for all } x,y \in U$$

Replacing x by x + z in above relation gives

$$\theta(z)A(x, y) + \theta(x)A(z, y) = 0 \quad \text{for all } x, y, z \in U$$

Left multiplication of the above relation by A(x, y), we get

$$A(x, y)\theta(z)A(x, y) = 0 \quad \text{for all } x, y, z \in U$$

Right multiplication of the above relation by r θ(z), r∈R and by primness of R, we get

$$A(x, y)\theta(z) = 0 \quad \text{for all } x, y, z \in U$$

By the surjectivity of θ and primness of R, we get (50)

In particular when y = x (50) reduces to

$$2T(x^2) = T(x)\theta(x) + \theta(x)T(x) \quad \text{for all } x \in U$$

By Theorem.1.1 in [11] it follows that T(xy)= T(x)θ(y) = θ(x)T(y) for all x,y∈U, which completes the proof. □

Corollary 1.

Let R be a 2-torsion free prime ring , U a square closed Lie ideal, and let T: R→R an additive mapping. Suppose that 3T(xyx) = T(x)yx + xT(y)x + xyT(x) and xT(xy+yx)x = $xT(y)x^2 + x^2T(y)x$ holds for all pairs x, y ∈ U , where T(u) ∈U, for all u∈U, then T(xy)= T(x)y = xT(y) for all x,y∈U.

References

[1] J. Bergen, I.N. Herstein, and J.W. Ker,: Lie ideals and derivation of prime rings , J. Algebra 71 (1981) , 259-267.

[2] M. Brešar, J. Vukman: Jordan derivations on prime rings, Bull. Austral. Math. Soc. 37(1988), 321-322.

[3] M. Brešar: Jordan derivations on semiprime rings, Proc. Amer. Math. Soc. 104 (1988), 1003-1006.

[4] M. Brešar: Jordan mappings of semiprime rings, J. Algebra 127 (1989), 218-228.

[5] J. Cusack: Jordan derivations on rings, Proc. Amer. Math. Soc. 53 (1975), 321-324.

[6] M.N.Daif, M.S.Tammam El-Sayiad, N.M.Muthana: An Equation Related to θ-Centralizers in Semiprime Rings , International mathematical forum , 3,2008. no. 24, 1165 – 1177.

[7] I.N. Herstein: Jordan derivations of prime rings, Proc. Amer. Math. Soc. 8 (1957), 1104-1110.

[8] A.H.Majeed, M.I.Meften: Jordan θ-Centralizers of Prime and Semiprime Rings. (preprint).

[9] A.H.Majeed, M.I.Meften: On θ-Centralizers of Prime and Semiprime Rings. (preprint).

[10] A.H.Majeed, M.I.Meften: Lie Ideals and Jordan θ-Centralizers Of Prime Rings. (preprint).

[11] A.H.Majeed, M.I.Meften: Lie Ideals and θ-Centralizers of Prime Rings. (preprint).

[12]A.H.Majeed, M.I.Meften: An Identity on θ-Centralizers of Prime and Semiprime Rings. (preprint).

[13]A.H.Majeed, M.I.Meften: θ-Centralizers of Lie Ideals in Prime Rings. (preprint).

[14] J. Vukman and I. Kosi-Ulbl, On centralizers of semiprime rings, Aequationes Math. 66, 3 (2003) 277 - 283.

[15] J.Vukman: An identity related to centralizers in semiprime rings, Comment. Math. Univ. Carolinae 40 (1999), 447-456.

[16] J.Vukman: Centralizers on semiprime rings, Comment. Math. Univ. Carolinae 42(2001),237-245.

[17] B. Zalar: On centralizers of semiprime rings, Comment. Math. Univ. Carolinae 32 (1991), 609-614