Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.17.(2) 2025,pp.Math 30–36

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

# Quasi-semi clean rings

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ARTICLE INFO

*Article history:*
Received: 06/03/2025
Rrevised form: 14/03/2025
Accepted : 17/04/2025
Available online: 30/06/2025

*Keywords:*

Clean rings, r-clean rings , quasi clean rings

ABSTRACT

Recently, Tang et al. [1] studied a broader concept than the concept of clean rings called quasi-clean rings. In this paper, we use the method of Tang et al. [1] in generalizing clean to construct and study a generalization of semiclean rings of Ye [3] to a broader class of rings called quasi-semiclean rings. Let $R$ be a ring with identity. Then, $R$ is said to be a quasi-semicelan ring if every element $r \in R$ can be expressed as $r = u + e$ where $u \in U(R)$ is a unit of $R$ and $e$ is a quasi-periodic element of $R$: that is $e$ satisfies the equation $e^n = ke^m$ where $n, m \in \mathbb{Z}^+$ and $k$ is a central unit, $k \in U(R) \cap C(R)$. We prove several properties of the class of quasi-semiclean rings similar to those of semiclean rings and also settle new properties and results. In particular, we show that clean rings and certain types of quasi-semiclean rings (quasi-clean rings to be precise) are the same concept and prove this equivalence. Furthermore, we investigate the quasi-semicleaness of various types of rings such as Morita context rings, trivial extensions of modules $R(M)$, and the Nagata ring $R(+)M$.

MSC..

https://doi.org/10.29304/jqcsm.2025.17.22204

## 1. Introduction

In the recent decades, the original definition of clean ring provided by Nicholson [2] witnessed a vast interest due to the algebraic richness of this type of rings. Recall that a ring $R$ is said to be a clean ring if we can write each $r \in R$ as $r = u + e$, where $u \in U(R)$ is a unit and $e \in Id(R)$ is an idempotent. The concept of clean ring was then extended to the concept of semiclean ring by Ye [3], where he replaced the idempotent by a general periodic element (an element $e$ that fulfills the condition $e^n = e^m$ where $n, m \in \mathbb{Z}^+$). Ye provided interesting classes of rings that are semiclean but not clean, see ([3], Example 3.1). Later, Fan and Yang [4] continued the work of Nicholson by defining $f(x)$-clean ring, where $f(x)$ is a polynomial with coefficients from the underlying ring $R$. Let $f(x) \in C(R)[x]$ (here $C(R)$ denotes the center of the ring $R$). They declare $R$ is an $f(x)$-clean ring if each $r \in R$ can be written as $r = u + e$

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Communicated by 'sub editor'

where $u \in U(R)$ and $e$ fulfills $f(e) = 0$. They gave a nice identification of clean rings in terms of certain types of $f(x)$-clean rings, see ([4], Theorem 3.2). Continuing on the previous work of Fan and Yang, El-Najjar and Akram [5-6] studied the concept of $f(x), g(x)$-clean ring where each element $r \in R$ can be written as $r = u + e_1 + e_2$, where $u \in U(R)$ and $e_1, e_2$ fulfills $f(e_1) = g(e_2) = 0$. They provided a characterization of feebly clean rings [7] in terms of certain $f(x), g(x)$-clean rings, see ([5], Theorem 2.3). More recently, Tang et al. [1] defined a new type of rings called quasi-clean rings. An element $e \in R$ is said to be a quasi-idempotent if $e^2 = ke$ where $k \in U(R) \cap C(R)$. A ring $R$ is said to be a quasi-clean ring if each element $r \in R$ can be written as $r = u + e$ where $u \in U(R)$ and $e \in QI(R)$ ($QI(R)$ is the set of all quasi-idempotents of $R$). They constructed an abundant of examples that properly generalize clean rings with multiple intriguing results.

In this paper, we investigate the concept of quasi-semiclean rings as a broader extension of quasi-clean rings. An element $e \in R$ is said to be a quasi-periodic element if $e^n = ke^m$ where $n, m \in \mathbb{Z}^+$ and $k \in U(R) \cap C(R)$. The set of all quasi-periodic elements of $R$ is denoted by $QP(R)$. An element $r \in R$ is said to be quasi-semicelan if $r = u + e$ where $u \in U(R)$ and $e \in QP(R)$ and $R$ is said to be quasi-semicelan ring if every element of $R$ is quasi-semicelan. Among many results, we show that for certain rings the quasi-semicleanness of them is equivalent to the quasi-semicleanness of their upper triangular matrices ring $T_n(R)$ for all $n \geq 1$, see Theorem 2.7. Moreover, we give a new definition of clean rings [2] in term of quasi-clean rings. In particular, we show in Theorem 2.11 that:

**Theorem:** Let $R$ be a ring, $k \in U(R) \cap C(R)$, and $a, b \in C(R)$ such that $b - a \in U(R)$. Then, the following statements are equivalent:

1. $R$ is a $k$-quasi-clean ring.
2. $R$ is a clean ring.
3. Every element of $R$ can be written as a sum of a unit element, and a root of the polynomial $(x - k^2a)(x - k^2b)$.

We also extend the quasi-semiclean concept to various rings such as the power series rings, the localization of rings, the quotient rings, and various other constructions. Throughout the paper, $R$ is a ring that posses a multiplicative identity.

## 2. Quasi-semiclean rings

**Definition 2.1:** Let $R$ be a ring with identity. An element $e \in R$ is said to be a quasi-periodic element if $e^n = ke^m$ where $n, m \in \mathbb{Z}^+$ and $k \in U(R) \cap C(R)$. The set of all quasi-periodic elements of $R$ is denoted by $QP(R)$.

**Definition 2.2:** Let $R$ be a ring with identity. An element $r \in R$ is said to be quasi-semicelan if $r = u + e$ where $u \in U(R)$ and $e \in QP(R)$. If every element of $R$ is quasi-semiclean element, then $R$ is said to be a quasi-semiclean ring.

It is clear that every semicelan ring is a quasi-semiclean ring. The converse needs not to hold as explained in the following example.

**Example 2.3:** Let us consider the ring $R$ which is defined by $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} \cap \mathbb{Z}_{(7)} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ such that } b \neq 0 \text{ and } 3,5,7 \text{ does not divides } b\right\}$. Using (1, Corollary 3.9(2)-(3)), we see that $R$ is a quasi-clean ring (hence a quasi-semiclean ring) but not a semiclean ring.

**Proposition 2.4:** Let $f: R \to S$ be a ring homomorphism. If $R$ is a quasi-semiclean ring, then so is $f(R)$.

**Proof:** Let $s \in f(R)$. By the assumption there exists an element $r \in R$ such that $s = f(r)$, and $r = u + e$ where $u \in U(R)$, and $e$ fulfills $e^n = ke^n$ for some $n, m \in \mathbb{Z}^+$ and $k \in U(R) \cap C(R)$. Consequently, $s = f(r) = f(u + e) = f(u) + f(e)$ where we have $f(u) \in U(f(R))$, and $f(e)^n = f(e^n) = f(ke^m) = f(k)f(e)^m$, and we see $f(k) \in U\big(f(R)\big) \cap C(f(R))$. Therefore, $f(R)$ is a quasi-semiclean ring.

**Corollary 2.5:** Let $R$ be a ring and $I$ be a two sided ideal of $R$. Then, $R$ is quasi-semiclean ring implies that $\frac{R}{I}$ is a quasi-semiclean ring.

**Proof:** Let $\pi: R \to \frac{R}{I}$ be the canonical homomorphism defined by $\pi(r) = r + I$. Since $\pi$ is an onto ring homomorphism, then $\frac{R}{I}$ is a quasi-semiclean ring by Proposition 2.4.

**Definition 2.6:** A ring $R$ is said to be of fixed quasi-periodicity if there exist fixed $n, m \in \mathbb{N}$ such that for all $e \in QP(R)$, we have $e^n = ke^m$ where $k \in U(R) \cap C(R)$..

**Theorem 2.7:** Let $R$ be a ring of fixed quasi-periodicity. Then, $R$ is quasi-semicelan ring if and only if $T_n(R)$ is a quasi-semicelan ring for all $n \geq 1$.

**Proof:** (⇒) Since $R$ is of fixed quasi-periodicity, there exist fixed $n', m' \in \mathbb{N}$ such that for all $e \in QP(R)$, we have $e^{n'} = ke^{m'}$ where $k \in U(R) \cap C(R)$. Let $M = (m_{ij}) \in T_n(R)$. Then, $m_{ij} = 0$ for all $1 \leq j < i \leq n$. By assumption, we have $m_{ii} = u_{ii} + e_{ii}$, where $u_{ii} \in U(R)$, and $e_{ii}$ fulfills $e_{ii}^{n_{ii}} = k_{ii}e_{ii}^{m_{ii}}$ for some $n_{ii}, m_{ii} \in \mathbb{N}$ and $k_{ii} \in U(R) \cap C(R)$ where $1 \leq i \leq n$. Thus, we can factorize $M$ as follows:

$$M = \begin{pmatrix} u_{11} & m_{12} & \cdots & m_{1n} \\ \vdots & & \ddots & \vdots \\ & & & m_{n-1n} \\ 0 & & \cdots & u_{nn} \end{pmatrix} + \begin{pmatrix} e_{11} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & e_{nn} \end{pmatrix}.$$

The first matrix is invertible, since its determinant is a unit element of $R$. Also, we see that

$$\begin{pmatrix} e_{11} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & e_{nn} \end{pmatrix}^{n'} = \begin{pmatrix} e_{11}^{n'} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & e_{nn}^{n'} \end{pmatrix} = \begin{pmatrix} k_{11}e_{11}^{m'} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & k_{nn}e_{nn}^{m'} \end{pmatrix} = \begin{pmatrix} k_{11} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & k_{nn} \end{pmatrix} \begin{pmatrix} e_{11}^{m'} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & e_{nn}^{m'} \end{pmatrix}$$
$$= \begin{pmatrix} k_{11} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & k_{nn} \end{pmatrix} \begin{pmatrix} e_{11} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & e_{nn} \end{pmatrix}^{m'}$$

because $e_{ii} \in QP(R)$ for all $1 \leq i \leq n$. Thus the $e$'s matrix lies in $QP(T_n(R))$ because the $k$'s matrix lies in $U(T_n(R)) \cap C(T_n(R))$. Consequently, $M$ is a quasi-semiclean element and so is $T_n(R)$.

(⇐) Define the map $f: T_n(R) \to R$ by the mapping $M = (m_{ij})$ to $m_{11}$ i.e. $f(M) = f\left((m_{ij})\right) = m_{11}$. Evidently, $f$ is a rings homomorphism and $f\left(T_n(R)\right) = R$, so $R$ is a quasi-semiclean ring by Proposition 2.4.

**Proposition 2.8**: Let $R$ be a ring, then the following statements hold

1. $R$ is quasi-semiclean if and only if $R[[x]]$ is quasi-semiclean.
2. If $R$ is commutative, then $R[x]$ is never a quasi-semiclean ring.

**Proof:** (1) (⇒) Let $f = \sum_{n\geq 0} a_n x^n \in R[[x]]$. By assumption, $a_0 = u + e$ where $u \in U(R)$, and $e$ fulfills $e^n = ke^m$ for some $m, n \in \mathbb{Z}^+$ and $k \in U(R) \cap C(R)$. It follows that

$$f = a_0 + \sum_{n\geq 1} a_n x^n = \left(u + \sum_{n\geq 1} a_n x^n\right) + e$$

where, the series in the brackets is a unit and so $R[[x]]$ is a quasi-semiclean ring.

(⇐) Define $\phi: R[[x]] \to R$ by the formula $\phi(f) = f(0)$. Since $\phi\left(R[[x]]\right) = R$, we conclude that $R$ is a quasi-semiclean ring by Proposition 2.4.

(2) We first observe that $U(R[x]) = U(R)$ and the periodic elements of $R[x]$ are those of $R$ hence $QP(R[x]) = QP(R)$. Suppose that $R[x]$ is a quasi-semiclean ring. Then, $x = u + e$ where $u \in U(R)$, and $e \in QP(R)$. Thus, $x - e = u \in U(R[x]) = U(R)$ which is absurd, thus $R[x]$ is not a quasi-semiclean ring.

**Definition 2.9:** Let $R$ be a quasi-semiclean ring and $T \subseteq U(R) \cap C(R)$. If we can write each $r \in R$ as $r = u + e$ where $u \in U(R)$, and $e$ fulfills $e^n = ke^m$ for some $m, n \in \mathbb{Z}^+$ and $k \in U(R) \cap C(R)$, where $k \in T$, we say that $R$ is a $T$-quasi-semiclean ring. In particular, if $T = \{k\}$ where $k \in U(R) \cap C(R)$, we say that $R$ is a $k$-quasi-semiclean ring.

**Definition 2.10:** If $R$ is a $k$-quasi-semiclean ring where the $e$'s are always quasi-idempotent elements (that is $n = 2$ and $m = 1$ so $e^2 = ke$), we say that $R$ is a $k$-quasi-clean ring.

**Theorem 2.11:** Let $R$ be a ring, $k \in U(R) \cap C(R)$, and $a, b \in C(R)$ such that $b - a \in U(R)$. Then, the following statements are equivalent:

1. $R$ is a $k$-quasi-clean ring.
2. $R$ is a clean ring.
3. Every element of $R$ can be written as a sum of a unit element, and a root of the polynomial $(x - k^2a)(x - k^2b)$.

**Proof:** (1) ⇒ (3) Let $r \in R$. Since $R$ is $k$-quasi-clean ring, we have that $\frac{r-k^2a}{k(b-a)} = u + e$, where $u \in U(R)$ and $e$ fulfills $e^2 = ke$. Therefore, $r = k(b-a)u + k(b-a)e + k^2a$, where $k(b-a)u \in U(R)$ and $k(b-a)e + k^2a$ fulfills

$$(k(b-a)e + k^2a - k^2a)(k(b-a)e + k^2a - k^2b) = k^2(b-a)^2e^2 - k^3(b-a)^2e = (b-a)^2k^2(e^2 - ke) = 0$$

thus, every element of $R$ is a sum of a unit element and a root of the polynomial $(x - k^2a)(x - k^2b)$ as required.

(3) ⇒ (1) Let $r \in R$. Then, $k(b-a)r + k^2a = u + e$, where $u \in U(R)$ and $(e - k^2a)(e - k^2b) = 0$. Then, $k(b-a)r = u + e - k^2a$, and thus

$$r = \frac{u}{k(b-a)} + \frac{e - k^2a}{k(b-a)}.$$

It is obvious that the first fraction is in $U(R)$, so we need only to show that the second fraction satisfies

$$\left(\frac{e - k^2a}{k(b-a)}\right)^2 = k\left(\frac{e - k^2a}{k(b-a)}\right).$$

We have

$$\left(\frac{e-k^2a}{k(b-a)}\right)^2 = \frac{(e-k^2a)(e-k^2a)}{k^2(b-a)^2} = \frac{(e-k^2a)(e-k^2b+k^2b-k^2a)}{k^2(b-a)^2} = \frac{(e-k^2a)(e-k^2b)+k^2(b-a)(e-k^2a)}{k^2(b-a)^2}$$
$$= \frac{e-k^2a}{b-a} = k\left(\frac{e-k^2a}{k(b-a)}\right)$$

as required. Hence $R$ is a $k$-quasi-clean ring.

(2) ⇔ (3) Since $a, b \in C(R)$, we see that $ka, kb \in C(R)$ with $kb - ka = k(b-a) \in U(R)$. Thus, the result follows by ([4], Theorem 2.3].

**Proposition 2.12:** Let $R$ be a commutative ring and $S$ a multiplicative subset of $R$. Then, the following statements hold:

1. If $R$ is a quasi-semiclean ring, then so is $S^{-1}R$.
2. If quasi-periodic elements of $R$ lift modulo $N(R)$, then $R$ is a quasi-semiclean ring if and only if $R/N(R)$ is a quasi-semiclean ring.

**Proof:** (1) Let $\frac{r}{s} \in S^{-1}R$. Since $R$ is quasi-semiclean ring, we have $r = u + e$ where $u \in U(R)$, and $e$ fulfills $e^n = ke^m$ for some $m, n \in \mathbb{Z}^+$ and $k \in U(R) \cap C(R)$. As a result, we see that $\frac{r}{s} = \frac{u+e}{s} = \frac{u}{s} + \frac{e}{s}$ where $\frac{u}{s} \in U(S^{-1}R)$, and

$\left(\frac{e}{s}\right)^n = \frac{e^n}{s^n} = \frac{ke^m}{s^n} = \frac{s^m k}{s^n} \cdot \frac{e^m}{s^m} = \frac{s^m k}{s^n} \cdot \left(\frac{e}{s}\right)^m$ where $\frac{s^m k}{s^n} \in U(S^{-1}R) \cap C(S^{-1}R)$. This shows that $\frac{r}{s}$ is a quasi-semiclean element of $S^{-1}R$ and so $S^{-1}R$ itself.

(2) (⇐) Let $r \in R$. Then, $r + N(R) = (u + N(R)) + (e + N(R))$ where $u + N(R) \in U(R/N(R))$, and $e + N(R)$ fulfilles $(e + N(R))^n = (k + N(R))(e + N(R))^m$ where $k + N(R) \in U(R/N(R)) \cap C(R/N(R))$. Since unit elements, and quasi-periodic lift modulo $N(R)$, we see that $e$ is a quasi-periodic element of $R$ and $u$ is unit element of $R$. Thus, $r - u - e = n \in N(R)$, and so $r = (n + u) + e$ which implies that $r$ is a quasi-semicelan element of $R$ because $n + u \in U(R)$ and so the ring $R$ itself is quasi-semiclean.

(⇒) This follows from Proposition 2.4, since $R/N(R)$ is a homomorphic image of $R$.

A Morita context is a sixtuple $(R, S, M, N, \eta, \theta)$ where $R$ and $S$ are rings, $M$ and $N$ are $(R, S)$-bimodule and $(S, R)$-bimodule respectively, with two bimodule homomorphism $\eta: M \otimes_S N \to R$ and $\theta: N \otimes_R M \to S$ that satisfies:

$$\eta(m \otimes n)m' = m\theta(n \otimes m'), \theta(n \otimes m)n' = n\eta(m \otimes n').$$

We can associate with such Morita context a ring

$$T = \begin{bmatrix} R & M \\ N & S \end{bmatrix} = \left\{\begin{bmatrix} r & m \\ n & s \end{bmatrix} \mid r \in R, m \in M, n \in N, s \in S\right\},$$

with the usual addition of matrices and a product defined by:

$$\begin{bmatrix} r & m \\ n & s \end{bmatrix}\begin{bmatrix} r' & m' \\ n' & s' \end{bmatrix} = \begin{bmatrix} rr' + \eta(m \otimes n') & rm' + ms' \\ nr' + sn' & \theta(n \otimes m') + ss' \end{bmatrix}.$$

The resulting ring is called a Morita context ring.

**Proposition 2.13:** Let $T = \begin{bmatrix} R & M \\ N & S \end{bmatrix}$ be a Morita context ring where $\eta, \theta = 0$. If $T$ is a quasi-semiclean ring, then so are $R$ and $S$.

**Proof:** Suppose that $T$ is quasi-semiclean with $\eta, \theta = 0$. Define $I = \begin{bmatrix} 0 & M \\ N & S \end{bmatrix}$ and $J = \begin{bmatrix} R & M \\ N & 0 \end{bmatrix}$. It is immediate that $I$ and $J$ are ideals of $T$ with $\frac{T}{I} \cong R$ and $\frac{T}{J} \cong S$ thus $R$ and $S$ are quasi-semiclean rings by Corollary 2.5.

As an immediate result from Proposition 2.13, we have the following result.

**Corollary 2.14:** Let $R$ and $S$ be rings and $M$ be an $(R, S)$-bimodule. Let $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ be the formal triangular matrix ring. If $T$ is a quasi-semiclean ring, then so are $R$ and $S$.

Let $R$ be a commutative ring and $M$ an $R$-module. The trivial extension of $R$ by $M$ is a commutative ring denoted by $R(M)$ whose underling set is

$$R(M) = \left\{\begin{bmatrix} r & m \\ 0 & r \end{bmatrix} \mid r \in R, m \in M\right\},$$

with usual matrix addition and multiplication.

**Proposition 2.15:** Let $R$ be a commutative ring and $M$ an $R$-module. Then, the trivial extension $R(M)$ is a quasi-semiclean ring if and only if $R$ is a quasi-semiclean ring.

Proof: (⇒) Suppose that $R(M)$ is a quasi-semiclean ring and consider the set $I = \left\{\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \mid m \in M\right\}$. We easily recognize $I$ as an ideal of $R(M)$ and additionally that $\frac{R(M)}{I} \cong R$, thus $R$ is a quasi-semiclean ring by Corollary 2.5.

($\Leftarrow$) Suppose that $R$ is a quasi-semiclean ring, and let $\begin{bmatrix} r & m \\ 0 & r \end{bmatrix} \in R(M)$. By the assumption, we can write $r = u + e$, where $u \in U(R)$, and $e$ fulfills $e^n = ke^m$ for some $m, n \in \mathbb{Z}^+$ and $k \in U(R) \cap C(R)$. Consequently, we have that

$$\begin{bmatrix} r & m \\ 0 & r \end{bmatrix} = \begin{bmatrix} u & m \\ 0 & u \end{bmatrix} + \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix},$$

where, $\begin{bmatrix} u & m \\ 0 & u \end{bmatrix} \in U(R(M))$, and

$$\begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}^n = \begin{bmatrix} e^n & 0 \\ 0 & e^n \end{bmatrix} = \begin{bmatrix} ke^m & 0 \\ 0 & ke^m \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} e^m & 0 \\ 0 & e^m \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}^m$$

so $\begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$ is a quasi-periodic element of $R(M)$ hence we have a quasi-semiclean decomposition of the chosen element and thus $R(M)$ is a quasi-semiclean ring.

Let $R$ be a commutative ring and $M$ and $R$-module. Then, Nagata ring is a commutative ring denoted by $R(+)M$ which is the abelian group $R \times M$ with multiplication defined as: $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$.

**Proposition 2.16:** Let $R$ be a commutative ring and $M$ an $R$-module. Then, the Nagata ring $R(+)M$ is a quasi-semiclean ring if and only if $R$ is a quais-semiclean ring.

**Proof:** ($\Rightarrow$) Suppose that $R(+)M$ is a quasi-semiclean ring and consider the set $0(+)M$. This is clearly an ideal of the ring $R(+)M$ with the additional property that $\frac{R(+)M}{0(+)M} \cong R$, so $R$ is a quasi-semiclean ring by Corollary 2.5.

($\Leftarrow$) Suppose that $R$ is a quasi-semiclean ring. Let $(r, m) \in R(+)M$. By the hypothesis, we write $r = u + e$, where $u \in U(R)$, and $e$ fulfills $e^n = ke^m$ for some $m, n \in \mathbb{Z}^+$ and $k \in U(R) \cap C(R)$. It follows that $(r, m) = (u + e, m) = (u, m) + (e, 0)$, which is a quasi-semiclean decomposition by ([8], Theorem 25.1(6)).

**Proposition 2.17** Let $R$ be a ring with $2 \in U(R)$ and $G$ be a group. Then, the following statements hold:

1. If $G = C_2$, then $R$ is a quasi-clean ring if and only if $RC_2$ is a quasi-clean ring.
2. If $G = C_4$, then $R$ is a quais-clean ring if and only if $RC_4$ is a quasi-clean ring.
3. If $R$ is a quasi-clean ring and $G = C_{2^k}$, then $RC_{2^k}$ is a quasi-clean ring for all $k \geq 0$

**Proof:** (1) Suppose that $R$ is a quasi-clean ring. Since $2 \in U(R)$, we know, form ([9], Proposition 3), that $RC_2 \cong R \times R$, thus $RC_2$ is a quasi-clean ring by ([1], Proposition 2.8(5)). The converse part is now clear since $R$ is a homomorphic image of $RG$ so the statement follows from Proposition 2.4.

(2) As above one direction is clear. Now, suppose that $R$ is a quasi-clean ring. Since $2 \in U(R)$, we see form ([10], Lemma 3.3), that

$$RC_4 \cong R \times R \times R[X]/(X^2 + 1).$$

But, again as $2 \in U(R)$, we see that $R[X]/(X^2 + 1) \cong RC_2 \cong R \times R$, thus we have that $RC_4 \cong R \times R \times R \times R$ and the statement follows from ([1], Proposition 2.8(5)).

(3) We know that $RC_{2^k} \cong (RC_k)C_2$, and it suffices to show that $RC_2$ is a quasi-clean ring which is by (1) and so the statement holds true.

An element $e$ of a ring $R$ is said to be a quasi-antiperiodic element, if $e^n = -ke^m$ for some $n, m \in \mathbb{Z}_+$ and $k \in U(R) \cap C(R)$.

**Theorem 2.18:** Let $R$ be a ring such that all quasi-periodic and quasi-antiperiodic elements satisfy $e^{2n} = ke$ and $e^{2m} = -k'e$ respectively, where $k, k \in U(R) \cap C(R)$ are arbitrary elements, and $n, m \in \mathbb{Z}_+$. Then, $R$ is a quasi-semiclean ring if and only if every element $r$ of $R$ can be written as $r = u + e$ where $u \in U(R)$, and $e$ is a quasi-antiperiodic element.

**Proof:** (⇒) Let $r \in R$. Then, $-r = u + e$, where $u \in U(R)$, and $e$ fulfills $e^{2n} = ke$ for some $k \in U(R) \cap C(R)$ and $n \in \mathbb{Z}_+$. Then, $r = -u + (-e)$, where $-u \in U(R)$, and we have $(-e)^{2n} = ke = -k(-e)$ so $-e$ is a quasi-antiperiodic element and the statement is satisfied.

(⇐) Let $r \in R$. Then, $-r = u + e$, where $u \in U(R)$, and $e$ fulfills $e^{2m} = -k'e$ for some $k \in U(R) \cap C(R)$ and $m \in \mathbb{Z}_+$. Then, $r = -u + (-e)$, where $-u \in U(R)$, and we have $(-e)^{2m} = e^{2m} = -k'e = k'(-e)$ so $-e$ is a quasi-periodic element and the statement is satisfied.

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