Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.17.(2) 2025,pp.Math 46–51

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

A Study Of Weak Attractor And Stability Theory Of Dynamical Systems

Ansam Abbass Ali[a], Haneen Zaghair Hasan[b]

[a]*Directorate of Education in Al-Qadisiyah,Iraq. Email: ansam.a.ali1987@gmail.com*

[b]*Al-Qadisiyah University, College of Law,Iraq. Email: haneen.zaghair.hasan@qu.edu.iq*

ARTICLE INFO

Article history:
Received: 13/03/2025
Rrevised form: 06/04/2025
Accepted : 20/04/2025
Available online: 30/06/2025

Keywords:

Dynamical systems

Attractor

Weak attractor

Stability theory Lyapunov stable

ABSTRACT

The topic of this paper is to concentrate on what is an attractor and show consider an exceptional class of this type of system and what is weak attractor and pullback , forward attractor and explain the difference between them and which one leads to the other through their definitions and do they have the same properties as the weak attractor. We concluded from this that every pullback attractor and forward attractor is weak attractor .Then we explain the stability theory of a dynamical system. We give characterizations of the stability theory in dynamical systems, We also study the Lyapunov stability and define this important theory, which is considered one of the most important theories , where it is ,Adynamical system is Lyapunov stable about an equilibrium point if state trajectories are confined to abounded region whenever the initial condition is chosen sufficiently close to equilibrium point .

MSC..

https://doi.org/10.29304/jqcsm.2025.17.22205

1. Introduction

Dynamical frameworks are the investigation of the drawn-out conduct of developing frameworks. The advanced hypothesis of dynamical frameworks began toward the finish of the nineteenth hundred years with central inquiries concerning the soundness and development of the planetary group. Endeavors to respond to those questions prompted the improvement of a rich and strong field with applications to physical science, science, meteorology, stargazing, financial matters, and different regions. (1)

One of the numerical fields of dynamical frameworks is an attractor. In systems thinking and psychology an attractor is a concept used to describe elements or forces that naturally draw other parts of a system toward them . These attractors guide behavior , moods , or states in a system , whether it be in ecosystems , Human relationship , or organizations. In essence attractors represent patterns or forces that tend to pull or influence individuals or entities toward a particular state . In systems , these attractors can stabilize the system (positive or stable attractors) or lead to destabilization and change (negative or chaotic attractors) (15) .In systems thinking attractors

∗Corresponding author: Ansam Abbass Ali

Email addresses: *ansam.a.ali1987@gmail.com*

Communicated by ‘sub etitor’

help explain why certain behavior or patterns are recurring within complex systems . They highlight forces within a system that pull other elements toward a particular behavior or state often leading to predictable outcomes . Numerous dynamical frameworks have attractors of a more muddled nature. An attractor is a state toward which a framework will in general develop for a wide assortment of beginning states of the framework. Framework esteems that draw near enough to the attractor values stay close regardless of whether marginally upset. [2] There are four principal kinds of attractors: point attractors, limit cycle attractors, Torus attractors, and Odd attractors. [1] In this review, we meaning of frail attractors and make sense of that pullback attractors and Forward attractors are feeble attractors. Then, at that point, we talk about the property of solidness is vital for the way of behaving of dynamical frameworks. Naturally, solidness can be perceived as the necessity that little annoyances of the framework cause just little changes in the framework conduct. As a rule, steadiness assumes a significant part in the hypothesis of dynamical frameworks and control. It describes the property of an unperturbed direction that all bothered directions beginning little irritations cause just little changes in the framework conduct. The main idea of strength has been presented by the Russian mathematician A. M. Lyapunov in 1892. In view of his well-known work a general (Lyapunove hypothesis) has been created to explore the security conduct of general dynamical frameworks. (4)

1.1 Definitions and Related Documentation.

Dynamical system is a system whos state changes over time .Mathematically , a dynamical system consists of a state space and the law of dynamics that allows determining the state that corresponds to the current state defined by linear differential equations of the first order of the later forms and definitions .

Definition 1.1 (3)

A dynamical system on X is the triple $(X.\mathbb{R}.\varphi)$ where φ is a map from the product space $X \times \mathbb{R}$ into the space X satisfying the following axioms: $\varphi(x.0) = x.\forall x \in X \cdots\cdots (1)$

$$\varphi(\varphi(x.t_1).t_2) = \varphi(x.t_1 + t_2).\forall x \in X \text{ and } t_1.t_2 \in \mathbb{R} \cdots\cdots (2)$$

φ is continuous $\cdots\cdots (3)$

Given a dynamical system acting on X. the space. X is called the phase space and the map φ is called the phase map. We will delete the symbol φ and denote the image $\varphi(x.t)$ of a point $(x.t)$ in $X \times \mathbb{R}$ as $t \cdot x$. The above identities will then read

$$0 \cdot x = x..\forall x \in X \cdots\cdots (4)$$

$$t_2 \cdot (t_1 \cdot x) = (t_{1+}t_2) \cdot x.\forall x \in X \text{ and } t_1.t_2 \in \mathbb{R} \cdots\cdots (5)$$

In according to this notation if $\mathcal{M} \subset X$ and $A \subset R$. then $A \cdot \mathcal{M}$ is the set $\{t \cdot x : x \in \mathcal{M} \text{ and } t \in A\}$. If either $\mathcal{M}$ or A is asingleton i.e $\mathcal{M} = \{x\}$ or $A = \{t\}$ .then we simply write $A \cdot x$ or $t \cdot M$ for $\{x\}A$ and $\mathcal{M}\{t\}$. Respectively .For any $x \in X$ the set $\mathbb{R} \cdot x$ is called the trajectory through x . The phase map determines two other maps when one of the variables x or t is fixed . Thus for fixed $t \in \mathbb{R}$. The map $\varphi^t : X \to X$ defined by $\varphi_x : \mathbb{R} \to X$ defined by $\varphi_x(t) = t \cdot x$ is called amotion (through x). Note that φ_x maps $\mathbb{R}$ onto $\mathbb{R} \cdot x$.

Definition 1.2 (2)

A set $\mathcal{M} \subset X$ is called invariant whenever $t \cdot x \in \mathcal{M}.\forall x \in \mathcal{M}$ and $t \in \mathbb{R} \cdots\cdots (6)$

It is called positively invariant whenever (6)holds with $\mathbb{R}$ replaced by $\mathbb{R}^+$ and is called negatively invariant if the same holds with $\mathbb{R}$ replaced by $\mathbb{R}^-$.

Theorem 1.3 :

Let $\{\mathcal{M}\}_i$ be a collection of positively invariant, negatively invariant, or invariant subsets of X . Then their intersection and their union have the same property.

Proof : To fix our ideas let the sets $\mathcal{M}_i$ be positively invariant. Let $M^{`} = \cup \mathcal{M}_i$, and $M^{``} = \cap \mathcal{M}_i$. For any $x \in M^{`}$, we have $x \in \mathcal{M}_i$ for some i .

Thus $xt \in \mathcal{M}_i$ for all $t \in R^+$ since $\mathcal{M}_i$ is positively invariant. Hence indeed $xt \in M^{`}$ for all $t \in R^+$ as $M^{`} \sqsupseteq \mathcal{M}_i$, $M^{`}$ is therefore positively invariant. Now let $x \in M^{``}$ . Then $x \in \mathcal{M}_i$ for every i and by positive invariant of each $\mathcal{M}_i$, $xt \in \mathcal{M}_i$ for each i and each $t \in R^+$ . Hence $xt \in \cap \mathcal{M}_i = M^{``}$ for each $t \in R^+$ and $M^{``}$ is positively invariant.

The proofs for negative invariance and for invariance are entirely analogous .

Theorem 1.4: (5)

Let $\mathcal{M} \subset X$ be positively invariant , negatively invariant or invariant then the closure $\overline{\mathcal{M}}$ has the same property.
Proof : Consider the case of invariant . Let $x \in \overline{M}$ and $t \in R$. Then there is a sequence $\{x_n\}$ in M such that $x_n \to x$ By invariance of M we have $x_n t \in M$ for each n . Since $x_n t \to xt$ we have $xt \in \overline{\mathcal{M}}$. Thus $\overline{\mathcal{M}}$ is invariant . The proofs for negative invariance and for invariance are entirely analogous .

Theorem 1.5 :
A set $\mathcal{M} \subset X$ is positively invariant if and only if the set $X \setminus \mathcal{M}$ is negatively invariant. $\mathcal{M}$ is invariant if and only if $X \setminus \mathcal{M}$ is invariant .
Proof : Let $\mathcal{M}$ be positively invariant . If $x \in X \setminus \mathcal{M}$ and $t \in R^-$ then we must show that $x \in X \setminus \mathcal{M}$. Suppose not . Then $xt \in \mathcal{M}$ and since $-t \in R^+$ we have $xt(-t) = x(t-t) = x0 = x \in M$ by positive invariance of M .
This contradiction shows that $X \setminus \mathcal{M}$ is negatively invariant .
As a final useful result on invariance we have .

Corollary 1.6:
A set $\mathcal{M} \subset X$ is invariant if and only if is both positively and negatively invariant.

Definition 1.7:(6)
We introduce the map $\Gamma . \Gamma^+ .$ and Γ^- from X into 2^x by :

$$\Gamma(x) = \{t \cdot x : t \in \mathbb{R}\} \cdots\cdots (7)$$
$$\Gamma^+(x) = \{t \cdot x : t \in \mathbb{R}^+\} \cdots\cdots (8)$$
$$\Gamma^{\setminus -}(x) = \{t \cdot x : t \in \mathbb{R}^-\} \cdots\cdots (9)$$

$\forall x \in X$ the sets $\Gamma(x). \Gamma^+(x).$ and $\Gamma^-(x)$ are respectively called the trajectory, the positive semi–trajectory through (x) . not that $\forall x \in X. \Gamma(x) = \mathbb{R} \cdot x$

Definition 1.8
A set $\mathcal{M} \subset X$ is invariant , positively invariant, or negatively invariant if and only if respectively
$\Gamma(\mathcal{M}) = \mathcal{M}. \Gamma^+(\mathcal{M}) = \mathcal{M}$ or $\Gamma^-(\mathcal{M}) = \mathcal{M}$.

Definition 1.9
A set $\mathcal{M} \subset X$ is invariant ,positively invariant or negatively invariant if and only if for each
$x \in \mathcal{M}$. Respectively, $\Gamma(x) \subset \mathcal{M}. \Gamma^+(x) \subset \mathcal{M}$ or $\Gamma^-(x) \subset \mathcal{M}$.

2. Stability Theory and Attraction.

This section is devoted to the study of stability and attractor , We start by some formal definitions about attractor and clarification global weak attractor, enlightening the various aspect of this notion .Then we introduce the Lyapunov method , after focusing on the case of linear systems . Stability in dynamical systems subject to some law of force is considered . This leads to a set of differential equations which govern the motion (10). The stability of an orbit of a dynamical systems characterizes whether nearly orbits will remain in a neighborhood of that orbit or be repelled away from it . The basic feature of the stability theory of a Lyapunov function is that one seeks to characterizes stability and a asymptotic stability of a given set in terms of a non-negative scalar function defined on a neighborhood of the given set and decreasing a long its trajectories . It is ingeneral not possible to characterizes stability and the various attractor properties by means of continuous functions .(9)

Definition 2.1 :(7)
With A given $\mathcal{M}$ we associate the sets

$$A_w(\mathcal{M}) = \{x \in X : A^+(x) \cap \mathcal{M} \neq \emptyset \cdots\cdots (10)\}$$
$$A(\mathcal{M}) = \{x \in X : A^+(x) \neq \emptyset \text{ and } A^+(x) \subset \mathcal{M} \cdots\cdots (11)\}$$
$$A_u(\mathcal{M}) = \{x \in X : J^+(x) \neq \emptyset \text{ and } J^+(x) \subset \mathcal{M} \cdots\cdots (12)\}$$

$J^+(x) = \{y \in X : \exists \{x_n\} \in X \text{ and } \{t_n\} \in R^+ \ s \cdot t \cdot x_n \to x. t_n \to +\infty. \text{and } x_n t_n \to y\} \cdots\cdots (13)$:
The sets

$$A_w(\mathcal{M}). A(\mathcal{M}). \text{and } A_u(\mathcal{M})$$

Are respectively called the region of weak attraction, attraction, and uniform attraction of set $\mathcal{M}$ .More ever any point x in $A_w(\mathcal{M}).A(\mathcal{M})$. or $A_u(\mathcal{M})$ may respectively be said to be weakly attracted, attracted, and uniformly attracted to $\mathcal{M}$.

Remark :2.2 (12) A weak attractor will be called a global weak attractor whenever $A_w(\mathcal{M}) = X$. Similarly for attractors , uniform attractors ,or asympototically stable sets , the adjective global is used to indicate that the corresponding region of attracting is the whole space .

Proposition 2.3:(8)
Given $\mathcal{M}$ appoint x is weakly attracted to $\mathcal{M}$ if and only if there is a sequence $\{t_n\} \in \mathbb{R}$ with $t_n \to +\infty$ and $K(t_n - x.\mathcal{M} \to 0)$ ⋯⋯ (14)
A point x is attracted to $\mathcal{M}$ if and only if $K(t.x.\mathcal{M}) \to 0$ as $t \to +\infty$ ⋯⋯ (15)
A point x is uniformly attracted to $\mathcal{M}$ if and only if every neighborhood V of $\mathcal{M}$ there is a neighborhood U of x and a $T > 0$ with $U_t \subset V$ for $t \geq T$ ⋯⋯ (16)

Theorem 2.4:
For any given $\mathcal{M}$, $A_w(\mathcal{M}) \supset A(\mathcal{M}) \supset A_u(\mathcal{M})$ ⋯⋯ (17)
the set
, $A_w(\mathcal{M}).A(\mathcal{M})$ and $A_u(\mathcal{M})$ ⋯⋯ (18) are invariant .

Definition 2.5: (7)
The family $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ is said to be :
1. Pullback attractor with respect to the process U ,if for all $t \in \mathbb{R}$ all bounded $D \subset X$ and all $\varepsilon > 0$ there exists $T_\varepsilon.D(t) > 0$ such that for all $c \geq T_\varepsilon.D(t)$, $dist_x(U(t.t-c)D.\mathcal{M}(t)) < \varepsilon$.

Definition 2.6 : (11)
The family of sets $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ is said to be forward attracting for U if for all $t \in \mathbb{R}$ it satisfies $\lim_{\tau\to\infty} dist_X(U(t.\tau)D.\mathcal{M}(t)) = 0$ for all bounded $D \subset X$

Definition 2.7 (13)
The family of sets $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ is said to be a global pullback attractor for $U(t.\tau)$ if :
The set $\mathcal{M}(t)$ is compact for each $t \in \mathbb{R}$
The family $\mathcal{M}(t)$ is pullback attractor , it means this family has the following property : $dist(U((t.\tau)B, \mathcal{M}(t)) \to 0$, as $\tau \to -\infty$, for all $B \in \mathfrak{R}, t \in \mathbb{R}$.
$\mathcal{M}(t) \subset \cup(t.\tau)\,\mathcal{M}(\tau)$ for all $-\infty < \tau \leq t < \infty$ negatively invariant .
The family $\mathcal{M}$ is the minimal closed family with property (2)

Definition 2.8 :
The forward attracting is said to be uniform if $\lim_{t\to\infty} \sup_{\tau\in\mathbb{R}} dist\, X(U(t+\tau.\tau)D.\mathcal{M}(t+\tau))\tau = 0$ for all bounded $D \subset X$.

Remark 2.9 :
Let the set $\mathcal{M}_1$ and $\mathcal{M}_2$ weak attractor then $\mathcal{M}_1 \cup \mathcal{M}_2$ is weak attractor .

Remark 2.10 :
Every Pullback attractor is weak attractor.
Remark 2.11 :
Every forward attractor is weak attractor.

Definition 2.12 (11)
A given set $\mathcal{M}$ is said to be a weak attractor if $A_w(\mathcal{M})$ is a neighborhood of $\mathcal{M}$ ⋯⋯ (19)
An attractor if $A(\mathcal{M})$ is a neighborhood of $\mathcal{M}$ ⋯⋯ (20)
A uniform attractor if $A_u(\mathcal{M})$ is a neighborhood of $\mathcal{M}$ ⋯⋯ (21)
Stable if every neighborhood U of $(\mathcal{M})$ has positively invariant neighborhood V of $\mathcal{M}$ ⋯⋯ (22)

asymptotically stable if it is stable and is an attractor ....(23)
Unstable if it is not stable ....(24)

Proposition 2.13:
Let X be a closed and stable subset of $\mathcal{M}$. Then X is positively invariant.

Theorem 2.14:
The set $\mathcal{M}$ is stable if and only if every component of $\mathcal{M}$ is stable.
Proof : Not that if $\mathcal{M}$ is compact , then every component of $\mathcal{M}$ is compact . Further if $\mathcal{M}$ is positively invariant , so is every one its components . Now let $\mathcal{M} = \cup \{\mathcal{M}_i : i \in \vartheta\}$ where ϑ is an index set , and $\mathcal{M}_i$ are components of $\mathcal{M}$. Let each $\mathcal{M}_i$ be stable , i.e $D^+(\mathcal{M}_i) = \mathcal{M}_i$. Then $D^+(\mathcal{M}) = \cup\ D^+(\mathcal{M}_i) = \cup\ \mathcal{M}_i = \mathcal{M}$ and $\mathcal{M}$ is stable . To see the convers , let $D^+(\mathcal{M}) = \mathcal{M}$ is stable . Let $\mathcal{M}_i$ be a component of $\mathcal{M}$. Then $D^+(\mathcal{M}_i)$ is a compact connected set and $D^+(\mathcal{M}_i) \subset \mathcal{M}$ since $\mathcal{M}_i$ is a component of $\mathcal{M}$. we have $D^+(\mathcal{M}_i) \subset \mathcal{M}_i$. Clearly then $D^+(\mathcal{M}_i) = (\mathcal{M}_i)$ as $\mathcal{M}_i \subset D^+(\mathcal{M}_i)$ holds always . Thus $\mathcal{M}_i$ is stable

Lemma 2:15 Suppose that $\mathcal{M}$ is positively invariant. Let $\mathcal{M}_1$ be a component of $\mathcal{M}$ then,
1. $\mathcal{M}_1$ is also positively invariant.
2. $A(\mathcal{M}_1) \cap (\mathcal{M} - \mathcal{M}_1) = \emptyset$.

Remark 2:16 (13) For any set $\mathcal{M} \subset X, x \in A(\mathcal{M})$ implies $A^+(x) \subset D^+(\mathcal{M})$.

Theorem 2:17 Let the set $\mathcal{M}$ be a stable , Then $\mathcal{M}$ is an attractor if and only if $\mathcal{M}$ is a weak attractor .
Proof : Let $\mathcal{M}$ be a weak attractor . Let $x \in A(\mathcal{M}) \setminus \mathcal{M}$. As $A^+(\mathcal{M}) \cap \mathcal{M} \neq \emptyset$,choose $W \in A^+(x) \cap \mathcal{M}$ then $A^+(x) \subset D^+(\mathcal{M})$. However $\mathcal{M}$ is staible , so $D^+(\mathcal{M}) = \mathcal{M}$ we have thus proved that $A^+(x) \subset \mathcal{M}$ for each $x \in A(\mathcal{M})$
i.e $\mathcal{M}$ is an attractor the converse is trivial , thus the theorem holds .

Remark 2:18 (3) If a compact invariant set $\mathcal{M}$ is a weak attractor then $D^+(\mathcal{M}) = \{y \in X: A^-(y) \cap \mathcal{M} \neq \emptyset\}$.

Theorem 2:19 A positive weak attractor $\mathcal{M}$ is also a negative weak attractor if and only if $D^+(\mathcal{M}) = A(\mathcal{M})$.
Proof : The set $A(\mathcal{M})$ is a neighborhood of $\mathcal{M}$, therefore if $D^+(\mathcal{M}) = A(\mathcal{M})$ we have $A^-(y) \cap \mathcal{M} \neq \emptyset$ for each $y \in A(\mathcal{M})$. Thus $\mathcal{M}$ is a negative weak attractor . If $D^+(\mathcal{M}) \neq A(\mathcal{M})$ then $D^+(\mathcal{M})$ cannot be a neighborhood of $\mathcal{M}$. Thus every neighborhood U of$\mathcal{M}$ meets $A(\mathcal{M}) \setminus D^+(\mathcal{M})$ .If however $y \in \cup \cap (A(\mathcal{M}) \setminus D^+(\mathcal{M}))$ then $A^-(y) \cap \mathcal{M} = \emptyset$ for otherwise $y \in D^+(\mathcal{M})$. hence $\mathcal{M}$ is not a negative weak attractor .

Definition 2.20:(6)
We say the invariant set $\mathcal{M}$ is Lyapunov stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that
$x(t.\mathcal{M}(\delta)) \subset \mathcal{M}(\varepsilon), \forall t \geq 0$.
Definition 2.21 (9)(14)
The solution $x(t.t_0.x_0)$ is said to be Lyapanov stable if for any $\varepsilon > 0$. and $t_0 \geq 0$. there exists
$\delta(\varepsilon.t_0)$ Such that:
1. All the solutions $x(t.t_0.y_0)$ satisfying the condition$|x_0 - y_0| \leq \delta$ and defined for $t \geq t_0$
2. For these solutions the inequality$|x(t.t_0.x_0) - x(t.t_0.y_0)| \leq \delta$ for $t \geq t_0$ is valid if $\delta(\varepsilon.t_0)$ is independent of t_0 the Lyopunov is called uniform .

Note 2:22
The definition of stability in the sense of Lyapanov is closely related to that of continuity of solutions . An equilibrium is stable if all solutions starting at nearby ,otherwise , it is un stable . It is asymptotically stable if all solutions starting at nearby , but also tend to the equilibrium point as time approaches infinity .

Conclusion:
The essential point of this study is to acquire a comprehension of what it method for frail attractor and soundness hypothesis. First and foremost we present the meaning of a dynamical system. Specifically, we investigate the connection between feeble attractors and dependability by certain definitions and properties , then we explored

some interesting theorems about stability , Where we proved that if $\mathcal{M}$ is stable then it is an attractor if and only if $\mathcal{M}$ is a weak attractor . And we provided an important definition is global pullback attractor .Then , at that point , We present meanings of pullback attractor and forward attractor then we reasoned that each pullback and forward attractor are powerless attractor . We also concluded that it is possible to call a weak attractor a global attractor under a special condition . What's more, we explain the most important part of this paper is Lyopunov stable we investigate the importance of steadiness in a dynamical system and afterward make sense of Laypnuov solidness.

References

[1] C.C Alan Fung , K. Y . Michael Wong , He Wang , and Si Wu.Dynamical systems Enhance Neural Information processing ;Gracefulness , Accuracy and Mobility . Neural Computation, 24(5) , (May 2012),pp. 1147 -1185.

[2] Mingshan Xue , Bassam V.Atallah , and Massimo Scanziani . Equalizing excitation – inhibition ratios across visual cortical neurons . Nature , 511(7511); (july 2014),pp. 596 -600 .

[3] H . Crauel . Random point attractors versus random set attractors Preprint , 1999 .

[4] Ralph Bourdoukan and Sophie Deneve . Enforcing balance allows local supervised learning in spiking recurrent networks . Advance in Neural Information processing Systems, (2015), pp. 9 .

[5] Welington de melo and Sebastian van strien. One dimensional dynamics .springer- verlag,Berlin,1993.

[6] D.N. cheban,Global Attractors of Non-autonomous Dissipative Dynamical systems.World Scientific,2004.

[7] M.Scheutzow , Attractors for ergodic and monotone random dynamical systems , in seminar on stochastic Analysis ,Random Fields and Applications V, R. Dalang ,M.Dozzi,and F. Russo (eds) , Birkhauser Verlag (2008). PP.331-344.

[8] L.F Abbott , Brian Depasquale , and Raoul –Martin Memmesheimer . Building functional networks of spiking model neurons . Nature Neuroscience , 19(3); (March 2016),pp. 350-355.

[9] Scheinerman ,Edward. Invitation to Dynamical systems . Web.12 may 2014.

[10] . Mehall , kevin . Vector Fields , Vector Field On line Graphing. Web.25, March 2015 .

[11] Gengshuo |Tian , Shangyang Li , Tiejun Huang , and Si Wu . Excitation – Inhibition Balanced Neural Networks for Fast Signal Detection . Frontiers in Computational Neuroscience , 14;79, September 2020 .

[12] V.V Chepyzhov , Approximating the trajectory attractor of the 3D Navier ,stokes system using various models of fluid dynamics , Sb . Math , 207(2016) ,pp.610 -638. Doi:10.1070 /Sm8549 .

[13] C.Amrouche ,L.C.Berselli , R.Lewandowski and D.D.Nguyen,Turbulent flows as generalized Kelvin /Voigh materials :Modeling and analysis , nonlinear Anal . 196 (2020), 111790,24 pp. doi :10.1016/j.na.2020.111790 .

[14] K.R.Verma,Modern Control theory,CBS publishers and Distributors Pvt ltd 2018.

[15] Chaoming Wang , Xiaoyu Chen , Tianqiu Zhaang , and Si Wu. Brainpy ;A fiexible ,integrative , efficient , and extensible framework towards general purpose brain dynamics programming ,preprint , Neuroscience , October 2022 .