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A Study Of Weak Attractor And Stability Theory Of Dynamical Systems

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ABSTRACT

This paper looks at (1) what an attractor is, (2) a rare type of attractor system, (3) weak attractor and pullback attractor and forward attractor, (4) the relationship between pullback, forward, and weak attractoress, (5) the stability of a dynamical system, and (6) Lyapunov stability as it relates to the stability of dynamical systems. To summarise our findings, we conclude that pullback and forward attractors are weak attractors. We justify this using stability theory in the context of dynamical systems. We observe that all pullback and forward attractors possess the same properties as weak attractors. We also provide characterisations of the stability of dynamical systems, including examining the Lyapunov stability of a dynamical system. A dynamical system is Lyapunov stable about its equilibrium point if, there exists a bounded set of trajectories that will remain in a bounded region, given sufficient proximity to the equilibrium points at the time of initial conditions.

MSC..

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Introduction:

Dynamical systems are the future behavior of evolving systems over long time scales. The modern dynamical systems theory began in the late 19th century with fundamental questions about the stability and evolution of the solar system. Trying to answer these questions led to the development of a very rich and robust theory that finds applications in many fields, including physics, biology, meteorology, astronomy, economics, etc. (1)

An example of a numerical field in dynamical systems is an attractor. Attractors are used in psychology and systems thinking to describe forces or elements that draw other elements of a system into their vicinity. Attractors influence moods, behaviors, or states of an ecosystem, a human relationship, or an organization, for example, guiding the system towards one of three states: an attractor stabilizes the system (positive or stable attractor); destabilizing and changing the system (negative, chaotic attractor). (15) As a result, attractors provide insight into the nature of developments in behavior or underlying structural issues in complex systems. Through the influence of various attractors within a system, many of the same behavior and development patterns happen repeatedly, indicating that predictable results are possible. Many dynamical systems have attractors which are more obscure in nature than the models themselves. An attractor is a general state towards to which all frameworks of various start states will

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approach. Frameworks do not lose touch with the attractor state no matter how they might develop or experience disturbance. [2] Attractors can be classified into four main categories: point attractors, limit cycle attractors, toroidal attractors, and odd attractors. [1] In this article, we will consider what weak attractors are, and subsequently state why pullback and forward attractors fall under this classification. Furthermore, we will discuss the significance of stability to the behaviour of dynamic systems. Stability is generally characterised by the property that small perturbations to a dynamic system will cause only small perturbations to the system's behaviour. Stability is an important aspect of theoretical dynamics and control and is described as the property of an unperturbed trajectory where all trajectories that are perturbed and initially very close will only produce small perturbations to the system's behaviour. The concept of stability was originally proposed by Russian mathematician A.M. Lyapunov in 1892. Following his well-established research, researchers have concentrated on developing the general theory of stability (Lyapunov's theory) to further examine the stability behaviour of general dynamic systems. (4)

1.1 Definitions and Related Documentation.

A dynamical system can be described mathematically as a state space together with a rule for how the state evolves over time, typically expressed by a set of first order linear differential equations.

Definition 1.1 (3)

A dynamical system on X is a triple $(X, \mathbb{R}, \varphi)$, where $\varphi: X \times \mathbb{R} \to X$ satisfies:

$$\varphi(x,0) = x \text{ for every } x \in X. \qquad (1)$$

$$\varphi(\varphi(x,t_1),t_2) = \varphi(x,t_1+t_2) \text{ for all } x \in X \text{ and } t_1,t_2 \in \mathbb{R}. \qquad (2)$$

$$\varphi \text{ is continuous.} \qquad (3)$$

The set X is called the *phase space* (the collection of all possible states). The map φ is referred to as the *phase flow*. Often we write $t \cdot x$ instead of $\varphi(x,t)$. Then the identities become:

$$0 \cdot x = x \, \forall x \in X, (4)$$

$$t_2 \cdot (t_1 \cdot x) = (t_1 + t_2) \cdot x \, \forall x \in X,\ t_1, t_2 \in \mathbb{R}. (5)$$

For a set $A \subset \mathbb{R}$ and a set $\mathcal{M} \subset X$, we define $A \cdot \mathcal{M} = \{t \cdot x : x \in \mathcal{M},\ t \in A\}$. If $A = \{t\}$ or $\mathcal{M} = \{x\}$, we write $t \cdot \mathcal{M}$ or $A \cdot x$ respectively.

For any point $x \in X$, the set $\mathbb{R} \cdot x = \{t \cdot x : t \in \mathbb{R}\}$ is called the *trajectory* (or orbit) through x.

For a fixed $t \in \mathbb{R}$, the map $\varphi^t : X \to X$ given by $\varphi^t(x) = t \cdot x$ is the *time-t map*. Note that the map $t \mapsto t \cdot x$ sends $\mathbb{R}$ onto the trajectory through x.

Definition 1.2 (2)

A set $\mathcal{M} \subset X$ is called *invariant* if $t \cdot x \in \mathcal{M}$ for every $x \in \mathcal{M}$ and every $t \in \mathbb{R}$. (6)
It is *positively invariant* if (6) holds with $\mathbb{R}$ replaced by $\mathbb{R}^+$, and *negatively invariant* if it holds with $\mathbb{R}$ replaced by $\mathbb{R}^-$.

Theorem 1.3

Let $\{\mathcal{M}_i\}$ be a family of subsets of X that are all positively invariant (resp. negatively invariant, invariant). Then their union and their intersection also enjoy the same invariance property.

Proof. We prove the positively invariant case. Set $M' = \bigcup_i \mathcal{M}_i$ and $M'' = \bigcap_i \mathcal{M}_i$.
Take any $x \in M'$. Then $x \in \mathcal{M}_i$ for some i. Because $\mathcal{M}_i$ is positively invariant, $t \cdot x \in \mathcal{M}_i$ for all $t \geq 0$. Hence $t \cdot x \in M'$ for all $t \geq 0$, so M' is positively invariant.
Now let $x \in M''$. Then $x \in \mathcal{M}_i$ for every i. Positive invariance of each $\mathcal{M}_i$ implies $t \cdot x \in \mathcal{M}_i$ for every i and all $t \geq 0$. Consequently $t \cdot x \in \bigcap_i \mathcal{M}_i = M''$ for all $t \geq 0$, so M'' is positively invariant.
The arguments for negative invariance and invariance are completely analogous.

Theorem 1.4 (5)

If $\mathcal{M} \subset X$ is positively invariant, negatively invariant, or invariant, then its closure $\bar{\mathcal{M}}$ has the same property.

Proof. Consider the case of invariance. Let $x \in \bar{\mathcal{M}}$ and $t \in \mathbb{R}$. There exists a sequence $\{x_n\} \subset \mathcal{M}$ with $x_n \to x$. By invariance of $\mathcal{M}$, we have $t \cdot x_n \in \mathcal{M}$ for each n. Since $t \cdot x_n \to t \cdot x$, we obtain $t \cdot x \in \bar{\mathcal{M}}$. Hence $\bar{\mathcal{M}}$ is invariant. The other cases are proved similarly.

Theorem 1.5
A set $\mathcal{M} \subset X$ is positively invariant iff its complement $X \setminus \mathcal{M}$ is negatively invariant. Moreover, $\mathcal{M}$ is invariant iff $X \setminus \mathcal{M}$ is invariant.

Proof. Assume $\mathcal{M}$ is positively invariant. Take $x \in X \setminus \mathcal{M}$ and $t \leq 0$. Suppose, for contradiction, that $t \cdot x \notin X \setminus \mathcal{M}$; then $t \cdot x \in \mathcal{M}$. Because $-t \geq 0$, positive invariance yields $(-t) \cdot (t \cdot x) = 0 \cdot x = x \in \mathcal{M}$, contradicting $x \in X \setminus \mathcal{M}$. Thus $X \setminus \mathcal{M}$ is negatively invariant. The converse and the statement about invariance are straightforward.

Corollary 1.6
$\mathcal{M} \subset X$ is invariant if and only if it is both positively and negatively invariant.

Definition 1.7 (6)
Define the maps $\Gamma, \Gamma^+, \Gamma^-: X \to 2^X$ (the power set of X) by:

$$\Gamma(x) = \{t \cdot x: t \in \mathbb{R}\}, (7)$$
$$\Gamma^+(x) = \{t \cdot x: t \geq 0\}, (8)$$
$$\Gamma^-(x) = \{t \cdot x: t \leq 0\}. (9)$$

For each x, $\Gamma(x)$ is the *trajectory*, $\Gamma^+(x)$ the *positive semi-trajectory*, and $\Gamma^-(x)$ the *negative semi-trajectory* through x. Note that $\Gamma(x) = \mathbb{R} \cdot x$.

Definition 1.8
$\mathcal{M} \subset X$ is invariant (resp. positively invariant, negatively invariant) if and only if

$$\Gamma(\mathcal{M}) = \mathcal{M} \text{ (resp. } \Gamma^+(\mathcal{M}) = \mathcal{M},\ \Gamma^-(\mathcal{M}) = \mathcal{M}).$$

Definition 1.9
$\mathcal{M} \subset X$ is invariant (resp. positively invariant, negatively invariant) if and only if for every $x \in \mathcal{M}$,

$$\Gamma(x) \subset \mathcal{M} \text{ (resp. } \Gamma^+(x) \subset \mathcal{M},\ \Gamma^-(x) \subset \mathcal{M}).$$

2. Stability Theory and Attraction.

The present section concerns itself with the investigation of stable attractors and their definitions, as well as some additional information on global weak attractors. The focus will then shift to an overview of the Lyapunov method, first for linear systems. The analysis of dynamical systems that are acting under the influence of some physical law, yields a set of differential equations, such as the governing equation of the dynamics (10). The characterization of an orbit's stability provides insight into whether trajectories that begin near that orbit will remain close to it or be pushed out away from it. Stability theory utilizes a Lyapunov function to define stability and asymptotic stability of trajectories with a non-negative-valued scalar function defined on a small neighborhood of the trajectory, which decreases as it moves along its trajectory. However, it is not possible to simply define stability and the other characteristics of attractors with continuous functions. (9)

Definition 2.1 :(7)

For a given set $\mathcal{M} \subset X$, we define the following three sets:

$$A_w(\mathcal{M}) = \{x \in X: A^+(x) \cap \mathcal{M} \neq \emptyset\}(10)$$
$$A(\mathcal{M}) = \{x \in X: A^+(x) \neq \emptyset \text{ and } A^+(x) \subset \mathcal{M}\}(11)$$
$$A_u(\mathcal{M}) = \{x \in X: J^+(x) \neq \emptyset \text{ and } J^+(x) \subset \mathcal{M}\}(12)$$

Here $J^+(x)$ is defined as:

$$J^+(x) = \{y \in X: \exists\{x_n\} \subset X,\ \exists\{t_n\} \subset \mathbb{R}^+ \text{ such that } x_n \to x,\ t_n \to +\infty, \text{ and } x_n t_n \to y\}(13)$$

The sets $A_w(\mathcal{M})$, $A(\mathcal{M})$, and $A_u(\mathcal{M})$are called, respectively, the region of weak attraction, the region of attraction, and the region of uniform attraction of $\mathcal{M}$.
Moreover, a point $x$belonging to $A_w(\mathcal{M})$, $A(\mathcal{M})$, or $A_u(\mathcal{M})$is said to be weakly attracted, attracted, or uniformly attracted to $\mathcal{M}$, respectively.

Remark :2.2 (12) Any attractor can be termed either local ($A_w(M) \subset X$) or global ($A_w(M) = X$). The term "global" is also used to denote uniform and asymptotic stability for any of the kinds of attractors described. Therefore, any attraction region (attractor region) is characterized by its entirety, as expressed through the general properties associated with "global" in terms of uniformity of multiple attraction points (attractor points) attracting towards the exact geometrically positioned attraction points (attractor point locations).

Proposition 2.3 (8)
Given a set $\mathcal{M}$, a point $x$is weakly attracted to $\mathcal{M}$if and only if there exists a sequence $\{t_n\} \subset \mathbb{R}$with $t_n \to +\infty$such that

$$\mathrm{dist}(t_n \cdot x, \mathcal{M}) \to 0. (14)$$

A point $x$is attracted to $\mathcal{M}$if and only if

$$\mathrm{dist}(t \cdot x, \mathcal{M}) \to 0 \text{as } t \to +\infty. (15)$$

A point $x$is uniformly attracted to $\mathcal{M}$if and only if for every neighbourhood $V$of $\mathcal{M}$there exist a neighbourhood $U$of $x$and a number $T > 0$such that $U \cdot t \subset V$for all $t \geq T$. \qquad\qquad (16)

Theorem 2.4
For any given set $\mathcal{M}$,

$$A_w(\mathcal{M}) \supset A(\mathcal{M}) \supset A_u(\mathcal{M}). (17)$$

Moreover, the sets $A_w(\mathcal{M})$, $A(\mathcal{M})$, $A_u(\mathcal{M})$are invariant. \qquad\qquad (18)

Definition 2.5 (7)
A family $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$is called a **pullback attractor** with respect to the process Uif, for every $t \in \mathbb{R}$, every bounded set $D \subset X$, and every $\varepsilon > 0$, there exists $T_{\varepsilon,D}(t) > 0$such that for all $c \geq T_{\varepsilon,D}(t)$,

$$\mathrm{dist}_X\ (U(t, t-c)D, \mathcal{M}(t)) < \varepsilon.$$

Definition 2.6 (11)
The family $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$is said to be **forward attracting** for Uif, for every $t \in \mathbb{R}$and every bounded $D \subset X$,

$$\lim_{\tau\to\infty} \mathrm{dist}_X\ (U(t,\tau)D, \mathcal{M}(t)) = 0.$$

Definition 2.7 (13)
The family $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$is called a **global pullback attractor** for $U(t,\tau)$if:

1. $\mathcal{M}(t)$is compact for each $t \in \mathbb{R}$.
2. The family $\mathcal{M}(t)$is pullback attracting, meaning that for every $t \in \mathbb{R}$and every bounded $B \subset X$(with $B \in \mathfrak{R}$),

$$\mathrm{dist}\ (U(t,\tau)B, \mathcal{M}(t)) \to 0 \text{as } \tau \to -\infty.$$

$\mathcal{M}(t) \subset U(t,\tau)\mathcal{M}(\tau)$for all $-\infty < \tau \leq t < \infty$(negative invariance).

3. The family $\mathcal{M}$is the smallest closed family satisfying property (2).

Definition 2.8
Forward attraction is called **uniform** if

$$\lim_{t\to\infty} \sup_{\tau\in\mathbb{R}} \mathrm{dist}_X\ (U(t+\tau,\tau)D, \mathcal{M}(t+\tau)) = 0$$

holds for every bounded $D \subset X$.

Remark 2.9
If $\mathcal{M}_1$and $\mathcal{M}_2$are weak attractors, then their union $\mathcal{M}_1 \cup \mathcal{M}_2$is also a weak attractor.

Remark 2.10
Every pullback attractor is a weak attractor.

Remark 2.11
Every forward attractor is a weak attractor.

Definition 2.12 (11)
A set $\mathcal{M}$ is called:

- a **weak attractor** if $A_w(\mathcal{M})$ is a neighbourhood of $\mathcal{M}$; \qquad (19)
- an **attractor** if $A(\mathcal{M})$ is a neighbourhood of $\mathcal{M}$; \qquad (20)
- a **uniform attractor** if $A_u(\mathcal{M})$ is a neighbourhood of $\mathcal{M}$; \qquad (21)
- **stable** if every neighbourhood U of $\mathcal{M}$ contains a positively invariant neighbourhood V of $\mathcal{M}$; \qquad (22)
- **asymptotically stable** if it is both stable and an attractor; \qquad (23)
- **unstable** if it is not stable. \qquad (24)

Proposition 2.13
Let X be a closed and stable subset of $\mathcal{M}$. Then X is positively invariant.

Theorem 2.14
$\mathcal{M}$ is stable if and only if each of its components is stable.

Proof. Observe that if $\mathcal{M}$ is compact, then every component of $\mathcal{M}$ is compact. Moreover, if $\mathcal{M}$ is positively invariant, so are all its components.
Write $\mathcal{M} = \bigcup_{i\in\vartheta} \mathcal{M}_i$, where ϑ is an index set and each $\mathcal{M}_i$ is a component of $\mathcal{M}$. Assume each $\mathcal{M}_i$ is stable, i.e. $D^+(\mathcal{M}_i) = \mathcal{M}_i$. Then

$$D^+(\mathcal{M}) = \bigcup_i D^+(\mathcal{M}_i) = \bigcup_i \mathcal{M}_i = \mathcal{M},$$

so $\mathcal{M}$ is stable.
Conversely, suppose $D^+(\mathcal{M}) = \mathcal{M}$ (i.e., $\mathcal{M}$ is stable). Let $\mathcal{M}_i$ be a component of $\mathcal{M}$. Then $D^+(\mathcal{M}_i)$ is a compact connected set and $D^+(\mathcal{M}_i) \subset \mathcal{M}$. Because $\mathcal{M}_i$ is a component, we must have $D^+(\mathcal{M}_i) \subset \mathcal{M}_i$. The reverse inclusion $\mathcal{M}_i \subset D^+(\mathcal{M}_i)$ always holds, hence $D^+(\mathcal{M}_i) = \mathcal{M}_i$. Thus $\mathcal{M}_i$ is stable.

Lemma 2.15
Suppose $\mathcal{M}$ is positively invariant and let $\mathcal{M}_1$ be a component of $\mathcal{M}$. Then:

1. $\mathcal{M}_1$ is also positively invariant.
2. $A(\mathcal{M}_1) \cap (\mathcal{M} \setminus \mathcal{M}_1) = \emptyset$.

Remark 2.16 (13)
For any set $\mathcal{M} \subset X$, if $x \in A(\mathcal{M})$ then $A^+(x) \subset D^+(\mathcal{M})$.

Theorem 2.17
Let $\mathcal{M}$ be stable. Then $\mathcal{M}$ is an attractor if and only if it is a weak attractor.

Proof. Assume $\mathcal{M}$ is a weak attractor. Take any $x \in A(\mathcal{M}) \setminus \mathcal{M}$. Since $A^+(x) \cap \mathcal{M} \neq \emptyset$, choose $W \in A^+(x) \cap \mathcal{M}$. Then $A^+(x) \subset D^+(\mathcal{M})$. Because $\mathcal{M}$ is stable, $D^+(\mathcal{M}) = \mathcal{M}$. Consequently $A^+(x) \subset \mathcal{M}$ for every $x \in A(\mathcal{M})$, which means $\mathcal{M}$ is an attractor. The converse direction is obvious.

Remark 2.18 (3)
If a compact invariant set $\mathcal{M}$ is a weak attractor, then

$$D^+(\mathcal{M}) = \{y \in X: A^-(y) \cap \mathcal{M} \neq \emptyset\}.$$

Theorem 2:19 If $A(\mathcal{M}) = D + (\mathcal{M})$, then $\mathcal{M}$ is both a positive and negative weak attractor therefore
Proof : Let N be a neighborhood of $\mathcal{M}$. Then $A(\mathcal{M})$ has $A^-(y) \cap \mathcal{M} \neq \emptyset$ for every $y \in A(\mathcal{M})$ i.e. $A(\mathcal{M})$ would be a positive weak attraction on A but $A(\mathcal{M})$ not equal $D^+(\mathcal{M})$, thus $D^+(\mathcal{M})$ cannot be a neighborhood

of $\mathcal{M}$. Thus $A(\mathcal{M}) \setminus D^{+}(\mathcal{M})$ will meet with every neighborhood U of $\mathcal{M}$. If y is in $\cup\cap$ $(A(\mathcal{M}) \setminus D^{+}(\mathcal{M}))$, then $A^{-}(y)\cap \mathcal{M} = \emptyset$ otherwise $y \in D^{+}(\mathcal{M})$. Thus, $\mathcal{M}$ will not be considered as a negative weak attraction.

Definition 2.20:(6)

We say the invariant set $\mathcal{M}$ is Lyapunov stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that
$x(t.\mathcal{M}\ (\delta)) \subset \mathcal{M}\ (\varepsilon), \forall t \geq 0$.

Definition 2.21 (9)(14)

The solution $x(t.t_0.x_0)$ is said to be Lyapanov stable if for any $\varepsilon > 0$. and $t_0 \geq 0$. there exists $\delta(\varepsilon.t_0)$ Such that:

1. All the solutions $x(t.t_0.y_0)$ satisfying the condition$|x_0 - y_0| \leq \delta$ and defined for $t \geq t_0$
2. For these solutions the inequality$|x(t.t_0.x_0) - x(t.t_0.y_0)| \leq \delta$ for $t \geq t_0$ is valid if $\delta(\varepsilon.t_0)$ is independent of t_0 the Lyopunov is called uniform.

Note 2:22

According to Lyapanov's definition of Stability the Continuity of solutions has been quite closely connected to the definition of Stability itself. The stability of the equilibrium point is based on all starting points having similar characteristics. If they do not, then we can say the equilibrium point is unstable. The asymptotic stability of the equilibrium point is tied to the fact that all of the solutions beginning near that point, asymptotically, will tend toward that same point as time becomes infinite.

Conclusion:

The study intends to comprehend how both weak attractors, and the concepts of attractors, are related to stability. The researchers began by defining dynamic systems specifically by looking at how weak attractors are related to stability through the use of definitions and properties from other researchers' work and theorems on stability. They also demonstrate that strongly stabilized objects will remain weakly attracted to the system, using their definition for weakly attracted local objects with their systems. Next, they will define both pullback and forward attractors, then show that both types of attractors can also be defined as weakly attracting. Finally, they note that under certain circumstances, a weakly attracting object can be regarded as a globally attractive object, as well as examining the significance of Lyapunov stability to dynamic systems and explaining how to become Lyapunov stable as per the reports of other researchers.

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