Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.17.(2) 2025,pp.Math 52–65

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

Adjacency matrices of Ideal based-zero divisor graphs

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ARTICLE INFO

Article history:
Received: 26/03/2025
Rrevised form: 01/04/2025
Accepted : 23/04/2025
Available online: 30/06/2025

Keywords:
Adjacency Matrix,
Eigenvalue,
Ideal based-zero Divisor graph,
Betti number,
Projective graph.

ABSTRACT

In this article, we investigate the adjacency matrix and the eigenvalues of the Ideal-based-zerodivisor graph $\mathcal{g}_J(R)$ for a given ideal J of the finite commutative ring R. Additionally, we define the notion of projection graph of the Idealbased-zerodivisor graph $\pi(\mathcal{g}_J(\mathbb{Z}_n))$ as a graph with vertices $s(d)$ for all d/n and edge connecting $s(d_i)$ with $s(d_j)$ if every element in $s(d_i)$ is connected with $s(d_j)$. We consider the ring Z_n for some special cases of n. If $n = p_1^{t_1} \dots p_s^{t_s}$, where $p_1, \dots, p_s$ (where $s \geq 2$) and I is non-prime ideal of $\mathbb{Z}_n$, then we determined the determinant of the adjacency matrices A of the Idealbased-zerodivisor graph of $\mathbb{Z}_n$ and satisfies $det\left(A\left(\mathcal{g}_J(\mathbb{Z}_n)\right)\right) = 0$. We showed that in the Ideal-based-zerodivisor graph $\left(\mathcal{g}_J(\mathbb{Z}_n)\right)$, where $n = \prod_{i=1}^{s} P_i^{t_i}$, J is not prime ideal of $\mathbb{Z}_n$ and not zero ideal of $\mathbb{Z}_n$, then $\left|\pi\left(\mathcal{g}_J(\mathbb{Z}_n)\right)\right| = \prod_{i=1}^{s}(t_i+1) - 3$ and whenever $n = \prod_{i=1}^{s} P_i^{t_i}$ then $rank\ A\left(\mathcal{g}_J(\mathbb{Z}_n)\right) = rank\ A(\pi(\mathcal{g}_J(\mathbb{Z}_n)))$.Finally, we investigate some basic properties of the edge ideal of the graph $\mathcal{g}_J(\mathbb{Z}_n)$. We calculated the graph's depth, girth, diameter, Betti number, regularity, and projective dimension.

MSC.. 13D40; 13D02; 05C90; 13F55

https://doi.org/10.29304/jqcsm.2025.17.22206

1. Introduction

If R is a commutative ring with unity. We denote the set of all zero divisors in R by $Z(R)$ and denote the non-zero zero divisors of R by $Z^*(R)$. According to Beck [1,3], the Zerodivisor graph is a finite graph $\mathcal{g}(R)$ defined on the set of all zero divisors $Z(R)$ of R and there is an edge between two different elements $a, b \in Z(R)$, if $ab = 0$. After this, Anderson and Naseer considered the graph with the set of all non-zero zero divisors $Z^*(R)$ of R as the set of all vertices $\mathcal{g}(R)$ and they considered the same condition of adjacency of Beck's definition [2,3]. For the finite graph $\mathcal{g}(R)$, Anderson and Livingston were obtained a lot of nice properties [3]. For instance, the most important result, for which they obtained is that, between any two different vertices of the finite graph $\mathcal{g}(R)$, there is a finite path that

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connect them. Anderson and Badawi were investigated the finite graph $\mathcal{g}(R)$ for different types of finite rings, for instance chained ring and a ring R for which $p \in Z(R)$ for all $p \in spec(R)$ [4]. The showed that, if $Z(R)^2 \neq 0$ and $p \in Z(R)$ for $p \in spec(R)$ are linearly ordered. Then $diam\left(\mathcal{g}(R)\right) = 2$. The Zerodivisor graph is currently applied to many combinatorial models. Park, et al. established a complete characterization of the diameter and girth of the zero-divisor graph $\mathcal{g}(Z_{\mathrm{n}})$. Additionally, they determined the chromatic number of $\mathcal{g}(Z_{\mathrm{n}})$ [15].

Consider $V(\chi) = \{t_1, \ldots, t_n\}$ as a set of vertices, and let $E(\chi)$ represent the set of edges. A finite simple graph, denoted by $\chi = (V(\chi), E(\chi))$, consists of the vertex set $V(\chi)$ and the edge set $E(\chi)$. A subset L of χ is defined as a subgraph of χ if, for any two vertices t_i and t_j in L, the condition $\{t_i, t_j\} \in E(L)$ holds if and only if $\{t_i, t_j\} \in E(\chi)$. Now, let $\chi = (V(\chi), E(\chi))$ and $\chi' = \left(V(\chi'), E(\chi')\right)$ be two finite simple graphs with vertex sets $V(\chi)$ and $V(\chi')$, respectively. The union of these graphs, written as $\chi \cup \chi'$, is the graph formed by combining the vertex sets $V(\chi) \cup V(\chi')$ and the edge sets $E(\chi) \cup E(\chi')$. A graph χ is said to be complete if every pair of distinct vertices t_i and t_j satisfies $\{t_i, t_j\} \in E(\chi)$. Such a graph is denoted as χ_n. A vertex cover of a graph χ is a subset $W \subseteq V(\chi)$ such that every edge $e \in E(\chi)$ has at least one endpoint in W. A vertex cover is minimal if no proper subset of W can also serve as a vertex cover [8]. An independent set is defined as a subset $D \subseteq V(\chi)$ in which no two vertices t_i and t_j are adjacent, i.e., $\{t_i, t_j\} \notin E(\chi)$ for any $t_i, t_j \in D$. In contrast, a clique is a subset $C \subseteq V(\chi)$ where every pair of vertices t_i and t_j satisfies $\{t_i, t_j\} \in E(\chi)$. The complement of a graph χ, denoted by χ^c, is the graph where $V(\chi^c) = V(\chi)$, but the edge set $E(\chi^c)$ consists of all pairs $\{t_i, t_j\} \notin E(\chi)$ for $1 \leq i, j \leq n$ and $i \neq j$. This can also be expressed as $E(\chi^c) = E(\chi_n) \setminus E(\chi)$, where χ_n is the complete graph. A graph χ is classified as chordal if every cycle of length greater than three contains at least one chord. Similarly, a cochordal graph is defined as one in which this property applies to its complement, χ^c. If a graph χ can be decomposed into two sets D and C, where D is an independent set and C is a clique, then χ is referred to as a divided graph. For more information on these types of graphs and their characteristics, refer to [5,6,7].

Consider an ideal J in a commutative ring with unity. Redmond extended the concept of the zerodivisor graph to introduce the ideal-based Zerodivisor graph [5]. This graph, denoted as $\mathcal{g}_J(R)$, is defined on the vertex set $\{a \in R - J : ab \in J \; for \; some \; b \in R - J\}$, where two distinct elements in this set are adjacent if their product ab lies in J. In his study, Redmond derived several interesting results related to this generalization of the zerodivisor graph. He examined properties such as connectivity, cut-points, and bridges, as well as the clique number and girth of these idealbased-zero divisor graphs. Additionally, he explored which graphs with t vertices can be represented as idealbased- zerodivisor graphs and characterized planar idealbased-zero divisor graphs up to isomorphism. For example, if J is a proper ideal of R, the following results were established:

- $\mathcal{g}_J(R) = \emptyset$ if and only if J is a prime ideal of R.
- $\mathcal{g}_J(R)$ is connected with $diam(\mathcal{g}_J(R)) \leq 3$.
- If $\mathcal{g}_J(R)$ contains a cycle, then $gr(\mathcal{g}_J(R)) \leq 7$.
- $\mathcal{g}_J(R)$ is infinite if $\mathcal{g}(R/J)$ is infinite.
- $\mathcal{g}_J(R)$ is a graph on $t.|J|$ vertices, where $\mathcal{g}(R/J)$ is a graph with t vertices.

Let $\mathcal{g}$ represent a graph. For any two vertices x and y in $\mathcal{g}$, let $d(x, y)$ denote the length of the shortest path between them. It is evident that $d(x, x) = 0$, and $d(x, y) = \infty$ if no path exists connecting x and y. The diameter of $\mathcal{g}$, denoted as $diam(\mathcal{g})$, is defined as $sup\{d(x, y): x \text{ and } y \text{ are vertices of } \chi\}$. A *clique* in a graph is a subset of vertices that are all mutually adjacent. The largest possible size of a clique in a graph $\mathcal{g}$, referred to as the clique number, is denoted by $\omega(\mathcal{g})$. In $\mathcal{g}$, a *stable set* is a group of vertices such that no two are adjacent. A stable set is considered maximum if no larger stable set exists within the graph. The size of the largest stable set in $\mathcal{g}$, known as the stability number, is denoted by $\alpha(\mathcal{g})$. Finally, the girth of $\mathcal{g}$, represented as $gr(\mathcal{g})$, is the length of the shortest cycle in $\mathcal{g}$. If $\mathcal{g}$ has no cycles, then $gr(\mathcal{g}) = \infty$.

The adjacency matrix of a graph $\mathcal{g}$ with t vertices, denoted by $\Lambda(\mathcal{g}) = (c_{uv})$, is an $t \times t$ matrix where c_{uv} represents the number of edges between vertices u and v. For loops, each is counted as two edges. In the case of a simple graph, $\Lambda(\mathcal{g})$ is a real and symmetric matrix with entries limited to 0 and 1, and all diagonal entries are zero. The eigenvalues of a square matrix Λ are defined as the roots of its characteristic polynomial, given by $\det(\Lambda - \lambda I)$, where I is the identity matrix. The characteristic polynomial of a graph Λ, denoted by $\mathcal{H}(\mathcal{g}, \lambda)$, is the characteristic polynomial of its adjacency matrix, expressed as $\mathcal{H}(\mathcal{g}, \lambda) = \det(\Lambda - \lambda I)$. For any natural number t, the function $g(t)$ determines the count of positive integers less than t that are relatively prime to t. Furthermore, $\phi_t(\mathcal{H})$ represents the vector space of all $t \times t$ square matrices whose entries belong to a field $\mathcal{H}$. A circulant matrix of size $t \times t$, with entries a and b (where $\mu, \gamma \in \mathbb{R}$), is represented by $C(\mu, \gamma, t)$ and is structured as follows:

$$C(\mu,\gamma,t) = \begin{bmatrix} \mu & \gamma & \gamma & \cdots & \gamma \\ \gamma & \mu & \gamma & \cdots & \gamma \\ \gamma & \gamma & \mu & \cdots & \gamma \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & \gamma & \gamma & \cdots & \mu \end{bmatrix}.$$

Magi, Jose, and Kishore [10], explored the adjacency matrix of certain types of Zerodivisor graphs. They specifically analyzed the eigenvalues of $\mathcal{g}(Z_n)$ for $n = p_1^2 p_2^2$, where p_1 and p_2 are distinct prime numbers. In their study, they also calculated key graph invariants, including the girth, diameter, clique number, and stability number of $\mathcal{g}(Z_n)$. Bajaj and Panigrahi [9], extended this work by examining the structural and spectral properties of the ZeroDivisor graph of the ring $\mathbb{Z}_n$. For any non-prime integer $n \geq 4$, with t denoting the number of proper divisors of n, they demonstrated that the adjacency spectrum of $\mathcal{g}(Z_n)$ is derived from the eigenvalues of a symmetric $t \times t$ matrix $C(F_n)$. These eigenvalues are restricted to 0 and -1 at most. They further determined the precise multiplicity of the eigenvalue 0 and established that all eigenvalues of $C(F_n)$ are nonzero by evaluating the rank and nullity of the adjacency matrix of $\mathcal{g}(Z_n)$. Additionally, they identified specific values of n for which the adjacency spectrum of $\mathcal{g}(Z_n)$ consists solely of nonzero eigenvalues. Finally, by computing the characteristic polynomial of $C(F_n)$, they derived the characteristic polynomial of $\mathcal{g}(Z_n)$ for cases where n is a prime power.

A free resolution of a finitely generated R-module Y is a chain of R-module homomorphisms:

$$\cdots \to F_i \overset{d_i}{\to} F_{i-1} \to \cdots \to F_1 \overset{d_1}{\to} F_0,$$

where each F_i is a finitely generated free R-module expressed as $F_i = \bigoplus_{y \in \mathbb{Z}} R(-y)^{b_{i,y}(Y)}$. The sequence is exact, and Y is isomorphic to the quotient $F_0/\mathrm{Im}(d_1)$. A more compact representation of this resolution can be written as:

$$\cdots \to F_i \overset{d_i}{\to} F_{i-1} \to \cdots \to F_1 \overset{d_1}{\to} F_0 \overset{d_0}{\to} Y \to 0.$$

The coefficients $b_{i,y}(Y)$ are known as the Betti numbers, where i represents the homological degree and y denotes the external degree. Two significant invariants associated with Y are the projective dimension, $pd(Y)$, and the Castelnuovo-Mumford regularity, $reg(Y)$. These are defined as:

$$pd(Y) = max\{i: b_{i,y}(Y) \neq 0\}, reg(Y) = max\{y - i: b_{i,y}(Y) \neq 0\}$$

Now, consider the polynomial ring $R = T[x_1, \ldots, x_n]$, where T is a field and $x_1, \ldots, x_n$ are variables. The edge ideal of a graph G, denoted by $I(\chi)$, is generated by all quadratic square-free monomials $x_i x_j \in R$ corresponding to edges $i \leftrightarrow j$ in G. The edge ring of the graph χ is then defined as $R/I(\chi)$, which is a finitely generated R-module. By the Hilbert-Syzygy theorem, this edge ring has a minimal free resolution (mf-resolution). Computing this resolution allows us to study the homological properties of the edge ideal of G. For a more detailed exploration of homological properties of modules, refer to [11].

According to the previous we generalize the results to the Ideal-based zerodivisor graphs. In this article, we study the adjacency matrix of the Ideal-based zerodivisor graph $\mathcal{g}_J(R)$ for a given ideal J of the finite commutative ring R. Especially, we consider the ring Z_n for some special cases of n. If $n = p_1^{t_1} \ldots p_s^{t_s}$, where $p_1, \ldots, p_s$ (where $s \geq 2$) and J is non-prime ideal of $\mathbb{Z}_n$, then we determined the determinant of the adjacency matrices A of the Idealbased-zerodivisor graph of $\mathbb{Z}_n$ and satisfies $det\left(A\left(\mathcal{g}_J(\mathbb{Z}_n)\right)\right) = 0$. Moreover, we define the notion of projection graph of the Idealbased-zerodivisor graph $\pi(\mathcal{g}_J(\mathbb{Z}_n))$ as a graph with vertices $s(d)$ for all d/n and edge connecting $s(d_i)$ with $s(d_j)$ if every element in $s(d_i)$ is connected with $s(d_j)$. We showed that the number of vertices in the graph $\left(\mathcal{g}_J(\mathbb{Z}_n)\right)$, where $n = \prod_{i=1}^{s} P_i^{t_i}$, J is not prime ideal of $\mathbb{Z}_n$ and not zero ideal of $\mathbb{Z}_n$, then $\left|\pi\left(\mathcal{g}_J(\mathbb{Z}_n)\right)\right| = \prod_{i=1}^{s}(t_i + 1) - 3$ and whenever $n = \prod_{i=1}^{s} P_i^{t_i}$ then $rank\ A\left(\mathcal{g}_J(\mathbb{Z}_n)\right) = rank\ A(\pi(\mathcal{g}_J(\mathbb{Z}_n)))$. On the other hand, we investigate some basic properties of the edge ideal of the graph $\mathcal{g}_J(\mathbb{Z}_n)$. We calculated the graph's depth, girth, diameter, Betti number, regularity, and projective dimension.

2. The adjacency matrices and eigenvalue of Idealbased-zerodivisor graphs $\mathcal{g}_J(R)$

The connections between Zerodivisors in a ring R with respect to an ideal J can be seen using the idealbased-zerodivisor graph$\mathcal{g}_J(R)$. These relationships are captured by its adjacency matrix, and significant patterns and characteristics can be found by examining its eigenvalues. Understanding these matrices and their importance in algebra and graph theory is the main goal of this section.

Theorem 2.1. Idealbased-zerodivisor graph of $\mathcal{g}_J(\mathbb{Z}_n)$ is totally disconnected graphs.

Proof. If n is prime number then the ideal of $\mathbb{Z}_n$ is zero ideal then every vertex of $\mathbb{Z}_n$ is a unite element and the product two element is unit. i.e $x.y = 1 \neq 0 \notin$ J which implies that x not adjacent y, thus the graph $\Gamma_{(0)}(\mathbb{Z}_n)$ is disconnected.

If n is not prime number, then the ideal is a zero ideal, prime ideal and non-prime ideal of $\mathbb{Z}_n$. If $J = (0) \rightarrow \mathcal{g}_J(\mathbb{Z}_n) = \mathcal{g}(\mathbb{Z}_n)$ means that disconnected, if J is prime ideal then the graph $\mathcal{g}_J(\mathbb{Z}_n)$ is null graph. Thus $\mathcal{g}_J(\mathbb{Z}_n)$ is disconnected, if J not prime ideal of $\mathbb{Z}_n$ $\exists\, x, y \in \mathbb{Z}_n - I$ then $x.y \notin$ J. This implies that x is not adjacent y .Hence $\mathcal{g}_J(\mathbb{Z}_n)$ is disconnected.

Example 2.2. Find adjacency matrices for the following: Suppose that $p = 2$ and $n = 4$, then $\mathbb{Z}_{p^n} = \mathbb{Z}_{2^4} = \mathbb{Z}_{16}$, if J =(4) is an ideal in $\mathbb{Z}_{16}$, then

$$A\left(\mathcal{g}_{(4)}(\mathbb{Z}_{16})\right) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Since the sub graph $A\left(\mathcal{g}_{(4)}(\mathbb{Z}_{16})\right)$ is K_4. Then one of the eigenvalues is $\lambda_1 = n - 1$ and all others are -1 [12]. Then, $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = \lambda_4 = -1$. The characteristic polynomial is $p(\lambda) = (\lambda - 3)(1 + \lambda)^3$. Then, Rank $A(K_4) = 4 - 1 = 3$.

Example 2.3. To find adjacency matrices $\boldsymbol{R} = \mathbb{Z}_{\boldsymbol{p^n}} = \mathbb{Z}_{\boldsymbol{2^5}} = \mathbb{Z}_{\boldsymbol{32}}$, $\boldsymbol{J}$ =(4).

This implies that $A\left(\mathcal{g}_{(4)}(\mathbb{Z}_{32})\right) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$. The eigenvalues are

$\lambda_1 = 7$, and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = -1$. Hence, the characteristic polynomial is

$$p(\lambda) = (\lambda - 7)(\lambda + 1)^7.$$

$det\left(A(K_8)\right) = (-1)^{8-1}(8 - 1) = (-1)^7.7 = -7$ [13], thus Rank $A(K_8) = 8 - 1 = 7$.

3. Projection graphs

Let $n = p_1^{t_1} \dots p_s^{t_s}$, where $p_1, \dots, p_s$ are distinct prime for any divisor d of n, we define $s(d) = \{k \in \mathbb{Z}_n - J : \gcd(k, n) = d\}$, where I is not prime ideal of $\mathbb{Z}_n$.

Proposition 3.1 Let $R = \mathbb{Z}_n$, where n is positive integer and J is an ideal of R that is not a prime ideal for each divisor d of n define the set $s(d) = \{k \in \mathbb{Z}_n - J : \gcd(k, n) = d\}$, then cardinality of $s(d)$ is given by:

$|s(d)| = \emptyset\,(\frac{n}{d})$, where $\emptyset$ is Euler totient function.

Proof. The elements $k \in s(d)$.Then k is not in J and $\gcd(k,n) = d$. i.e. k can be expressed as $k = d.m$ for some integer m ,where $1 \leq m < \frac{n}{d}$, $k = d.m \notin$ J which implies that m must be co-prime to $\frac{n}{d}$,therefore k is co-prime with $\frac{n}{d}$ if and only if m is co-prime to $\frac{n}{d}$, then $m = \emptyset(\frac{n}{d})$. Thus $|s(d)| = \emptyset(\frac{n}{d})$.

Example 3.2 We illustrate proposition 3.1 with ideal-based-zero divisor graph $\mathcal{g}_J(\mathbb{Z}_{18})$ and it is adjacency matrix where $J = (6)$. Then

$\mathbb{Z}_{18} = \{0,1,2,3,4,5,7,8,9,10,11,13,14,15,16,17\}$ and $J = \{0,6,12\}$

$s(d) = \{k \in \mathbb{Z}_n - J: \gcd(k,n) = d\}$

$s(2) = \{k \in \mathbb{Z}_{18} - (6): \gcd(k, 18) = 2\}$

$s(2) = \{2,4,8,10,14,16\}$

$|s(d)| = \emptyset\,(\frac{n}{d})$

$|s(2)| \neq \emptyset\,(\frac{18}{2}) = \emptyset(9) = 9(1 - \frac{1}{3}) = 6.$

$$A\left(\mathcal{g}_{(6)}(\mathbb{Z}_{18})\right) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \text{ then } det\left(A\left(\mathcal{g}_{(6)}(\mathbb{Z}_{18})\right)\right) = 0.$$

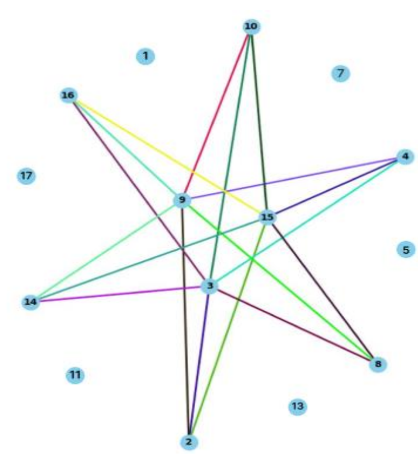

Figure 3.1: $\boldsymbol{\mathcal{g}_{(6)}(\mathbb{Z}_{18})}$

Theorem 3.3 Let $n = p_1^{t_1} \dots p_s^{t_s}$, where $p_1, \dots, p_s$ (where $s \geq 2$) and let $\mathbb{Z}_n$ denote the ring of composite integers modulo n if J non-prime ideal of $\mathbb{Z}_n$, then the determinant of the adjacency matrices A of the Idealbased-zerodivisor graph of $\mathbb{Z}_n$ satisfies $det\left(A\left(\mathcal{g}_J(\mathbb{Z}_n)\right)\right) = 0$.

Proof. Since n is composed so $\mathbb{Z}_n$ is not an integral domain, there are Zerodivisor in $\mathbb{Z}_n$,which contribute to the vertices of the graph. Let V denoted the set of vertices of the graph $\mathbb{Z}_n$ .This implies that $V = \{x \in \mathbb{Z}_n - J: xy \in J$ for some $y \in \mathbb{Z}_n - J\}$,two vertices $x, y \in v$ are adjacency if $xy \in J$. Assume that A be the adjacency matrix of the Idealbased-zerodivisor graph of $\mathbb{Z}_n$ if J is not a prime ideal of $\mathbb{Z}_n$ to show $det(A) = 0$.Since J not prime Ideal of $\mathbb{Z}_n$ this implies that the multiplication operation in $\mathbb{Z}_n - J$ is not cancellation, and $\mathbb{Z}_n - J$ becomes more interconnected and since the adjacency matrix A is symmetric i.e $A = A^T$ and the graph $\mathcal{g}_J(\mathbb{Z}_n)$ is undirected then the row of adjacency are linearly dependent means that at least one row can be expressed as a linear combination of other rows. This implies that $det(A) = 0$ when J not a prime ideal of $\mathbb{Z}_n$. Thus $det\left(A\left(\mathcal{g}_J(\mathbb{Z}_n)\right)\right) = 0$.

Remark 3.4. The sets $s(d)$ for all devisors d of a given integer n are an equitable partition of the set of vertices $V\ \mathcal{g}_J(\mathbb{Z}_n))$.That is any two vertices in $s(d_i)$ have the same number of neighbors in $s(d_j)$ for all divisor d_i, d_j of n .This allow as to define a projection graph of the Idealbased-zerodivisor graph .

Definition 3.5. A projection graph of the Idealbased-zerodivisor graph $\pi(\mathcal{g}_J(\mathbb{Z}_n))$ as a graph with vertices $s(d)$ for all d/n and edge connecting $s(d_i)$ with $s(d_j)$ if every element in $s(d_i)$ is connected with $s(d_j)$.

Remark 3.6. The size of the adjacency matrix depends on the number on the numbers of vertices in the projection graph.

Example 3.7. The projection graph of $\mathbb{Z}_{15}$ if $\boldsymbol{J} = (\boldsymbol{0})$, $\boldsymbol{J}$ must be not prime ideal of $\mathbb{Z}_{15}$, since if $J = (3), J = (5)$ is a prime ideal of $\mathbb{Z}_{15}$ then the graph is null graph.

$s(d) = \{k \in \mathbb{Z}_n - J: \gcd(k, n) = d\}$

$s(3) = \{k \in \mathbb{Z}_{15} - (0): \gcd(k, 15) = 3\}$

$s(3) = \{3,6,9,12\}$

$s(5) = \{k \in \mathbb{Z}_{15} - (0): \gcd(k, 15) = 5\}$

$s(5) = \{5,10\}$.

Since every element in $s(3)$ is connected with every element in $s(5)$ in $\mathcal{g}_{(0)}(\mathbb{Z}_{15})$, then $s(3)$ is adjacent of $s(5)$ in $\mathcal{g}_{(0)}(\mathbb{Z}_{15})$. This implies that the projection graph $\pi(\mathcal{g}_{(0)}(\mathbb{Z}_{15}))$.

Figure 3.2 : $\boldsymbol{\pi}(\boldsymbol{\mathcal{g}}_{(\mathbf{0})}(\mathbb{Z}_{\mathbf{15}}))$

$|s(d)| = \emptyset(\frac{n}{d})$, $|s(3)| = \emptyset(\frac{15}{3}) = 4$, $|s(5)| = \emptyset(\frac{15}{5}) = 2$.

$A\left(\pi\left(\mathcal{g}_{(0)}(\mathbb{Z}_{15})\right)\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $det\left(A\left(\pi\left(\mathcal{g}_{(0)}(\mathbb{Z}_{15})\right)\right)\right) = -1$.

Example 3.8. We study the projection graph of $\mathbb{Z}_{\mathbf{12}}$, where $\boldsymbol{J} = (\mathbf{6})$.

$s(d) = \{k \in \mathbb{Z}_n - J: \gcd(k, n) = d\}$, $J = \{0,6\}$ and $V(\mathbb{Z}_{12}) = \{1,2,3,4,5,7,8,9,10,11\}$

$s(1) = \{k \in \mathbb{Z}_{12} - \{0,6\}: \gcd(k, 12) = 1\} = \{1,5,7,11\}$

$s(2) = \{k \in \mathbb{Z}_{12} - \{0,6\}: \gcd(k, 12) = 2\} = \{2,10\}$

$s(3) = \{k \in \mathbb{Z}_{12} - \{0,6\}: \gcd(k, 12) = 3\} = \{3,9\}$

$s(4) = \{k \in \mathbb{Z}_{12} - \{0,6\}: \gcd(k, 12) = 4\} = \{4,8\}$.

Since every element in $s(2)$ is not connected with every element in $s(4)$ i.e $s(2)$ is not adjacent $s(4)$ because $2 \times 4 = 8 \notin$ J. Every element in $s(2)$ is connected with every element in $s(3)$ then $s(2) \leftrightarrow s(3)$ hence $s(2)$ is adjacent to $s(3)$. Every element in $s(3)$ is connected with every element in $s(4)$, then

$s(3) \leftrightarrow s(4)$. Therefore, $s(3)$ is adjacent of $s(4)$.The projection graph $\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12})\right)$ is

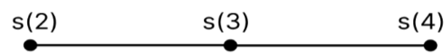

Figure 3.3: $\boldsymbol{\pi}\left(\boldsymbol{\mathcal{g}_{(6)}}(\mathbb{Z}_{\mathbf{12}})\right)$

$|s(2)| = \emptyset(\frac{12}{2}) = \emptyset(6) = 6(1 - \frac{1}{2}).\left(1 - \frac{1}{3}\right) = 2$

$|s(3)| = \emptyset(\frac{12}{3}) = \emptyset(4) = 4(1 - \frac{1}{2}) = 2$

$|s(4)| = \emptyset(\frac{12}{4}) = \emptyset(3) = 3 - 3^0$

$$A\left(\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12})\right)\right) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\left.\begin{array}{l} row1 \to s(2) \\ \\ row2 \to s(3) \\ \\ row3 \to s(4) \end{array}\right] \longrightarrow A\left(\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12})\right)\right) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$det\left(A\left(\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12})\right)\right)\right) = -1 * \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = -1 * 0 = 0.$$

Example 3.9. The projection graph of $\mathbb{Z}_{\mathbf{20}}$, where $\boldsymbol{J} = (\mathbf{10})$. Then,

$s(d) = \{k \in \mathbb{Z}_n - J: \gcd(k, n) = d\}$

$J = \{0,10\}$

$V(\mathbb{Z}_{20}) = \{1,2,3,4,5,6,7,8,9,11,12,13,14,15,16,17,18,19\}$

$s(2) = \{k \in \mathbb{Z}_{20} - \{0,10\}: \gcd(k, 20) = 2\}$

$s(2) = \{2,6,14,18\}$

$s(4) = \{k \in \mathbb{Z}_{20} - \{0,10\}: \gcd(k, 20) = 4\}$, then $s(4) = \{4,8,12,16\}$

$s(5) = \{k \in \mathbb{Z}_{20} - \{0,10\}: \gcd(k, 20) = 5\}$, then $s(5) = \{5,15\}$

$|s(d)| = \emptyset(\frac{n}{d})$

$|s(2)| = \emptyset\left(\frac{20}{2}\right) = \emptyset(10) = 4$

$|s(4)| = \emptyset(\frac{20}{4}) = \emptyset(5) = 4$, thus $|s(5)| \neq \emptyset(\frac{20}{5}) = \emptyset(4) = 2$. Since every element in $s(2)$ is connected with every element in $s(5)$ then $s(2) \leftrightarrow s(5)$ and since every element in $s(4)$ is connected with every element in $s(5)$ then $s(4) \leftrightarrow s(5)$

The projection graph $\pi\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20})\right)$

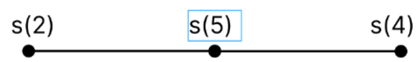

Figure 3.4: $\boldsymbol{\pi}\left(\boldsymbol{\mathcal{g}}_{(\mathbf{10})}(\mathbb{Z}_{\mathbf{20}})\right)$

$A\left(\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12})\right)\right) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\left.\begin{array}{l} row1 \to s(2) \\ \\ row2 \to s(4) \\ \\ row3 \to s(5) \end{array}\right\} \longrightarrow A\left(\pi\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20})\right)\right) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$det\left(A\left(\pi\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20})\right)\right)\right) = 0.$

Theorem 3.10. The number of vertices in the graph $\left(\pi\mathcal{g}_J(\mathbb{Z}_n)\right)$, where $n = \prod_{i=1}^{s} P_i^{t_i}$, J neither prime nor zero ideal of $\mathbb{Z}_n$, is $\left|\pi\left(\mathcal{g}_J(\mathbb{Z}_n)\right)\right| = \prod_{i=1}^{s}(t_i + 1) - 3$.

Proof. The vertices of $\pi\left(\mathcal{g}_J(\mathbb{Z}_n)\right)$ are the set $s(d) = \{k \in \mathbb{Z}_n - J : \gcd(k, n) = d\}$. And the vertices of the projection graph correspond to the divisors d of n. If the prime decomposition for $n = \prod_{i=1}^{s} P_i^{t_i}$ and the prime decomposition divisor

$$d = \prod_{i=1}^{s} P_i^{a_i}.$$

In an idealbased-zerodivisor graph, some divisors of n are excluded due to the definition of J (which is not prime ideal of $\mathbb{Z}_n$) and any divisor related to J this implies that

$\left|\pi\left(\mathcal{g}_J(\mathbb{Z}_n)\right)\right| = \prod_{i=1}^{s}(t_i + 1) - 3.$

c We apply Theorem 3.10 for $\mathbb{Z}_{12}$,where $\boldsymbol{J} = (\boldsymbol{6})$.

$\left|V(\pi\left(\mathscr{g}_J(\mathbb{Z}_n\ )\right))\right| = \prod_{i=1}^{s}(t_i + 1) - 3$

$\left|V(\pi\left(\mathscr{g}_J(\mathbb{Z}_n\ )\right))\right| = \prod_{i=1}^{s}(t_i + 1) - 3 = 3$. See Example 3.8.

$\left|V(\pi\left(\mathscr{g}_J(\mathbb{Z}_n\ )\right))\right| = s(2),\ s(3),\ s(4) = 3$

Remark 3.12 The number of vertices in the graph $\left|V(\pi\left(\mathscr{g}_J(\mathbb{Z}_n\ )\right))\right| = \prod_{i=1}^{s}(t_i + 1) - 2$ where J is a Zerodivisor ideal of $\mathbb{Z}_n$,see example 3.7

$\left|V(\pi\left(\mathscr{g}_{(0)}(\mathbb{Z}_{15}\ )\right))\right| = 2$

$\left|V(\pi\left(\mathscr{g}_{(0)}(\mathbb{Z}_{15}\ )\right))\right| = \mathrm{s}(2), \mathrm{s}(5) = 2$

This concept mentioned in [6] where $J = \{0\}$ implies that $\mathscr{g}_J(\mathbb{Z}_n\ ) = \mathscr{g}(\mathbb{Z}_n\ )$

$$\left|V(\pi\left(\mathscr{g}_J(\mathbb{Z}_n\ )\right))\right| = \prod_{i=1}^{s}(t_i + 1) - 2$$

Theorem 3.13. Let $n = \prod_{i=1}^{s} P_i^{t_i}$ then $rank\ A\big(\mathscr{g}_I(\mathbb{Z}_n\ )\big) = rank\ A(\pi\big(\mathscr{g}_J(\mathbb{Z}_n\ )\big))$.

Proof. Recall that vertices in $V\pi\left(\mathscr{g}_J(\mathbb{Z}_n\ )\right)$ correspond to sets $s(d)$,where d/n since each element of $s(d)$ contribute exactly the same row to the adjacency matrix $A\left(\mathscr{g}_J(\mathbb{Z}_n\ )\right)$

It follows that rank $A\left(\mathscr{g}_J(\mathbb{Z}_n\ )\right) \leq \left|V(\pi\left(\mathscr{g}_J(\mathbb{Z}_n\ )\right))\right|$.

On the other hand, $rank\ A(\pi\big(\mathscr{g}_J(\mathbb{Z}_n\ )\big)) \leq$ rank $A\left(\mathscr{g}_J(\mathbb{Z}_n\ )\right)$,since we just remove repeted rows and columns to get $rank\ A(\pi\big(\mathscr{g}_J(\mathbb{Z}_n\ )\big)) = rank\ A\left(\mathscr{g}_J(\mathbb{Z}_n\ )\right)$.

Example 3.14 We demonstrate theorem 3.13 with the idealbased-zero divisor graph $\mathscr{g}_I(\mathbb{Z}_{12}\ )$, $\pi\big(\mathscr{g}_I(\mathbb{Z}_{12}\ )\big)$ where $I = (6)$ and its to show $rank\ A(\pi\big(\mathscr{g}_{(6)}(\mathbb{Z}_{12}\ )\big)) = rank\ A\left(\mathscr{g}_{(6)}(\mathbb{Z}_{12}\ )\right)$

$J = \{0{,}6\}$, $\mathscr{g}_J\,(\mathbb{Z}_{12}\ ) = \{2{,}3{,}4{,}8{,}9{,}10\}$

$$A\left(\mathscr{g}_{(6)}(\mathbb{Z}_{12}\ )\right) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Because those columns 3,4,5,6 are reputed, we can remove them.

Then the adjacency matrix $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

In this adjacency matrix the two columns are clearly independent as neither column is a scalar multiple of the other. This implies that $rank\ A = 2$ or perform Gaussian elimination to reduce a to its row – echelon form.

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Swap row 1 and row 2 to get $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The third, fourth and sixth rows are repeated to the second row, we can remove row 3, row 4, and row 5 is repeated row 1 we can remove row 5 then we get $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then the reduced matrix has two non-zero rows, meaning two linearly independent row this implies that rank of the matrix of $A = 2$. Thus $rank\ A\left(\mathscr{g}_{(6)}(\mathbb{Z}_{12})\right) = 2$. And we can find rank $rank\ A(\pi(\mathscr{g}_{(6)}(\mathbb{Z}_{12})))$. In Example 3.8 $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ we can find rank A. Since row 3 is redundant row 1. We can remove row 3 then we get $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. In this adjacency matrix the two row are linearly independent because no row is a linear combination of the others. Thus, the rank of A is 2. Hence $rank\ A(\pi(\mathscr{g}_{(6)}(\mathbb{Z}_{12}))) = 2$. This implies that rank $A(\pi(\mathscr{g}_{(6)}(\mathbb{Z}_{12}))) = rank\ A(\pi(\mathscr{g}_{(6)}(\mathbb{Z}_{12}))) = 2$

4. Betti number, projective dimension with some other properties of the projective graphs

This section examines the projective graph's adjacency matrices while calculating the graph's depth, girth, diameter, Betti number, regularity, and projective dimension. The foundation of this study is the interaction between graph-theoretic measures and algebraic invariants. Exploring links between combinatorial structures and algebraic properties is made easier by the projective graph, which is derived from projective geometry. Regularity and the Betti number are algebraic invariants that capture the intricacy of minimal free resolutions of the related edge ideals. The structural features of these ideals are further disclosed by properties like projective dimension and related prime ideals, which provide a fuller understanding of their behavior. However, graph-theoretic ideas like dimension (the greatest distance between two vertices) and girth (the length of the shortest cycle) capture the graph's inherent combinatorial characteristics. To bridge the gap between algebraic and combinatorial viewpoints, these aspects can be analyzed using the projective graph's adjacency matrices. By combining these two areas, this study seeks to offer a thorough overview. We demonstrate the deep linkages between abstract algebra and graph theory by looking at the connections between combinatorial metrics like girth and width and algebraic features like related prime ideals or depth. The findings have potential uses in computer algebra, optimization, and geometry in addition to advancing our theoretical knowledge of projective graphs.

Example 4.1. Let $= \mathbb{Z}_{12}, J = (6)$, we study the minimal free resolution and find the betti numbers of $\pi\left(\mathscr{g}_{(6)}(\mathbb{Z}_{12})\right)$,we have

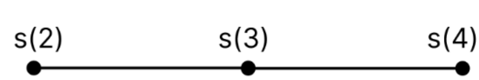

Figure 3.5: $\boldsymbol{\pi}\left(\boldsymbol{\mathscr{g}_{(6)}}(\mathbb{Z}_{12})\right)$

This implies that edge ideal of $\pi\left(\mathscr{g}_{(6)}(\mathbb{Z}_{12})\right) = \{x_2x_3, x_3x_4\}$. We can find minimal free resolution of $\frac{A}{B}$ over A:

Step 0: set $F_0 = A$ and let $d_0: A = F_0 \to \frac{A}{B}$.

Step 1: to show $er\ (d_0) = (x_2x_3, x_3x_4)$. Let $Z \in\ Ker\ (d_0)$ this implies that $d_0(Z) = \frac{A_0}{B} = B = (x_2x_3, x_3x_4)$ This implies that $Z + B = B \quad \Rightarrow\ = (x_2x_3, x_3x_4) \Rightarrow\ Ker\ (d_0) \subseteq (x_2x_3, x_3x_4)$ .To show $(x_2x_3, x_3x_4) \subseteq\ Ker\ (d_0)$ .Let $P \in (x_2x_3, x_3x_4) = B\ \Rightarrow \mathrm{P} + \mathrm{B} = \mathrm{B} \quad \Rightarrow d_0 = B \Rightarrow \mathrm{P} \in\ Ker\ (d_0)$ .Hence $Ker\ (d_0) = B = (x_2x_3, x_3x_4)$ .Then homogenous generator of $Ker\ (d_0)$ are x_2x_3, x_3x_4 with degree 2,2. We set $F_1 = A(-2) \oplus A(-2)$.Let f_1 be the generator of $A(-2)$, degree (f_1)= 2. Let f_2 be the generator of $A(-2)$, degree (f_2)= 2 .Then we have the resolution $A(-2) \oplus A(-2) \xrightarrow{d_1} F_0 = A \xrightarrow{d_0} \frac{A}{B} \to 0$. To find the matrix of d_1 where d_1 is defined by $d_1(f_1) = x_2x_3, d_1(f_2) = x_3x_4$. Let $P \in A(-2) \oplus A(-2)$ then $P = \propto f_1 + \beta f_2$ then $\propto, \beta \in K$, $d_1(P) = \propto d_1(f_1) + \beta d_1(f_2) = \propto x_2x_3 + \beta x_3x_4 = (x_2x_3, x_3x_4)\begin{pmatrix}\propto \\ \beta\end{pmatrix}$, so the matrix d_1 is (x_2x_3, x_3x_4). Then, the resolution has the form: $F_1 = A(-2) + A(-2) \xrightarrow{(x_2x_3, x_3x_4)} F_0 = A \xrightarrow{d_0} \frac{A}{B} \to 0$.

Step 2: to find the generator of $Ker\ (d_1)$ .Let $P \in Ker\ (d_1) \subseteq\ A(-2) \oplus A(-2)$.And f_1, f_2 are basis for $A(-2), A(-2)$. Then $P = \propto f_1 + \beta f_2$, $\propto, \beta \in K$.Then $d_1(P) = d_1(\propto f_1 + \beta f_2\ ) = \propto x_2x_3 + \beta x_3x_4\ \to \propto = x_4, \beta = -x_2$. Since $x_4f_1 - x_2f_2$ is generator of $Ker\ (d_1)$, $\deg(x_4f_1 - x_2f_2) = 3$. Then we set

$$F_2 = A(-3) \xrightarrow{\begin{pmatrix} x_4 \\ -x_2 \end{pmatrix}} A(-2) \oplus A(-2) \xrightarrow{(x_2x_3, x_3x_4)} A \xrightarrow{d_0} \frac{A}{B} \to 0.$$

Then the minimal free resolution:

$$0 \to A(-3) \xrightarrow{\begin{pmatrix} x_4 \\ -x_2 \end{pmatrix}} A(-2) \oplus A(-2) \xrightarrow{(x_2x_3, x_3x_4)} A \xrightarrow{d_0} \frac{A}{B} \to 0.$$

Hence, $0 \to A(-3) \to A^2(-2) \to A \to \frac{A}{I\left(\pi\left(\Gamma_{(6)}(\mathbb{Z}_{12}\ )\right)\right)} \to 0$.

Then,$F_0 = A, F_1 = A(-2) \oplus A(-2) = A^2(-2)$,

$F_3 = A(-3) \to F_0 = A(0) \Rightarrow b_{0,0} = 1 \to b_0 = 1$

$F_1 = A^2(-2) \to b_{1,2} = 2$. $F_2 = A(-3) \to b_{2,3} = 1$. Then, $Pd\left(\frac{A}{B}\right) = 2$.

Also, we can find the adjacency matrix $A\left(\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12}\ )\right)\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Remark 4.2. The edge ideal $I(G)$ of a finite simple graph G has regularity 2 if and only if G is co-chordal graph [6]. Since the compliment of a graph $\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12}\ )\right)$ is

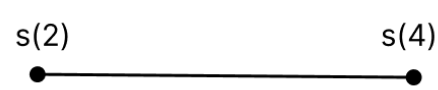

Figure 3.6: Compliment of graph $\boldsymbol{\pi}\left(\boldsymbol{\mathcal{g}}_{(6)}(\mathbb{Z}_{12}\ )\right)$.

Which has only two vertices connected by an edge then Compliment of graph $\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12}\ )\right)$ is trivially chordal (because there are no cycles of length four or more). Then the graph $\pi\left(\boldsymbol{\mathcal{g}}_{(6)}(\mathbb{Z}_{12}\ )\right)$ is co-chordal graph. This implies that regularity $\left(I(G)\right) = 2$ .The minimal free resolution $M = A/B$ where $A = k[x_1, x_2, x_3, x_4, x_6, x_{12}]$ .The Auslander-Buchesbum formular [14].

$Pd(M) + depth(M) = depth\ (A)\ .\ depth\ (A) = 6, Pd(M) = 2.$

$Pd(M) + depth(M) = depth\,(A)$. Therefore $2 + depth(M) = 6 \Rightarrow depth(M) = 4$.Hence for a graph $\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12}\ )\right)$. $s(3)$ is a minimal vertex cover, as it is the smallest set that covers all edges. By $Ass(I)$ we mean$Ass\left(\frac{S}{I}\right)$ then the primary decomposition of $I(G) =<\ x_2x_3, x_3x_4 \geq (x_2, x_3x_4) \cap (x_3) = (x_2, x_3) \cap (x_2, x_4) \cap (x_3) = (x_3) \cap (x_2, x_4)$. Then $Ass\left(\frac{S}{\mathrm{I}}\right) = \{(x_3), (x_2, x_4)\}$. $girth\left(\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12}\ )\right)\right) = \infty$ (because the graph $\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12}\ )\right)$ is a straight line) it mean there are no cycle. distance $\left(s(2), s(3)\right) = 1$, distance $\left(s(3), s(4)\right) = 1$, distance $\left(s(2), s(4)\right) = 2$, where $G = \pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12}\ )\right)$. $dim\left(\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12}\ )\right)\right) = \max\{d_G(s_1, s_2); s_1, s_2 \in V(G)\}$, $dim(G) = \max\{1,1,2\} = 2$.

The eigenvalues $of \left(\mathrm{A}\left(\pi\left(\mathcal{g}_{(6)}(\mathbb{Z}_{12}\ )\right)\right)\right)$ are $\sqrt{2}$, $-\sqrt{2}$ and 0, and the characteristic polynomial is $\mathrm{P}(\lambda) = \lambda^3 - 2\lambda$.

Example 4.4. Consider $R = \mathbb{Z}_{20}$, $I = (10)$ and $\pi\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20}\ )\right) = G$, we start with the minimal free resolution $\pi\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20}\ )\right)$,with edge ideal $= \{x_2x_5, x_4x_5\}$

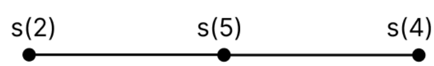

Figure 3.7: $\boldsymbol{\pi}\left(\boldsymbol{\mathcal{g}}_{(\mathbf{10})}(\mathbb{Z}_{\mathbf{20}}\ )\right)$

Then the minimal free resolution

$$0 \to A(-3) \xrightarrow{\binom{x_4}{-x_2}} A(-2) \oplus A(-2) \xrightarrow{(x_2x_5, x_4x_5)} A \xrightarrow{d_0} \frac{A}{B} \to 0.$$

Hence, $0 \to A(-3) \to A^2(-2) \to A \to \frac{A}{I\left(\pi\left(\Gamma_{(10)}(\mathbb{Z}_{20}\ )\right)\right)} \to 0$. Then, $Pd\left(\frac{A}{B}\right) = 2$.

Now, since the compliment of a graph $\pi\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20}\ )\right)$ is

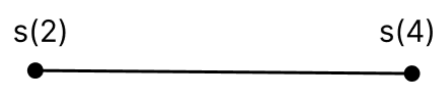

Figure 3.8: Compliment of graph $\boldsymbol{\pi}\left(\boldsymbol{\mathcal{g}}_{(\mathbf{10})}(\mathbb{Z}_{\mathbf{20}}\ )\right)$.

Which has only two vertices connected by an edge then Compliment of graph $\pi\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20}\ )\right)$ is trivially chordal (because there are no cycles of length four or more). Then the graph $\pi\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20}\ )\right)$ is co-chordal graph .This implies that regularity $\left(I(G)\right) = 2$ and $Pd(M) + depth(M) = depth\,(A)$, since $depth\,(A) = 6$ and $Pd(M) =, 2$ then $Pd(M) + depth(M) = depth\,(A)$, then $2 + depth(M) = 6 \Rightarrow depth(M) = 4$. It shows that for a graph $\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20}\ )\right)$, $s(5)$ is a minimal vertex cover, as it is the smallest set that covers all edges. The primary decomposition of $I(G) =<\ x_2x_5, x_4x_5 > \quad = (x_2, x_4x_5) \cap (x_5) = (x_2, x_5) \cap (x_2, x_4) \cap (x_5) = (x_5) \cap (x_2, x_4)$. Then $Ass\left(\frac{A}{\mathrm{I}}\right) = \{(\,x_5\,), (x_2, x_4)\}$.

$girth\left(\pi\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20}\ )\right)\right) = \infty$

distance $(s(2), s(5) = 1)$

distance $(s(5), s(4) = 1)$

distance $(s(2), s(4) = 2)$

$dim\,\pi\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20}\ )\right) = \max\{d_G(s_1, s_2); s_1, s_2 \in V(G)\}$

$dim(G) = \max\{1,1,2\} = 2$

The eigenvalues of $\left(\mathrm{A}\left(\pi\left(\mathcal{g}_{(10)}(\mathbb{Z}_{20}\ )\right)\right)\right)$ are $\sqrt{2}$, $-\sqrt{2}$ and 0 and the characteristic polynomial is, $\mathrm{P}(\lambda) = \lambda^3 - 2\lambda$.

Conclusion. In this work, we consider the Idealbased-zerodivisor graph. We determined the adjacency and eigenvalues of the Idealbased-zerodivisor graph of the ring $\mathbb{Z}_n$ in some special cases. We define the notion of projection graph of the Idealbased-zerodivisor graph $\pi(\mathcal{g}_J(\mathbb{Z}_n\ ))$ as a graph with vertices $s(d)$ for all d/n and edge connecting $s(d_i)$ with $s(d_j)$ if every element in $s(d_i)$ is connected with $s(d_j)$. If $n = p_1^{t_1} \dots p_s^{t_s}$, where $p_1, \dots, p_s$ (where $s \geq 2$) and I is non-prime ideal of $\mathbb{Z}_n$, then we determined the determinant of the adjacency matrices A of the IdealBased-Zerodivisor graph of $\mathbb{Z}_n$ and satisfies $det\left(A\left(\mathcal{g}_J(\mathbb{Z}_n\ )\right)\right) = 0$. We showed that in the Ideal-based zerodivisor graph $\left(\mathcal{g}_J(\mathbb{Z}_n\ )\right)$, where $n = \prod_{i=1}^{s} P_i^{t_i}$, I is not prime ideal of $\mathbb{Z}_n$ and not zero ideal of $\mathbb{Z}_n$, then $\left|\pi\left(\mathcal{g}_J(\mathbb{Z}_n\ )\right)\right| = \prod_{i=1}^{s}(t_i + 1) - 3$ and whenever $n = \prod_{i=1}^{s} P_i^{t_i}$ then $rank\ A\left(\mathcal{g}_J(\mathbb{Z}_n\ )\right) = rank\ A(\pi(\mathcal{g}_J(\mathbb{Z}_n\ )))$. Let $R = \mathbb{Z}_n$, where n is positive integer and I is an ideal of R that is not a prime ideal for each divisor d of n define the set $s(d) = \{k \in \mathbb{Z}_n - I : \gcd(k, n) = d\}$, then $|s(d)| = \emptyset\ (\frac{n}{d})$, where $\emptyset$ is Euler totient function.

Acknowledgements

The authors are grateful to the referee for valuable comments.

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