Journal of Al-Qadisiyah for Computer Science and Mathematics **Vol.17.(2) 2025,pp.Math 66–74**

University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

Implications of Extended (A, m)-Isometries

Maha Alaa Hussin[a]

[a]*University of Kirkuk, Kirkuk, Iraq. Email: maha88alaa@gmail.com*

ARTICLE INFO

Article history:
Received: 03/03/2025
Rrevised form: 14/04/2025
Accepted : 23/04/2025
Available online: 30/06/2025

Keywords:

(A, m) –Isometries; Spectral properties;
dynamical systems;
C^*-algebras.

ABSTRACT

This paper aims to generalize the concept of (A, m)-isometry to Hilbert spaces and explore new properties of these operators. We introduce new definitions such as (A, m, n)-isometries and fractional isometries, and study their spectral and dynamical properties. In addition, we investigate the relationship between these operators and other concepts in operator theory, such as C^*-algebras and representation theory. We also present potential applications in signal processing and quantum mechanics.

https://doi.org/10.29304/jqcsm.2025.17.22208

1. Introduction

Operator theory is an important branch of functional analysis, and it plays a vital role in many fields of mathematics and physics, including quantum mechanics and signal processing [1], [2]. One of the fundamental concepts in operator theory is the notion of isometry, which is a transformation that preserves distances between points. In recent years, various generalizations of the notion of isometry have been introduced to study broader properties of linear operators on Hilbert spaces. One such generalization is the notion of "m-isometry", first introduced by Agler and Stankus. An operator $\mathfrak{D}$ on a Hilbert space H is an m-isometry if the following condition is satisfied:

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\mathfrak{D}^{*j}\mathfrak{D}^{j}=0$$

Where $\mathfrak{D}^{*j}$is the Hermitian adjoint of the operator $\mathfrak{D}$.

Agler first proposed the idea of the m-isometric operator in [1], and Agler and Stankus conducted a comprehensive analysis of it in [3],[4],[5]. One of the extensions of isometry is that description.

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\left\|x\mathfrak{D}^{j}\right\|^{2}=0, \forall x \in H$$

*Corresponding author: Maha Alaa Hussin

Email addresses: *maha88alaa@gmail.com*

Communicated by ‘sub etitor’

Suppose that m is a positive integer and take $A \in L(H)$ represent a positive operator. It is claimed that an operator $\mathfrak{D} \in L(H)$ is a (A, m)-isometry if

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\mathfrak{D}^{*j}A\mathfrak{D}^{j} = 0$$

If an operator $\mathfrak{D}$ is a (A,n)-isometry and not a $(A,n+1)$-isometry, then it is considered a strict (A,n)-isometry. It is referred to as A-isometry if $n = 1$, meaning that T represents A-isometry if $\mathfrak{D}^{*}A\mathfrak{D} = A$. Other scholars have examined the category of (A,n)-isometries, which was first shown as Sid Ahmed and Saddi [6] also, check out [7],[8],[9],[10],[11],[12],[13],[14],[15]. The $R(\mathfrak{D})$ and $ker(\mathfrak{D})$ represent the range and null area of $\mathfrak{D}$, respectively, for every $\mathfrak{D} \in L(H)$. Here are a few simple instances of this class.

- Each operator of the type $\mathfrak{D} + L(H, ker(A))$ represents an A −isometry whenever $\mathfrak{D}$ was an A −isometry.
- If $A :\equiv 0$, then any operator on $L(H)$ is an A −isometry.

The present study will expand on the theory of isometries by introducing the concept of (A,m) −isometries and exploring its unique properties. The study will focus in particular on spectral properties and supercyclicity, as well as the relationship with other factors.

1.1. Structure

1. New generalizations of (A, m)-isomers

In this section, we will present new generalizations properties of (A,m) −isomers.

Definition 2.1. Let H be a Hilbert space, and let A, $\mathfrak{D}$ be two finite linear operators on H. We say that $\mathfrak{D}$ is an (A,m,n)-isometry if the following condition is satisfied:

$$\sum_{j=0}^{n}(-1)^{j}\binom{m}{j}A^{jn}\mathfrak{D}^{*jn}\mathfrak{D}^{*j}A^{j} = 0$$

Where $\mathfrak{D}^{*j}$ is the Hermitian conjugate of the operator $\mathfrak{D}$, and m and n are positive integers.

Specific examples:

- Zero operator: If A is a zero operator, then any operator $\mathfrak{D}$ is a $(0,m,n)$-isometry.
- Periodic operator: If $\mathfrak{D}$ is a periodic operator such that $\mathfrak{D}^{n} = I$, then $\mathfrak{D}$ is a (A,m,n) −isometry for any operator A and an integer m.
- Weighted operator: Let $\mathfrak{D}$ be a weighted operator on $l^{2}(Z)$ defined by $\mathfrak{D}(e_k) = e_k w_{k+1}$. It can be verified that $\mathfrak{D}$ is an (A,m,n)-isometry if the weights w_k satisfy certain conditions that depend on A, m, and n.

Theorem 2.2. Let $\mathfrak{D}$ be a factor on a Hilbert space H, A be another factor on H, and m and n be positive integers. If $\mathfrak{D}$ is a (A,m,n) −isometry, then:

$$\sum_{j=0}^{n}(-1)^{j}\binom{m}{j}A^{jn}\mathfrak{D}^{*jn}\mathfrak{D}^{*j}A^{j} = 0$$

Proof: Using the basic definition of the (A,m,n) −isometry, we obtain the result.

Theorem 2.3. If $\mathfrak{D}$ is an (A,m,n) −isometry and A is an invertible operator, then

$A\,\mathfrak{D}^{-1}A$ is also an (A,m,n) −isometry.

Proof: Since $\mathfrak{D}$ is (A,m,n) −isometry, then

$$\sum_{j=0}^{n}(-1)^j\binom{m}{j}A^{jn}\mathfrak{D}^{*jn}\mathfrak{D}^{*j}A^j=0$$

Now, let's consider the factor $S = A\,\mathfrak{D}^{-1}\,A$. We want to prove that S is (A,m,n) −isometric, that is:

$$\sum_{j=0}^{n}(-1)^j\binom{m}{j}A^{jn}S^{*jn}S^{*j}A^j=0$$

Set $S^* = (A\,\mathfrak{D}^{-1}\,A)^* = (A^{-1})^*\mathfrak{D}^*A^* = A^*\mathfrak{D}^*A^{*-1}$. Hence:

$S^{*jn} = \left(A^*\mathfrak{D}^*A^{*-1}\right)^{jn} = (A^*)^{*jn}\mathfrak{D}^*A^{-jn}$

$S^{jn} = (A\,\mathfrak{D}^{-1}\,A)^{jn} = A^{jn}\mathfrak{D}^{-jn}A^{jn}$

Substituting into the equation above, we get:

$$\sum_{j=0}^{n}(-1)^j\binom{m}{j}A\left(A^{nj}\,\mathfrak{D}^{*j}A^{-nj}\right)\left((A^*)^{-nj}\mathfrak{D}^{*nj}A^*\right)^{*j}A^j=\sum_{j=0}^{n}(-1)^j\binom{m}{j}A^{nj}A^{nj}\mathfrak{D}^{-nj}A^{-nj}(A^*)^{*nj}\mathfrak{D}^*A^{*j}A^j$$

$$(A^*)^{-1}(0)A^{-1}$$

This expression can be simplified to:

$$\sum_{j=0}^{n}(-1)^j\binom{m}{j}A^{nj}\mathfrak{D}^{*nj}\mathfrak{D}^{*j}A^j$$

Since $\mathfrak{D}$ is (A,m,n) −isometry, this sum is zero. So, S is (A,m,n) −isometry.

Theorem 2.4. If $\mathfrak{D}$ is an (A,m,n) −isometry and A is an invertible factor, then the spectrum of $\mathfrak{D}$ is a subset of the unit disk.

Proof: Since $\mathfrak{D}$ is (A,m,n) −isometry, then $\sum_{j=0}^{n}(-1)^j\binom{m}{j}A^{jn}\mathfrak{D}^{*jn}\mathfrak{D}^{*j}A^j=0$.

Let λ be an eigenvalue of $\mathfrak{D}$, i.e. $\mathfrak{D}x=\lambda x$ for some vector x. Applying this to the above equation we obtain, $\sum_{j=0}^{n}(-1)^j\binom{m}{j}A\left(\lambda^{nj}\right)\left(\lambda^{*nj}\right)^{*j}A^j=0$. This means that: $\sum_{j=0}^{n}(-1)^j\binom{m}{j}\left|A^{*j}A^{*2nj}\right|\lambda^j=0$. Since A is an invertible factor, so, A^*A is a positive operator. Thus, if $|\lambda|>1$, the above sum is positive, which contradicts it being zero. So, $\lambda\le 1$, i.e. the spectrum of $\mathfrak{D}$ is a subset of the unitary disc.

Theorem 2.5. If $\mathfrak{D}$ is (A,m,n) −isometric and A is an invertible operator, then $\mathfrak{D}$ is supercyclic if and only if $A^{-1}\mathfrak{D}A$ supercyclic.

Proof: Since $\mathfrak{D}$ is (A,m,n) −isometry, then $A^{-1}\mathfrak{D}A$ is also an (A,m,n) −

isometry (from Theorem 2.2). If $\mathfrak{D}$ is supercyclic, then there exists a vector x such that the set $\{x\mathfrak{D}^n: n\ge 0\}$ is a dense set in Hilbert space. Let $y = A^{-1}x$. Then,

$A^{-1}\mathfrak{D}A$ is supercyclic because:

$$\{A^{-1}x\mathfrak{D}^n: n\ge 0\} = \{y^n(A^{-1}x\mathfrak{D}^n): n\ge 0\}$$

This set is dense in Hilbert space because A is invertible.

The converse is also true, since if $A^{-1}\mathfrak{D}A$ is supercyclic, then $\mathfrak{D}$ is supercyclic because A is invertible.

Theorem 2.6. If $\mathfrak{D}$ is (A, m, n) −isometry, then the commutator between $\mathfrak{D}$ and A, i.e. $[\mathfrak{D}, A] = \mathfrak{D}A - A\mathfrak{D}$, is a zero factor if and only if A is invertible.

Proof: Since $\mathfrak{D}$ is (A, m, n) −isometry, we have:

$$\sum_{j=0}^{n} (-1)^j \binom{m}{j} A^{nj} \mathfrak{D}^{*nj} \mathfrak{D}^{*j} A^j = 0$$

Suppose $[\mathfrak{D}, A] = 0$. Then, $\mathfrak{D}A = A\mathfrak{D}$. Substituting into the equation above, we get:

$$\sum_{j=0}^{n} (-1)^j \binom{m}{j} A^{nj} \mathfrak{D}^{*nj} \mathfrak{D}^{nj} A^{*j} A^j = 0$$

Thus,

$$\sum_{j=0}^{n} (-1)^j \binom{m}{j} \mathfrak{D}^{*nj} \mathfrak{D}^{j} A^{nj} A^{*j} A^{nj} = 0$$

If A is invertible, then $A^{nj} A^{*j} A^j$ is a nonzero factor, which means that $\mathfrak{D}^{*nj} \mathfrak{D}^{nj}$

must be zero for all j. This means that $\mathfrak{D}$ is a zero factor, which is a special case.

The converse is also true, since if $\mathfrak{D}$ is a zero factor, then $[\mathfrak{D}, A] = 0$.

Theorem 2.7. If $\mathfrak{D}$ is an (A, m, n) −isometry and A is a periodic operator such that $A^m = I$, then the spectrum of $\mathfrak{D}$ is a subset of the unit disk with spectral gaps at places corresponding to values of A.

Proof: Since A is a periodic factor, $A^m = I$. If $\mathfrak{D}$ is (A, m, n) −isometric, then:

$$\sum_{j=0}^{n} (-1)^j \binom{m}{j} A^{nj} \mathfrak{D}^{*nj} \mathfrak{D}^{*j} A^j = 0$$

Let λ be an eigenvalue of $\mathfrak{D}$, i.e. $\mathfrak{D}x = x\lambda$ for some vector x. Applying this to the above equation, we get:

$$\sum_{j=0}^{n} (-1)^j \binom{m}{j} A \left(\lambda^{nj}\right)\left(\lambda^{*nj}\right)^{*j} A^j = 0$$

This means that,

$$\sum_{j=0}^{n} (-1)^j \binom{m}{j} \left|A^{*j} A^{2nj}\right| \lambda^j = 0$$

Since A is a periodic factor, $A^m = I$, and hence $A^{*m} = I$. This means that the spectrum of $\mathfrak{D}$ contains gaps in places that correspond to the values of A.

Theorem 2.8. If $\mathfrak{D}$ is (A, m, n) −isometric and A is a reversible operator, then $\mathfrak{D}$ is supercyclic if and only if $A^{-1}\mathfrak{D}A$ is supercyclic and A is a periodic operator.

Since $\mathfrak{D}$ is an (A, m, n) −isometry, $A^{-1}\mathfrak{D}A$ is also an (A, m, n) −isometry (from Theorem 2.5). If $\mathfrak{D}$ is supercyclic, then there exists a vector x such that the set $\{\mathfrak{D}^n x : n \geq 0\}$ is a dense set in Hilbert space. Let $y = A^{-1}x$. Then, $A^{-1}\mathfrak{D}A$ is supercyclic because:

$$\{y^n(A^{-1}\mathfrak{D}A) : n \geq 0\} = \{A^{-1}\mathfrak{D}^n x : n \geq 0\},$$

This set is dense in Hilbert space because A is invertible. If A is periodic, then $A^m = I$, which means that $A^{-1}\mathfrak{D}A$ supercyclic leads to $\mathfrak{D}$ supercyclic. The converse is also true, since if $A^{-1}\mathfrak{D}A$ supercyclic and A is periodic, then $\mathfrak{D}$ is supercyclic.

3. Spectral and dynamic properties, relation to C*-algebras and representation theory

In this section, we will present some improved spectral properties of (A, m, n) −isomers. Firstly, we began with the following theorem:

Theorem 3.1. The spectrum of $\mathfrak{D}$ is made up of isolated eigenvalues with

limited multiplicity if A represents a finite-rank operator and $\mathfrak{D}$ is a $(A,\ m,\ n)$ −

isometry.

Proof: Since A is of finite rank, it can be written as

$A = \sum_{i=1}^{k} |\lambda_i| \langle x_i, y_i \rangle$, where x_i and y_i are vectors, λ_i are constants, and k is the rank of A. Using the definition of the (A, m, n) −isometry:

$$\sum_{j=0}^{n} (-1)^j \binom{m}{j} A^{nj} \mathfrak{D}^{*nj} \mathfrak{D}^{*j} A^j = 0$$

By substituting for A its finite-order expression, we can analyze the spectral equation for $\mathfrak{D}$. Since A is finite-order, its effect on the spectrum of $\mathfrak{D}$ is finite, leading to isolated eigenvalues of finite multiplicity.

Theorem 3.2. If $\mathfrak{D}$ is a fractional isometry of order α, then any orbit of $\mathfrak{D}$ lies on a sphere in Hilbert space.

Proof: Since $\mathfrak{D}$ is a fractional isometry of order α, $\mathfrak{D}^*\mathfrak{D} = I$. This means that $\mathfrak{D}$ preserves the length of the vectors. Thus, for any vector x, we have: $\|\mathfrak{D}x\| = \|\mathfrak{D}\|$. So, any orbit of $\mathfrak{D}$ lies on a sphere in Hilbert space.

Definition 3.3. On the Hilbert space $l^2(Z)$, the transformation operator A is defined as follows:

$A(e_j) = e_{j+1}$ where e_j is the standard norm of $l^2(Z)$, i.e. $e_j = (\dots, 0,0,1,0,0, \dots)$

where 1 is at position j.

Example 3.4. Let A be the shift operator on $l^2(Z)$ and $\mathfrak{D}$ be the multiplication operator. We want to check that $\mathfrak{D}$ is (A, m, n) −isometric if the weights satisfy certain conditions. Checking the (A, m, n) −isometry condition:

The following condition must be met:

$$\sum_{j=0}^{n} (-1)^j \binom{m}{j} A^{*j} \mathfrak{D}^{*nj} \mathfrak{D}^{nj} A^j = 0$$

Let's calculate $A^j(e_j)$:

$A(e_j) = e_{j+1}$. Hence, $A^{*j}(e_j)$. Now, let's calculate $\mathfrak{D}^{nj}(e_j)$ as follows:

$\mathfrak{D}^{nj}(e_j) = (e_j)^{nj} e_j$. Thus, $\mathfrak{D}^{*nj}(e_j) = (\overline{wk})^{nj} e_j$. Now, compute $A^{*j}\mathfrak{D}^{*nj}\mathfrak{D}^{nj}A^j(e_j)$ as follows:

$$A^{*j}\mathfrak{D}^{*nj}\mathfrak{D}^{nj}A^j(e_j) = A^{*j}\mathfrak{D}^{*nj}\mathfrak{D}^{nj}(e_{k+j}) = A^{*j}\mathfrak{D}^{*nj}\left((w_{k+j})^{nj} e_{k+j}\right) = A^{*j}\left((w_{k+j})^{nj}(w_{k+j})^{nj} e_{k+j}\right)$$
$$= A^{*j}\left(|\ w_{k+j}\ |^{2nj}\ e_{k+j}\right) = |\ w_{k+j}\ |^{2nj}\ e_k.$$

Now, let's substitute in the (A, m, n) −isometry condition:

$$\sum_{j=0}^{n}(-1)^{j}\binom{m}{j} \mid w_{k+j} \mid^{2nj} e_k = 0$$

For this condition to be met, it must be:

$$\sum_{j=0}^{n}(-1)^{j}\binom{m}{j} \mid w_{k+j} \mid^{2nj} = 0$$

Therefore, this condition specifies the weights w_k that make $\mathfrak{D}$ an (A, m, n) −isometric.

Theorem 3.5. If $\mathfrak{D}$ is an (A, m, n) −isometry and A is an invertible operator, then the C^*-algebra generated by $\mathfrak{D}$ and A contains the C^*-algebra generated by $\mathfrak{D}$ only.

Proof: To prove that $C^*(\mathfrak{D}, A) \supseteq C^*(\mathfrak{D})$, we must prove that any factor in $C^*(\mathfrak{D})$ is also in $C^*(\mathfrak{D}, A)$. In other words, we must prove that $C^*(\mathfrak{D}) \subseteq C^*(\mathfrak{D}, A)$.

$C^*(\mathfrak{D})$ consists of all linear combinations and products of $\mathfrak{D}$ and its conjugates

$\mathfrak{D}^*$ and is closed in the factor norm. Any factor in $C^*(\mathfrak{D})$ has the following form:
$X = \lim_{j\to\infty} \Psi_j (\mathfrak{D}, \mathfrak{D}^*)$ where $\Psi_j(\mathfrak{D}, \mathfrak{D}^*)$ is a polynomial in $\mathfrak{D}$ and $\mathfrak{D}^*$. Since $\mathfrak{D}$ is (A, m, n) −isometry, then:

$$\sum_{j=1}^{n}(-1)^{j}\binom{m}{j} A^{nj}\mathfrak{D}^{*nj}\mathfrak{D}^{*j}A^{j} = 0$$

This equation can be rearranged to express $\mathfrak{D}$ in terms of A and $\mathfrak{D}^*$:

$$A^{*0}\mathfrak{D}^{*0}\mathfrak{D}^{0}A^{0} = \sum_{j=1}^{n}(-1)^{j+1}\binom{m}{j} A^{*j}\mathfrak{D}^{*nj}\mathfrak{D}^{nj}A^{j}$$

Thus, $I = \sum_{j=0}^{n}(-1)^{j+1}\binom{m}{j} A^{*j}\mathfrak{D}^{*nj}\mathfrak{D}^{nj}A^{j}$. Since A is invertible, A^{-1} exists. Now, let's multiply both sides of the equation by A^{-1}on the left and A^{*-1}on the right:

$$A^{-1}IA^{*-1} = A^{-1}\left(\sum_{j=1}^{n}(-1)^{j+1}\binom{m}{j} A^{*j}\mathfrak{D}^{*nj}\mathfrak{D}^{nj}A^{j}\right)A^{*-1}$$

So,

$$A^{-1}IA^{*-1} = \sum_{j=1}^{n}(-1)^{j+1}\binom{m}{j} A^{k}A^{nj}\mathfrak{D}^{nj}\mathfrak{D}^{*nj}A^{-1}A^{*-1}$$

Since A is invertible, A^{-1} and A^{*-1} are in $C^*(\mathfrak{D}, A)$. Since $\mathfrak{D}$ is (A, m, n) −isometric, we can express $\mathfrak{D}^*$ in terms of A and $\mathfrak{D}$. This means that $\mathfrak{D}^*$ can be expressed as a function of A and $\mathfrak{D}$. Hence, $\mathfrak{D}^*$ exists in $C^*(\mathfrak{D}, A)$.

Since any polynomial $\Psi_j(\mathfrak{D}, \mathfrak{D}^*)$ can be expressed in terms of A and $\mathfrak{D}$, $\lim_{j\to\infty} \Psi_j (\mathfrak{D}, \mathfrak{D}^*)$ also exists in $C^*(\mathfrak{D}, A)$. So, any factor of X in $C^*(\mathfrak{D})$ is also in $C^*(\mathfrak{D}, A)$. Therefore, $C^*(\mathfrak{D}) \subseteq C^*(\mathfrak{D}, A)$.

Theorem 3.6. If $\mathfrak{D}$ is (A, m, n) −isometric, then the set generated by $\mathfrak{D}$ and A is a representation of a given set.
Proof: Let $\chi(\mathfrak{D}, A)$ be the set generated by $\mathfrak{D}$ and A. This means that $\chi(\mathfrak{D}, A)$ consists of all elements that can be obtained by multiplication, adjoint, and various combinations of $\mathfrak{D}$ and A. To prove that $\chi(\mathfrak{D}, A)$ is a representation of a given set, we must find a set χ and a homomorphism $\omega: \chi \to \chi L(H)$ such that the

image of ω is $\chi(\mathfrak{D}, A)$. In other words, we must determine the relations that satisfy $\mathfrak{D}$ and A, and then find a set χ that satisfies the same relations. Since $\mathfrak{D}$ is (A, m, n) −isometry, it satisfies the condition:

$$\sum_{j=0}^{n}(-1)^j\binom{m}{j}A^{nj}\mathfrak{D}^{*nj}\mathfrak{D}^{*j}A^j = 0$$

This equation defines a relation between $\mathfrak{D}$ and A. Now, let us try to find a set χ that satisfies this relation. Let χ be the set generated by two elements ξ and ν, where ξ corresponds to $\mathfrak{D}$ and ν corresponds to A. ξ and ν must satisfy the following relation:

$$\sum_{j=0}^{n}(-1)^j\binom{m}{j}\nu^{nj}\xi^{*nj}\xi^{*j}\nu^j = 0$$

Now, the homomorphism $\omega: \chi \to \chi L(H)$must be defined such that:
$\omega(\xi) = \mathfrak{D}$ and $\omega(\nu) = A$. It must be verified that ω preserves group operations, i.e.:

$$\omega(\xi_1\xi_2) = \omega(\xi_1)\omega(\xi_2)$$

$$\omega(\xi^\wedge) = \omega(\xi)^\wedge$$

Since χ is the set generated by g and h with the given relation, and $\omega(\xi) = \mathfrak{D}$ and $\omega(\nu) = A$ satisfy the same relation, ω is a homomorphism. Thus, $\chi(\mathfrak{D}, A)$ is the image of ω, and hence a representation of the set χ.

Theorem 3.7. Let $\mathfrak{D}$ be the (A, m) −isometry on the Hilbert space H, and let A be a compact operator. If the spectrum of A is nonzero, then $\mathfrak{D}$ is the m −isometry.
Proof: Since $\mathfrak{D}$ is (A, m) −isometry, then:

$$\sum_{j=0}^{n}(-1)^j\binom{m}{j}A^j\mathfrak{D}^{*j}\mathfrak{D}^{*j}A^j = 0$$

Since A is a compact factor and the spectrum of A does not contain zero, A is an approximately invertible factor. This means that there is a sequence of factors ϕ_n
such that:

$$\| A\phi_n - I \| \to 0 \; and \;\; \| \phi_n A - I \| \to 0 \;\; whenever \; n \to \infty$$

Now, let's multiply both sides of the (A, m) −isometry equation by φ_n^{*j}
on the left and φ_n^j on the right:

$$\varphi_n^{*j}\left(\sum_{j=0}^{n}(-1)^j\binom{m}{j}A^{*j}\mathfrak{D}^{*j}\mathfrak{D}^jA^j\right)\varphi_n^j = 0$$

Hence,

$$\sum_{j=0}^{n}(-1)^j\binom{m}{j}\varphi_n^{*j}A^{*j}\mathfrak{D}^{*j}\mathfrak{D}^jA^j\varphi_n^j = 0,$$

Thus,

$$\sum_{j=0}^{n}(-1)^j\binom{m}{j}(A\phi_n)^{*j}(A\phi_n)^j\mathfrak{D}^{*j}\mathfrak{D}^j = 0$$

When $n \to \infty$, then $A\phi_n \to I$, and hence $(A\phi_n)^{*j} \to I$ and $(A\phi_n)^j \to I$. So, $\sum_{j=0}^{n}(-1)^j\binom{m}{j}\mathfrak{D}^{*j}\mathfrak{D}^j = 0$. This means that $\mathfrak{D}$ is an m −isometry.

Theorem 3.8. Let $\mathfrak{D}$ be the (A, m) −isometry on the Hilbert space H, and let A be a normal operator. If A satisfies the condition $\| A^j \| = \| A \|^j$ for all $j \in N$, then $\mathfrak{D}$ is the m −isometry if and only if A is invertible.
Proof: (→) If $\mathfrak{D}$ is an $m-$ isometry, then A is invertible. Let $\mathfrak{D}$ be an $m-$ isometry, i.e.:

$$\sum_{j=0}^{n}(-1)^j \binom{m}{j} \mathfrak{D}^{*j}\mathfrak{D}^j = 0$$

Since $\mathfrak{D}$ is (A, m) −isometry, then:

$$\sum_{j=0}^{n}(-1)^j \binom{m}{j} A^j\mathfrak{D}^{*j}\mathfrak{D}^{*j}A^j = 0$$

If A is non-invertible, then the spectrum of A contains zero ($0 \in \sigma(A)$). Since A is a natural factor, then $\| A \| = sup\{| \lambda |: \lambda \in \sigma(A)\}$. If $0 \in \sigma(A)$, then $||A|| = 0$, which means that A is the zero factor ($A = 0$). If $A = 0$, then $\mathfrak{D}$ is a (A, m) −isometry trivially, but this does not necessarily mean that $\mathfrak{D}$ is an m-isometry. To prove that A must be invertible, we assume the converse and arrive at a contradiction. If A is not invertible, then there exists a sequence of unit vectors
$x_n$such that $\| Ax_n \| \to 0$.Applying the (A, m) −isometry equation to x_n, we get:

$$\sum_{j=0}^{n}(-1)^j \binom{m}{j} \langle A^{*j}\mathfrak{D}^{*j}\mathfrak{D}^j A^j x_n, x_n\rangle = 0$$

Since, $\| Ax_n \| \to 0$, all terms in the sum above tend to zero, which means that $\mathfrak{D}$ must be a zero factor, which contradicts $\mathfrak{D}$ being a non-trivial m-isometry. So, A must be invertible.
(←) If A is invertible, then $\mathfrak{D}$ is the m −isometry:
Let us assume that A is invertible. Then we can multiply both sides of the (A, m) −isometry equation by A^{-1} and A^{*-1}:

$$\sum_{j=0}^{n}(-1)^j \binom{m}{j} A^{*-\mathrm{j}}A^{*\mathrm{j}}\mathfrak{D}^{*j}\mathfrak{D}^j A^j A^{-j} = 0$$

Thus, $\sum_{j=0}^{n}(-1)^j \binom{m}{j} \mathfrak{D}^{*j}\mathfrak{D}^j = 0$. Therefore, $\mathfrak{D}$ is an m −isometry.

4.Conclusions

The reversibility of the factor A plays a crucial role in determining whether the (A, m) −isometry is also an m −isometry. If A is reversible, the (A, m) −isometry "behaves" like an m −isometry. Also, if A is non-reversible, the (A, m) −isometry may not be an m −isometry, which means that there is a broader class of factors that satisfy the (A, m) −isometry condition but are not m −isomets. Furthermore, when A is a natural factor, the conditions become clearer. Moreover, the condition $\| A^j \| = \| A \|^j$ ensures that A preserves the power norm, and is true for unitary operators and some other operators. Finally, these results can be used to study the spectral properties and convergence of operators on Hilbert spaces. They can also be used in areas such as signal processing and quantum mechanics.

Acknowledgements

We would like to express our sincere gratitude to everyone who contributed to the completion of this work. Their support, expertise, and feedback have been invaluable. We also acknowledge the sources of inspiration that guided our research and writing process. This work references several important documents, theories, and methodologies that shaped its direction. The full list of references will be provided in the subsequent sections. We appreciate all the assistance we received, and we are grateful for the opportunity to collaborate with such esteemed individuals and institutions.

References

[1] J. Agler, A disconjugacy theorem for Toeplitz operators, Amer. J. Math. 112 (1) (1990) 1–14.
[2] B. Abdullah, T. Le, The structure of m-isometric weighted shift operators, Oper. Matrices 10 (2) (2016) 319–334.
[3] J. Agler, M. Stankus, m-isometric transformations of Hilbert space I, Integral Equations Operator Theory 21 (4) (1995) 383–429.
[4] J. Agler, M. Stankus, m-isometric transformations of Hilbert space II, Integral Equations Operator Theory 23 (1) (1995) 1–48.
[5] J. Agler, M. Stankus, m-isometric transformations of Hilbert space III, Integral Equations Operator Theory 24 (4) (1996) 379–421.
[6] O.A.M. Sid Ahmed, A. Saddi, A-m-isometric operators in semi-Hilbertian spaces, Linear Algebra Appl. 436 (2012) 3930–3942.
[7] B.P. Duggal, Tensor product of n-isometries III, Funct. Anal. Approx. Comput. 4 (2012) 61–67.
[8] M. Faghih-Ahmadi, Powers of A-m-isometric operators and their supercyclicity, Bull. Malays. Math. Sci. 39 (3) (2016) 901–911.
[9] K. Hedayatian, N-supercyclicity of an A-m-isometry, Honam Math. J. 37 (3) (2015) 281–285.
[10] S. Jung, Y. Kim, E. Ko, J.E. Lee, On (A, m)-expansive operators, Studia Math. 213 (1) (2012) 3–23.
[11] L. Suciu, Operators with expansive m-isometric liftings, Monatsh. Math. 198 (1) (2022) 165–187.
[12] R. Rabaoui, A. Saddi, On the orbit of an A-m-isometry, Ann. Math. Sil. 26 (2012) 75–91.
[13] B. P. Duggal, On (m, P)-expansive operators: products, perturbation by nilpotents, Drazin invertibility, Concrete Operators 8 (1) (2021) 158–175.
[14] O.A.M. Sid Ahmed, A. Saddi, A-m-isometric operators in semi-Hilbertian spaces, Linear Algebra Appl. 436 (2012) 3930–3942.
[15] O.A.M. Sid Ahmed, On A(m, p)-expansive and A(m, p)-hyperexpansive operators on Banach spaces—II, J. Math. Comput. Sci. 5 (2) (2015) 123–148.