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On a Generalization of IF-Rings

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ABSTRACT

In this paper, we present and investigate the notion of a right IREGF-ring as a proper generalization of the concept of a right IF-ring. A ring R is defined as a right IREGF-ring if every injective right R-module is Reg-flat. We provide numerous characterizations and explore various properties of right IREGF-rings.

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1. Introduction

Throughout this paper, all modules are unitary R-modules, where R is an associative ring with identity. The class of right R-modules is denoted by Mod-R and the class of left R-modules is denoted by R-Mod. We will denote the finitely generated by the symbol f.g. A submodule N of M_R is said to be pure if the tensor-induced map $N\otimes_R A \longrightarrow M\otimes_R A$ is injective, for any left R-module A [6]. According to [5], a left R-module M is called regular if all its submodules are pure. The sum of all regular submodules of $M \in$ Mod-R is denoted by Reg(M). For a submodule N of a module M, the notations $N \leq M$, (resp., $N \leq^p M$, $\mathrm{N} \leq^{\mathrm{reg}} M$, $\mathrm{N} \leq^{\mathrm{fgreg}} M$) means that N is a submodule (resp. pure, regular, finitely generated regular) submodule of M. The module $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, known as the character module. The symbol c.u.d.p. means closed under direct products. If $M \in$ Mod-R is pure in all modules that include it as a submodule, then M is called FP-injective [9]. A left R-module M is said to be injective, if for every left R-homomorphism $f: A \longrightarrow B$ (where A and B are left R-modules) and every left R-homomorphism $g: A \longrightarrow M$, there exists left R-homomorphism $h: B \longrightarrow M$ such that $g = hf$ [1]. In [8], the concept of Reg-N-injective modules was introduced as a proper generalization of injective modules, where a left R-module M is said to be Reg-N-injective

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(where $N \in R$-Mod) if every left R-homomorphism from any $A \leq^{reg} N$ into M extends to N. A module M is said to be Reg-injective, if M is Reg-R-injective. In [8], the concept of Reg-N-flat (resp., Reg-flat) modules were introduced as a proper generalization of N-flat (resp., flat) module. A module $M \in$ Mod-R is named Reg-N-flat (where $N \in R$-Mod) if for every $B \leq^{reg} N$, exactness holds for the sequence $0 \longrightarrow M \otimes_R B \longrightarrow M \otimes_R N$. A module M is said to be Reg-flat if it is Reg-R-flat. We use $(\text{Reg-}\mathcal{F})_R$ (resp., ${}_R(\text{Reg-}\mathbb{I})$) to denote the class of Reg-flat right R-modules (resp. the class of Reg-injective left R-modules). We use E(M) to denote the injective envelope of $M \in$ Mod-R. Colby in [2], introduced the concept of right IF-rings. If all injective right R-modules are flat, then ring R is referred to be a right IF-ring.

In this paper, we present and examine the idea of a right IREGF-ring as a proper generalization of a right IF-ring. It is said that a ring R is a right IREGF-ring if all injective right R-modules are Reg-flat. Many examples of IREGF-rings are given. Many characterizations of IREGF-rings are given, for example, we show in Proposition 2.3 that for a given ring R, the following assertions are all equivalent: (1) R is a right IREGF-ring; (2) M is embedded in a Reg-flat module, for any right R-module M; (3) A Reg-flat module contains M embedded in it; for each injective right R-module M; (4) E(M) is embedded in a Reg-flat module, for any $M \in$ Mod-R; (5) For any right R-module M, E(M) is a Reg-flat module. Also, we prove in Proposition 2.4 that a ring R is a right IREGF-ring $\Leftrightarrow$ All FP-injective right R-modules are Reg-flat $\Leftrightarrow$ If an FP-injective right module M has an FP-injective submodule N, then M / N is a Reg-flat module for that module$\Leftrightarrow$ The injective envelope of every finitely presented right R-module is Reg-flat $\Leftrightarrow$ For any free left R-module F, F* is Reg-flat. In Corollary 2.5, we prove that if under direct products, $(\text{Reg-}\mathcal{F})_R$ is closed, then R is a right IREGF-ring if and only if $i_K^*: ({}_R R)^* \longrightarrow K^*$ is an $(\text{Reg-}\mathcal{F})_R$-precover of K^*, for every f.g. regular left ideal K of R, where $i_K: K \longrightarrow R$ is the inclusion mapping if and only if $({}_R R)^*$ is a Reg-flat right R-module. Moreover, we prove in Proposition 2.10. that if a ring R is a right IREGF-ring and ${}_R(\text{Reg-}\mathbb{I})$ is closed under pure submodules, then ${}_R R$ is a Reg-injective left R-module. Finally, in Proposition 2.13, we prove that if $(\text{Reg-}\mathcal{F})_R$ is c.u.d.p. and there is a pure exact sequence $0 \longrightarrow {}_R R \rightarrow N \rightarrow L \longrightarrow 0$ with N^* is Reg-flat, then R is a right IREGF-ring.

2. Rings over which every injective module is Reg-flat

In this section, as a generalization of right IF-ring, we introduce the concept of right IREGF-ring.

Definition 2.1. A ring R is said to be a right IREGF-ring if every injective right R-module is Reg-flat.

Examples and Remarks 2.2.

(1) If $\text{Reg}({}_R R) = 0$, then R is a right IREGF-ring.

Proof. Since $\text{Reg}_R(R) = 0$, every right R-module is Reg-flat and so every injective right R-module is Reg-flat. Hence R is a right IREGF-ring.

(2) It is clear that every right IF-ring is a right IREGF-ring.

(3) The converse of (2) is not true in general, for Example: $\mathbb{Z}$ is an IREGF-ring (by (1) above). But $\mathbb{Z}$ is not an IF-ring, since $\mathbb{Q}/\mathbb{Z}$ is a injective $\mathbb{Z}$-module but it is not flat, by [4, Example (3), p. 401]. So, right IREGF-ring is a proper generalization of right IF-ring.

(4) Every regular ring is a right IREGF-ring, where a ring R is said to be regular if for each $a \in R$, we have $a = aba$ for some $b \in R$ [6, p.38].

Proof. Let R be a regular ring. By [6, Theorem 10.4.9, p.262], all right R-module is flat. Hence all injective right R-module is Reg-flat. So R is an IREGF-ring.

In the following proposition, we will introduce some characterizations of a right IREGF-ring.

Proposition 2.3. Let R be a ring. Then the following statements are equivalent:

(1) R is a right IREGF-ring.

(2) M is embedded in a Reg-flat module, for any right R-module M.

(3) M is embedded in a Reg-flat module, for any injective right R-module M.

(4) $E(M)$ is embedded in a Reg-flat module, for any right R-module M.

(5) $E(M)$ is a Reg-flat module, for any right R-module M.

Proof. $(\mathbf{2}) \Longrightarrow (\mathbf{3})$ and $(\mathbf{3}) \Longrightarrow (\mathbf{4})$ are obvious.

$(\mathbf{1}) \Longrightarrow (\mathbf{2})$. Let M be a right R-module. Thus, there is a right R-monomorphism $\alpha: M \longrightarrow E(M)$, where E($M$) is the injective envelope of M. By hypothesis, $E(M)$ is a Reg-flat module. Thus M is embedded in a Reg-flat module.

$(\mathbf{4}) \Longrightarrow (\mathbf{5})$. Let M be a right R-module. By hypothesis, there is a right R-monomorphism $\alpha: E(M) \longrightarrow L$, where L is Reg-flat and hence $E(M) \cong \alpha\big(E(M)\big)$. Since $E(M)$ is injective, by [1, Proposition 5.1.2, p.135], $\alpha(E(M))$ is a summand of L. Since L is Reg-flat, we have that $\alpha(E(M))$ is Reg-flat. Hence $E(M)$ is a Reg-flat module.

$(\mathbf{5}) \Longrightarrow (\mathbf{1})$. Let $M \in$ Mod-R with M is an injective right R-module. Since M is injective, then $M = E(M)$. By (5), $E(M)$ is a Reg-flat module and hence M is a Reg-flat module. Thus, R is a right IREGF-ring. □

We provide more characterizations of IREGF-ring in the following result.

Proposition 2.4. For a ring R, the next conditions are equivalent.

(1)R is a right IREGF-ring.

(2) $E(M)$ is Reg-flat, for any finitely presented right R-module M.

(3) F/K is a submodule of a f.g. free module, for any f.g. free right R-module F and a cyclic regular submodule K of F.

(4) All FP-injective right R-modules are Reg-flat.

(5)M/L is Reg-flat, for any FP-injective right module M and an $L \leq^p M$.

(6) M/L is Reg-flat, for any FP-injective right module M and any FP-injective submodule L of M.

(7) F^* is Reg-flat, for any free left R-module F.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$. Let $M \in$ Mod-R with M is finitely presented. Thus, $E(M)$ is a Reg-flat module, by Proposition 2.3.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$. Let $M = F/K$, where F is a f.g. free right R-module and $K \leq^{reg} F$ with K is cyclic. Thus, there is a monomorphism $\alpha: M \longrightarrow E(M)$. By (2), $E(M)$ is a Reg-flat module. Thus, α factors through a module, say F_1, is f.g. free and hence there is a $f \in \mathrm{Hom}_R(M, F_1)$, $g \in \mathrm{Hom}_R(F_1, E(M))$ such that $\alpha = gf$. Since α is a monomorphism, f is a monomorphism and hence $M \leq F_1$.

(3) $\Longrightarrow$ (4). Given a right R-homomorphism $\alpha: F/K \longrightarrow M$ with an FP-injective right R-module M, where F is any f.g. free right R-module and K is any cyclic regular submodule of F. By hypothesis, $F/K \leq F_1$, with F_1 is f.g. free. Thus, we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & F/K \arrow[r, "i"] \arrow[d, "\alpha"'] & F_1 \arrow[r, "\pi"] \arrow[dl, dashed, "\lambda"] & F_1/(F/K) \arrow[r] & 0 \\
& M & & &
\end{tikzcd}$$

where i (resp. π) is the inclusion (resp. natural epimorphism). Since F_1 is a f.g. free and F/K is a f.g. module, we have $F_1/(F/K)$ is a finitely presented module. Since M is FP-injective, it follows from [12, 35.1(c), p.297] that M is injective with respect to the sequence $0 \longrightarrow F/K \xrightarrow{i} F_1 \xrightarrow{\pi} F_1/(F/K) \longrightarrow 0$ and hence there is a homomorphism $\lambda: F_1 \longrightarrow M$ such that $\lambda i = \alpha$. Thus, α factors through a f.g. free module F_1 and hence M is a Reg-flat module.

$(\mathbf{4}) \Longrightarrow (\mathbf{5})$. Let M be an FP-injective right R-module and $L \leq^p M$. By hypothesis, M is Reg-flat. By [8, Corollary 2.5], M/L is a Reg-flat module.

$(\mathbf{5}) \Longrightarrow (\mathbf{6})$. Let L be an FP-injective submodule of an FP-injective right R-module M. By [9, p.561], $L \leq^p E(L)$ (resp. $M \leq^p E(M)$). Since $E(L)$ is an injective submodule of $E(M)$, we have from [1, Proposition 5.1.2, p.135] that $E(L)$ is a summand of $E(M)$ and hence $E(L) \leq^p E(M)$. Since $L \leq^p E(L)$, we have from [12, 33.3(1), p.276] that $L \leq^p E(M)$. Since $L \leq M \leq E(M)$, we have from [12, 33.3(2), p. 276] that $L \leq^p M$. By (5), M/L is Reg-flat.

$(\mathbf{6}) \Longrightarrow (\mathbf{7})$. If F is a free left R-module, we have from [7, Theorem, p. 239] that F^* is an injective right R-module. Since $< 0 >$ is an injective module, we have $< 0 >$ and F^* are FP-injective modules. By (6), $F^*/< 0 >$ is a Reg-flat module. Since $F^* \cong F^*/< 0 >$, we have that F^* is a Reg-flat module.

$(\mathbf{7}) \Longrightarrow (\mathbf{1})$. Let M be any injective right R-module. Thus, M^* is a left R-module. By [11, Proposition 2.5, p. 10], $M^* \cong F/K$, where F is a free left R-module. Thus, there is an epimorphism $\alpha: F \longrightarrow M^*$. By hypothesis, F^* is Reg-flat. By [3, Lemma 17-1.7(i), p. 361] $\alpha^*: M^{**} \longrightarrow F^*$ is a monomorphism. By [3, Corollary 17-1.5, p. 360], there a monomorphism $\beta: M \longrightarrow M^{**}$. Hence $\alpha^*\beta: M \longrightarrow F^*$ is a monomorphism. Since M is injective, $\alpha^*\beta(M)$ is a direct summand of F^* and hence $\alpha^*\beta(M)$ is a Reg-flat module. Since $M \cong \alpha^*\beta(M)$, we have that M is a Reg-flat module. Thus R is a right IREGF-ring. □

A homomorphism $\alpha: A \longrightarrow M$ is called $\mathcal{F}$-precover of a right R-module M where $\mathcal{F} \subseteq$ Mod-R and $A \in \mathcal{F}$ if, for any $g \in \mathrm{Hom}_R(K, M)$ such that $K \in \mathcal{F}$, there is a $h \in \mathrm{Hom}_R(L, A)$ with $\alpha h = g$ [1, p.244].

Corollary 2.5. If $(\text{Reg-}\mathcal{F})_R$ is c.u.d.p., then the next statements are equivalent:

(1) R is a right IREGF-ring.

(2) $({}_RR)^*$ is Reg-flat.

(3) $i_K^*: ({}_RR)^* \longrightarrow K^*$ is a $(\text{Reg-}\mathcal{F})_R$-precover of K^*, for every f.g. regular left ideal K of R, where $i_K: K \longrightarrow R$ is the inclusion mapping.

Proof. (1) ⟹ (2). Since ${}_RR$ is a free left R-module, $({}_RR)^*$ is Reg-flat (by Proposition 2.4).

(2) ⟹ (3). Let K be a f.g. regular left ideal of R. By [3, Lemma 17-1.7(ii), p. 361], $i_K^*: ({}_RR)^* \longrightarrow K^*$ is an epimorphism. By (2), $({}_RR)^*$ is Reg-flat. Let B is a Reg-flat right R-module, then the sequence $0 \longrightarrow B\otimes_R K \xrightarrow{I_B\otimes_R i_K} B\otimes_R R$ is exact. By [3, Lemma 17-1.7(ii), p. 361], the sequence $(B\otimes_R R)^* \xrightarrow{(I_B\otimes_R i_K)^*} (B\otimes_R K)^* \longrightarrow 0$ is exact. By [10, Theorem 2.75, p. 92], the sequence $\text{Hom}_R(B, ({}_RR)^*) \longrightarrow \text{Hom}_R(B, K^*) \longrightarrow 0$ is exact. Thus $i_K^*: ({}_RR)^* \longrightarrow K^*$ is a $(\text{Reg-}\mathcal{F})_R$-precover of K^*.

(3) ⟹ (1). Suppose that $i_K^*: ({}_RR)^* \longrightarrow K^*$ is a $(\text{Reg-}\mathcal{F})_R$-precover of K^*, for every $K \leq^{fgreg} {}_RR$. Thus, $({}_RR)^*$ is a Reg-flat right R-module. Let F be a free left R-module. Thus $F \cong {}_RR^{(I)}$, for some index set I. Thus $F^* \cong ({}_RR^{(I)})^* \cong (({}_RR)^*)^I$ by [7, Lemma 4.3.3, p. 86]. By hypothesis, $(({}_RR)^*)^I$ is Reg-flat and hence F^* is Reg- flat. By Proposition 2.4, R is a right IREGF-ring. □

It is essay to prove the following lemma:

Lemma 2.6. The class $(\text{Reg-}\mathbb{F})_\text{R}$ is c.u.d.p. if and only if $({}_\text{R}(\text{Reg-}\mathbb{I}))^* \subseteq (\text{Reg-}\mathbb{F})_\text{R}$.

Corollary 2.7. If ${}_RR$ is a Reg-injective left R-module and $(\text{Reg-}\mathcal{F})_R$ is c.u.d.p., then R is a right IREGF-ring.

Proof. Let ${}_RR$ be a Reg-injective left R-module. Since $(\text{Reg-}\mathcal{F})_R$ is c.u.d.p. (by hypothesis), $({}_\text{R}\text{R})^*$ is Reg-flat (by Lemma 2.6). By Corollary 2.5, R is a right IREGF-ring. □

Let $N \in R$-Mod and $M \in$ Mod-R. For any index set I, define $\varphi_N: M^I\otimes_R N \longrightarrow (M\otimes_R N)^I$ by $\varphi((m_\alpha)_{\alpha\in I}\otimes_R n) = (m_\alpha\otimes_R n)_{\alpha\in I}$, for any $n \in N$, $(m_\alpha)_{\alpha\in I} \in M^I$. Thus φ_N is a natural homomorphism, by [2, p. 241].

Proposition 2.8. Let U_R be a Reg-flat module. Then the following statements are equivalent:

(1) U^S is a Reg-flat right R-module, for every index set S.

(2) For any index set S and $K \leq^{fgreg} {}_RR$, the natural homomorphism $\varphi_K: U^S\otimes_R K \longrightarrow (U\otimes_R K)^S$ is an isomorphism.

Proof. (1) ⟹ (2). Let $K \leq^{fgreg} {}_RR$. Since U and U^S are Reg-flat right R-modules, we have from [8, Corollary 2.7] that the sequences: $0 \longrightarrow U \otimes_R K \xrightarrow{I_U\otimes_R i_K} U \otimes_R R \xrightarrow{I_U\otimes_R \pi_K} U \otimes_R (R/K) \longrightarrow 0$ and $0 \longrightarrow U^S \otimes_R K \xrightarrow{I_{U^S}\otimes_R i_K} U^S \otimes_R R \xrightarrow{I_{U^S}\otimes_R \pi_K} U^S \otimes_R (R/K) \longrightarrow 0$ are exact, where I_U, i_K and π_K are the identity homomorphism, the inclusion mapping and natural epimorphism, respectively. Thus, we get the next commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & U^S\otimes_R K \arrow[r, "I_{U^S}\otimes_R i_K"] \arrow[d, "\varphi_K"] & U^S\otimes_R R \arrow[r, "I_{U^S}\otimes_R \pi_K"] \arrow[d, "\varphi_R"] & U^S\otimes_R (R/K) \arrow[r] \arrow[d, "\varphi_{(R/K)}"] & 0 \\
0 \arrow[r] & (U\otimes_R K)^S \arrow[r, "\prod_S(I_U\otimes_R i_K)"] & (U\otimes_R R)^S \arrow[r, "\prod_S(I_U\otimes_R \pi_K)"] & (U\otimes_R (R/K))^S \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since ${}_RR$ and R/K are finitely presented left R-modules, it follows from [1, Proposition 5.3.15, p. 161] that φ_R and $\varphi_{(R/K)}$ are isomorphisms. By [1, Problem 12(b), p.88], φ_K is an epimorphism. Since $I_{U^S}\otimes i_K$ and φ_R are monomorphism, we have φ_K is a monomorphism. Thus, φ_K is an isomorphism.

(2) ⟹ (1). Let K be any f.g. regular left ideal of R. Thus, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & U^S\otimes_R K \arrow[r, "I_{U^S}\otimes_R i_K"] \arrow[d, "\varphi_K"] & U^S\otimes_R R \arrow[d, "\varphi_R"] \\
0 \arrow[r] & (U\otimes_R K)^S \arrow[r, "(I_U\otimes_R i_K)^S"] & (U\otimes_R R)^S
\end{tikzcd}$$

Since U is Reg-flat, we have from [8, Corollary 2.7] that the sequence $0 \to U\otimes_R K \xrightarrow{I_U\otimes i_K} U\otimes_R R$ is exact and hence the sequence: $0 \to (U\otimes_R K)^S \xrightarrow{(I_U\otimes_R i_K)^S} (U\otimes_R R)^S$ is exact. Since ${}_RR$ is a finitely presented left R-module, we have from [1, Proposition 5.3.15, p.161] that φ_R is an isomorphism. By hypothesis, φ_K is an isomorphism. Thus, the sequence $0 \to U^S\otimes_R K \xrightarrow{I_{U^S}\otimes_R i_K} U^S\otimes_R R$ is exact and so for any index set S, U^S is Reg-flat (by [8, Corollary 2.7]). □

Proposition 2.9. If R is a right IREGF-ring, then for every $K \leq^{fgreg} {}_RR$ and for any index set I, the natural homomorphism $\varphi_K: (({}_RR)^*)^I\otimes_R K \longrightarrow (({}_RR)^*\otimes_R K)^I$ is an isomorphism.

Proof. Suppose that R is a right IREGF-ring. By Proposition 2.4, $({}_RR^{(I)})^*$ is a Reg-flat right R-module, for any index set I. Since $(({}_RR)^*)^I \cong ({}_RR^{(I)})^*$ (by [6, Lemma 4.3.3, p.86]), it follows that $(({}_RR)^*)^I$ is a Reg-flat right R-module. Hence, the natural homomorphism $\varphi_K: (({}_RR)^*)^I\otimes_R K \longrightarrow (({}_RR)^*\otimes_R K)^I$ is an isomorphism, for any index set I, by Proposition 2.8. □

The following proposition discuss the converse of Proposition 2.9.

Proposition 2.10. If the right R-module $({}_RR)^*$ is Reg-flat and the natural homomorphism $\varphi_K: (({}_RR)^*)^I\otimes_R K \longrightarrow (({}_RR)^*\otimes_R K)^I$ is an isomorphism, for any $K \leq^{fgreg} {}_RR$ and for an index set I, then R is a right IREGF-ring.

Proof. Suppose that F is a free left R-module, thus $F \cong ({}_RR)^{(I)}$, for an index set I. By hypothesis, the natural homomorphism $\varphi_K: (({}_RR)^*)^I\otimes_R K \longrightarrow (({}_RR)^*\otimes_R K)^I$ is an isomorphism, for each $K \leq^{fgreg} {}_RR$. Since $({}_RR)^*$ is Reg-flat, $(({}_RR)^*)^I$ is Reg-flat (by Proposition 2.8). Since $F^* \cong (({}_RR)^{(I)})^* \cong (({}_RR)^*)^I$ (by [6, Lemma 4.3.3, p. 86]), it follows that F^* is a Reg-flat right R-module. By Proposition 2.4, R is a right IREGF-ring. □

Proposition 2.11. If a ring R is a right IREGF-ring and the class of Reg-injective left R-modules ${}_R$(Reg-$\mathbb{I}$) is closed under pure submodules, then ${}_RR$ is a Reg-injective left R-module.

Proof. Suppose that R is a right IREGF-ring. By Proposition 2.4, $({}_RR)^*$ is a Reg-flat right R-module. By [8, Theorem 2.3], $({}_RR)^{**}$ is a Reg-injective left R-module. Since ${}_RR \leq^p ({}_RR)^{**}$ and ${}_R$(Reg-$\mathbb{I}$) is closed under pure submodule, we have ${}_RR$ is Reg-injective. □

Proposition 2.12. If $(\text{Reg-}\mathcal{F})_R$ is c.u.d.p. and there is a pure exact sequence $0 \longrightarrow {}_RR \overset{\alpha}{\longrightarrow} A \overset{\beta}{\longrightarrow} K \longrightarrow 0$ with A^* is Reg-flat, then R is a right IREGF-ring.

Proof. Let $0 \longrightarrow {}_RR \overset{\alpha}{\longrightarrow} A \overset{\beta}{\longrightarrow} K \longrightarrow 0$ be a pure exact sequence, with A^* is Reg-flat. By [3, 18-2.13, p. 378], the sequence $0 \longrightarrow K^* \overset{\beta^*}{\longrightarrow} A^* \overset{\alpha^*}{\longrightarrow} ({}_RR)^* \longrightarrow 0$ is split. By [3, 1-4.4, p.11], $A^* = \beta^*(K^*) \oplus D$, for some $D \leq A^*$ with $D \cong ({}_RR)^*$. Since A^* is Reg-flat (by hypothesis), we get from [8, Corollary 2.5] that D is Reg-flat. Since $D \cong ({}_RR)^*$, we have $({}_RR)^*$ is Reg-flat. By Corollary 2.5, R is a right IREGF-ring. □

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