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Binomial Transform Technique for Solving Second Linear Difference Equations

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ABSTRACT

Our main goal in this research is to study the binomial transformation to solve second-order linear difference equations. It has been concluded that the binomial transformation method is an easy and simple way to solve second-order difference equations after the initial conditions are met.

The research recommended using the binomial transformation to solve partial difference equations.

MSC..

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1. Introduction and main result.

The binomial transformation is an important and simple method for solving linear difference equations in general. This was noted in previous research presented by researcher Asma Salih Jasim[1] on solving first-order linear equations using the binomial transformation method. The results were completely correct. Therefore, we will generalize the method to solving second-order difference equations.

Difference equations are models of the world around us. From clocks to computers to chromosomes, dealing with discrete objects in discrete steps is a common topic. They are the discrete equivalent of differential equations[21] and arise whenever the independent variables can only have discrete values [18,19,20]. Difference equations are used in real life and in various sciences (population models, genetics, psychology, economics, sociology, random time series, combinatorial analysis, queuing problems, number theory, geometry, radiation quantum and electrical networks).

Second-order linear difference equations appear in mathematics and science, both pure and applied. For example, homogeneous equations with constant coefficients have given rise to a large number of products in terms of multiple series, such as Horadam, Fibonacci, Lucas, Pell, Jacobsthal, etc. Furthermore, general second-order homogeneous linear difference equations also appear to be relevant to combinatorial problems, including recursions that give rise to many well-known combinatorial numbers, such as Fine, central Delannoy, Schröder, Motzkin, directed number counting, and scrambled numbers, which play an important role in enumerative

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combinatorics, and compute several combinatorial objects, see examples and references to the origin and meaning of these series [13,14,15,16,17].

The binomial transformation is a discrete transformation of one sequence into another, with many interesting applications in combinatorics and analysis. The theory of binomial transformations belongs to the growing field of discrete mathematics. The material in this book is useful for researchers interested in enumerative combinatorics, special numbers, and classical analysis. The book can also serve as a valuable reference book. It can be used as a lecture note for courses on binomial identities, binomial transformations, and Euler series transformations. The binomial transformation leads to various combinatorial and analytic identities with binomial coefficients. Many interesting identities can be written as binomial transformations and vice versa.

The binomial transformation is useful in many applications, both in applied and pure mathematics. A generalization of the binomial transformation was first proposed by Prodinger (1994). The binomial transformation is a discrete transformation of one series into another and has many interesting applications in combinatorics and analysis. This transformation is very useful for researchers interested in numerical aggregation, special numbers, and classical analysis. The binomial transformation is closely related to the Euler transformation. The binomial transformation is often used to speed up a series or, conversely, to simplify the structure of the hypergeometric terms of a series. This transformation has the elegant property of being self-invertible. This transformation is one of the most common transfers. For more information on this transformation, see [4,5,6,7,8,9,10,11].

2 - Second-Order Linear Difference Equations [12]

We now introduce the theory of second-order linear difference equations. Unlike the first-order case, there is no general formula that provides solutions to all such equations. Additional conditions must be imposed to obtain the general formula.

The general form of a second order linear difference equation is

$$x(n+2)+b(n)x(n+1)+c(n)x(n)=0\,,\ \ n\in N_0.$$

Here $b(n), c(n), f(n)$are given sequences. If $f(n)=0$for all n, then the equation is homogeneous, viz.

$$x(n+2)+b(n)x(n+1)+c(n)x(n)=0\,,\ \ n\in N_0.$$

If we define the operator

$$(Lx)(n)=x(n+2)+b(n)x(n+1)+c(n)x(n).$$

then $L\colon S(N_0)\to S(N_0)$is a linear operator,

3. Binomial transform

From the basic knowledge of the binomial coefficients, that is:

$$\binom{m}{j}=\frac{m(m-1)\dots(m-j+1)}{j!}$$

and throughout the use the agreement was held that:

$$\binom{m}{j}=0\ if\ j<0.$$

Now, the binomial transform is defined as:

For given a sequence $\{O_j\}$, $j=0,1,2,\ldots$, its binomial transform is the new sequence $\{B(O_n)\}$, $n=0,1,2\ldots$ and can be generated by the formula;[3]

$$B(O_n) \equiv b(O_n) = \sum_{j=0}^{n} \binom{m}{j} (-1)^{j-1} O_j$$

The inverse binomial transform was defined as:

$$O_n = \sum_{j=0}^{n} \binom{m}{j} (-1)^{j} B(O_n)$$

The following properties of binomial transform are listed in Table 1.

Table-1 : The Properties of Binomial Transform[1]

N	O	$\boldsymbol{B(O_n)}$
1.	$O_j = 1$	$B(O_n) = \sum_{j=0}^{n} \binom{m}{j} (-1)^{j-1} 1$ $= \binom{m}{0}(-1)^{-1} + \binom{m}{1}(-1)^{0} + \binom{m}{2}(-1)^{1} + \binom{m}{3}(-1)^{2} + \cdots$ $= -1 + n - \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots$
2.	$O_j = a^j$	$B(O_n) = \sum_{j=0}^{n} \binom{m}{j} (-1)^{j-1} a^j$ $= \binom{m}{0}(-1)^{-1}a^0 + \binom{m}{1}(-1)^{0}a^1 + \binom{m}{2}(-1)^{1}a^2 + \binom{m}{3}(-1)^{2}a^3 + \cdots$ $= -a^0 + na^1 - a^2\frac{n(n-1)}{2!} + a^3\frac{n(n-1)(n-2)}{3!} + \cdots$
3.	$O_j = j$	$B(O_n) = \sum_{j=0}^{n} \binom{m}{j} (-1)^{j-1} \mathrm{j}$ $= \binom{m}{0}(-1)^{-1}0 + \binom{m}{1}(-1)^{0}1 + \binom{m}{2}(-1)^{1}2 + \binom{m}{3}(-1)^{2}3 + \cdots$ $= 0 + n - 2\frac{n(n-1)}{2!} + 3\frac{n(n-1)(n-2)}{3!} + \cdots$ $n - n(n-1) + \cdots$

4.	O_{j+1}	$B(O_{n+1}) = \sum_{j=0}^{n} \binom{m}{j+1} (-1)^{j} O_{j+1}$ $= \binom{m}{1} (-1)^0 O_1$ $+ \binom{m}{2} (-1)^1 O_2$ $+ \binom{m}{3} (-1)^2 O_3$
5.	O_{j+2}	$B(O_{n+2}) = \sum_{j=0}^{n} \binom{m}{j+2} (-1)^{j+1} O_{j+2}$ $= \binom{m}{2} (-1)^1 O_2$ $+ \binom{m}{3} (-1)^2 O_3$
6.	O_{j-1}	$B(O_{n-1}) = \sum_{j=0}^{n} \binom{m}{j-1} (-1)^{j-2} O_{j-1}$ $= \binom{m}{0} (-1)^{-1} O_0$ $+ \binom{m}{1} (-1)^0 O_1$
7.	O_{j-2}	$B(O_{n-2}) = \sum_{j=0}^{n} \binom{m}{j-2} (-1)^{j-3} O_{j-2}$ $= \binom{m}{-1} (-1)^{-2} O_{-1}$ $+ \binom{m}{0} (-1)^{-1} O_0$

4. Application

Example 1 solve the first order difference equation by The binomial transform

$O_{j+2} - O_j = 0 \qquad (1),$

with initial boundary

$O(0) = 2, \quad O(1) = 0, \quad O(2) = 2$

Now, we will take the binomial transform of O_1 equation (1)

$$B(O_m) = \sum_{j=1}^{n} \binom{m}{j} (-1)^{j-1} O_j \quad = \binom{m}{1} (-1)^0 O_1 + \binom{m}{2} (-1)^1 O_2 + \cdots$$

$$= -2\frac{m(m-1)}{2!}$$

$$= -m(m-1)$$

$$= -m^2 + m \qquad (2)$$

Now, we will taking the inverse (2)

$$O_m = \sum_{j=1}^{n} \binom{m}{j} (-1)^j B(O_m)$$

$$= \sum_{j=1}^{n} \binom{m}{j} (-1)^j [-j^2 + j \quad]$$

$$\binom{m}{1} (-1)^0 [-1+1] + \binom{m}{2} (-1)^2 [-(2)^2 + 2]$$

$$= 0 - \frac{m(m-1)}{2!}[-2]$$

$$= -m(m-1)$$

We take the square root of the coefficient m without any sign.

$$m = \sqrt{1} = \mp 1$$

$$\therefore O_m = 1^j + (-1)^j$$

Example 2 solve the first order difference equation by The binomial transform

$O_{j+2} - 4O_j = 0 \qquad (1),$

with initial boundary

$O(0) = 2, \quad O(1) = 0, \quad O(2) = 8$

Now, we will take the binomial transform of O_1 equation (1)

$$B(O_m) = \sum_{j=1}^{n} \binom{m}{j} (-1)^{j-1} O_j \quad = \binom{m}{1} (-1)^0 O_1 + \binom{m}{2} (-1)^1 O_2 + \cdots$$

$$= -8\frac{m(m-1)}{2!}$$

$$= -4m(m-1)$$

$$= -4m^2 + 4m \qquad (2)$$

Now, we will taking the inverse (2)

$$O_m = \sum_{j=1}^{n} \binom{m}{j} (-1)^j B(O_m)$$

$$= \sum_{j=1}^{n} \binom{m}{j} (-1)^j [-4j^2 + 4j \quad]$$

$$\binom{m}{1} (-1)^0 [-4+4] + \binom{m}{2} (-1)^2 [-(4)^2 + 4]$$

$$= 0 - \frac{m(m-1)}{2!}[-8]$$

$$= -4m(m-1)$$

We take the square root of the coefficient m without any sign.

$$m = \sqrt{4} = \mp 2$$

$$\therefore O_m = 2^j + (-2)^j$$

Example 3 solve the first order difference equation by The binomial transform

$O_{j+2} - 16O_j = 0 \qquad (1),$

with initial boundary

$O(0) = 2, \quad O(1) = 0, \quad O(2) = 32$

Now, we will take the binomial transform of O_1 equation (1)

$$B(O_m) = \sum_{j=1}^{n} \binom{m}{j} (-1)^{j-1} O_j \quad = \binom{m}{1} (-1)^0 O_1 + \binom{m}{2} (-1)^1 O_2 + \cdots$$

$$= -32 \frac{m(m-1)}{2!}$$

$$= -16m(m-1)$$

$$= -16m^2 + 16m \qquad (2)$$

Now, we will taking the inverse (2)

$$O_m = \sum_{j=1}^{n} \binom{m}{j} (-1)^j B(O_m)$$

$$= \sum_{j=1}^{n} \binom{m}{j} (-1)^j [-16j^2 + 16j \ ]$$

$$\binom{m}{1} (-1)^0 [-16 + 16] + \binom{m}{2} (-1)^2 [-(16)^2 + 16]$$

$$= 0 - \frac{m(m-1)}{2!} [-32]$$

$$= -16m(m-1)$$

We take the square root of the coefficient m without any sign.

$$m = \sqrt{16} = \mp 4$$

$$\therefore O_m = 4^j + (-4)^j$$

Conclusions

1- Second-order linear difference equations cannot be solved unless the initial conditions are met.

2. We concluded that the binomial transformation is simple and important for solving second-order linear difference equations in a simpler way From the Laplace transform.

Recommendations

1. It is recommended to use the binomial transformation to solve a system of nonlinear difference equations.

2. I RECOMMEND USING THE BINOMIAL TRANSFORMATION TO SOLVE THE SYSTEM OF PARTIAL DIFFERENCE EQUATIONS.

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