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Triple Zero on Weakly Nearly Primary 2-Absorbing submodule

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ABSTRACT

In this paper, we introduce and investigate the concept of weakly nearly primary 2-absorbing triple zero submodule. A formal definition is proposed, and several theorems are established to describe the algebraic behavior of this type of submodule. The focus is placed on deriving theoretical results without the use of illustrative examples. Moreover, properties of weakly nearly primary 2-absorbing triple zero submodule are explored in relation to different module structures.

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1. Introduction

Many authors have studied various generalizations of prime and primary submodule. Among these generalizations is the concept of 2-absorbing submodule. introduced by Badawi [2], which has opened the door for further developments such as primary 2-absorbing submodule, nearly primary 2-absorbing submodule and more recently, weakly nearly primary 2-absorbing submodule.

The following are the fundamental definitions upon which our study is based:

*A proper submodule T of an R-module D is said to be a 2-absorbing submodule if $fhd \in T$ ($0 \neq fhd \in T$) for some $f, h \in R$, $d \in D$, then $fd \in T$ or $hd \in T$ or $fh \in [T:_R D]$ [3].

*A proper submodule T of an R-module D is called a primary 2-absorbing submodule if $fhd \in T$,for $f, h \in R, d \in D$, implies that either $fd \in rad_D(T)$ or $hd \in rad_D(T)$ or $fh \in [T + J(D):_R D]$ [3].

*Let R be a ring and D a left R-module. A submodule T of D is called maximal or cosimple if the quotient D/T is a simple module. The radical of the module D is the intersection of all maximal submodules of D,

rad(D)=∩{T |T is a maximal submodule of D} [3].

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*Recalling that the Jacobson radical of an R-module D is defined to be the intersection of all maximal submodule of D, denoted by J(D) or defined to be the sum of all small submodule of D [3].

*A proper submodule T of an R-module D is called a nearly primary 2-absorbing submodule if $fhd \in T$, for $f, h \in R$, $d \in D$, implies that either $fd \in rad_D(T) + J(D)$ or $hd \in rad_D(T) + J(D)$ or $fh \in [T + J(D):_R D]$ [4].

*A proper submodule T of an R-module D is said to be a weakly nearly primary 2-absorbing submodule if $0 \neq fhd \in T$, for $f, h \in R, d \in D$, implies that either $fd \in rad_D(T) + J(D)$ or $hd \in rad_D(T) + J(D)$ or $fh \in [T + J(D):_R D]$ [5].

Based on these concepts, this paper introduces and investigates a new type of submodule. called the weakly nearly primary 2-absorbing triple zero submodule. This work focuses on formally defining this new class of submodule. and presenting a series of theorems and propositions that describe its algebraic behavior. No examples are provided, and the emphasis is placed entirely on theoretical analysis.

A submodule, $T \subseteq D$ is said to be a weakly nearly primary 2-absorbing triple zero sub module if for some $f, h \in R$, $d \in D$, the condition fhd=0 implies that one of several specific annihilation conditions holds, such as:

$fd \notin rad_D(T) + J(D)$, $hd \notin rad_D(T) + J(D)$ and $fh \notin [T + J(D):_R D]$.

This work focuses on formally defining this new type of submodule and presenting a series of theorems and propositions that describe its algebraic behavior. No examples are provided, and the emphasis is placed entirely on theoretical analysis.

2. Weakly Nearly Primary 2-Absorbing Triple Zero

Definition 1

Let T be a weakly nearly primary2-absorbing of an R-module D, and $f, h \in R$, $d \in D$.We say that (f, h, d) is a WNP2-Absorbing triple zero of T if $fhd = 0$, $fd \notin rad_D(T) + J(D)$, $hd \notin rad_D(T) + J(D)$ and $fh \notin [T + J(D):_R D]$.

Example: Suppose that $R = Z_3$, $D = Z_{81}$, and the submodule T=(0), (9,9,1) is weakly nearly primary2-absorbing triple zero of T since T is weakly nearly primary2-absorbing by definition with 9.9.1=0 ∈ T, 9.1=9$\notin rad_D(T) + J(D) = 0 + (27) = (27)$ and $9.9 = 81 \notin [T + J(D):_R D] = 27Z_3$.

Proposition 2

If T is a WNP2-Absorbing submodule with (f, h, d) is a WNP2-Absorbing triple zero of T ther exists $f, h \in R$, $d \in D$. then $fhT = (0)$.

Proof:

Suppose that $fhT \neq (0)$, then $fht \neq 0$ for some $t \in T$. Since (f, h, d) is a WNP2-Absorbing triple zero of T then $fhd = 0$, $fd \notin rad_D (T) + J(D)$, $hd \notin rad_D(T) + J(D)$ and $fhD \notin T + J(D)$. Since $0 \neq fht \in T$ and T is a WNP-2 AB submodule of D, $fhD \subseteq T + J(D)$, then either $ft \in rad_D (T) + J(D)$ or $ht \in rad_D (T) + J(D)$. Now , $0 \neq fh(d + t) = fhd + fht = fht \in T$, and $fh \notin [T + J (D):_R D]$, then either $f(d + t) = fd + ft \in rad_D (T) + J(D)$ or $h(d + t) = hd + ht \in rad_D (T) + J(D)$, $ft \in rad_D (T) + J(D)$ we get $fd \in rad_D (T) + J(D)$ a contradiction . If $hd + ht \in rad_D (T) + J(D)$, $ht \in rad_D (T) + J(D)$ we get $hd \in rad_D (T) + J(D)$ a contradiction .Hence $fhT = 0$.

Proposition 3

If T is a WNP2-Absorbing submodule of D with (f, h, d) is WNP2-Absorbing triple zero there exists $f, h \in R$, $d \in D$ then $[T:_R D]hd = (0)$.

Proof:

Suppose that $[T:_R D]hd \neq (0)$, then $bhd \neq (0)$, for some $b \in [T:_R D]$. Since (f, h, d) is WNP2-Absorbing triple zero of T, $fhd = 0$, $fd \notin rad_D(T) + J(D)$, $hd \notin rad_D(T) + J(D)$ and $fhD \nsubseteq T + J(D)$. Since $0 \neq bhd \in T$, T is a WNP2-Absorbing submodule of D, then either $bd \in rad_D(T) + J(D)$ or $hd \in rad_D(T) + J(D)$ or $bh \in [T + J(D):_R D]$. Now, $0 \neq (f + b)hd = fhd + bhd = bhd \in T$ and T is a WNP-2AB sub mod. of D, then either $(f + b)d = fd + bd \in rad_D(T) + J(D)$ or $hd \in rad_D(T) + J(D)$ or $(f + b)h \in [T + J(D):_R D]$. Since $bd \in rad_D(T) + J(D)$ and if $fd + bd \in rad_D(T) + J(D)$, it follows that $fd \in rad_D(T) + J(D)$ a contradiction.

If $(f + b)h = fh + bh \in [T + J(D):_R D]$, $bh \in [T + J(D):_R D]$ then $fh \in [T + J(D):_R D]$ a contradiction. Thus $[T:_R D]hd = (0)$.

Corollary 4

If T is a WNP2-Absorbing submodule of D, with (f, h, d) is a WNP2-Absorbing triple zero of T, there exists $f, h \in R$, $d \in D$, then $[T:_R D]fd = (0)$.

Proposition 5

If is a WNP2-Absorbing submodule of D, with (f, h, d) is a WNP2-Absorbing triple zero of T ,there exists $f, h \in R$, $d \in D$, then $f[T:_R D]d = (0)$ and $h[T:_R D]d = (0)$

Proof:

Suppose that $f[T:_R D]d \neq (0)$, then $fmd \neq 0$ there exists $m \in [T:_R D]$. Since (f, h, d) is WNP2 -Absorbing triple zero of T, $fhd = 0$, $fd \notin rad_D(T) + J(D)$ or $hd \notin rad_D(T) + J(D)$ and $fhD \nsubseteq T + J(D)$. Now, since $0 \neq fmd \in T$, T is a WNP2-Absorbing submodule of D then either $fd \in rad_D(T) + J(D)$ or $hd \in rad_D(T) + J(D)$ or $(h + m)d = hd + md \in rad_D(T) + J(D)$ or $f(h + m) = fh + fm \in [T + J(D):_R D]$. Hence $fd \in rad_D(T) + J(D)$ or $hd \in rad_D(T) + J(D)$ which is a contradicition. If $(fh + fm)D \subseteq T + J(D)$, $fhD \subseteq T + J(D)$ a contradiction. Thus $f[T:_R D]d = (0)$.

Similarly, way can show that $h[T:_R D]d = (0)$.

Proposition 6

If T is a WNP2-Absorbing submodule of D with (f, h, d) is a WNP2-Absorbing triple zero of T there exists $f, h \in R$, $d \in D$, then $[T:_R D]hT = (0)$ and $[T:_R D]fT = (0)$.

Proof:

Suppose that $[T:_R D]hT \neq (0)$, then $mht \neq (0)$, for some $m \in [T:_R D]$, $t \in T$. Since (f, h, d) is a WNP2-Absorbing triple zero of T, $fhd = 0$, $fd \notin rad_D(T) + J(D)$, $hd \notin rad_D(T) + J(D)$ and $fhD \nsubseteq T + J(D)$.Now, $0 \neq mht \in T$ and T is a WNP2-Absorbing submodule of D then either $mt \in rad_D(T) + J(D)$ or $ht \in rad_D(T) + J(D)$ or $mhD \subseteq [T + J(D):_R D]$.Now,$(f + m)h(t + d) = fht + fhd + mht + mhd \neq 0 \in T$(because $fhd = 0$) and from Proposition 2 $fht = 0$ and from Proposition5, $fhd = 0$. That is $0 \neq (f + m)h(t + d) \in T$ and T is a WNP2-Absorbing submodule of D, then either $(f + m)(t + d) = ft + fd + mt + md \in rad_D(T) + J(D)$, $fd \in rad_D(T) + J(D)$ a contradiction, or $h(t + d) = ht_1 + hd \in rad_D(T) + J(D)$, then $fhD \subseteq T + J(D)$ a contradiction, thus $[T:_R D]hT = (0)$.

Similarly, we can prove that $[T:_R D]fT = (0)$.

Corollary to Proposition (6). 7

If T is a WNP2-Absorbing submodule of an R-module D, with (f, h, d) is a WNP-2AB triple zero of T there exists $f, h \in R$, $d \in D$ then $f [T:_R D]T = (0)$ and $h[T:_R D]T = (0)$.

Proof:

Suppose that $f[T:_R D]T \neq (0)$,then $fmt \neq 0$ there exists $m \in [T:_R D]$ and some $t \in T$ .Hence, the proof follows as in Proposition 5.

Similarly, for $h[T:_R D]T = (0)$.

Proposition 8

If T is a WNP2-Absorbing submodule of D with (f, h, d) is a WNP2-Absorbing triple zero of T there exists $f, h \in R, d \in D$, then $[T:_R D][T:_R D]d = (0)$.

Proof:

Suppose that $[T:_R D][T:_R D]d \neq 0$, then $0 \neq tkd$ there exists $f, h \in [T:_R D]$.Since (f, h, d) is a WNP2-Absorbing triple zero of T, then $fhd = 0$, $fd \notin rad_D(T) + J(D)$or $hd \notin rad_D(T) + J(T)$ and $fhD \nsubseteq T + J(D)$Now,$0 \neq tkd \in T$ and T is a WNP2-Absorbing submodule of D, then either $td \in rad_D(T) + J(D)$or $kd \in rad_D(T) + J(D)$ or $tkD \subseteq T + J(D)$.

Now, $0 \neq (f + t)(h + k)d = fhd + fkd + thd + tkd = tkd \in T$ (since $fhd = 0$, $fkd = 0$, $thd = 0$ by Proposition 5 .But T is WNP2-Absorbing submodule of D ,then either $(f + t)d = fd + td \in rad_D(T) + J(D)$,then $fd \in rad_D(T) + J(D)$ a contradiction, or $(h + k)d = hd + kd \in rad_D(T) + J(D)$, then $hd \in rad_D(T) + J(D)$ a contradiction. Or $(f + t)(h + k) = (fh + fk + th + tk)$.

then $fhD \subseteq T + J(D)$ a contradiction. Hence, $[T:_R D][T:_R D]d = (0)$.

Corollary to Proposition (8) 9

If T is WNP2-Absorbing submodule of D with (f, h, d) is WNP2-Absorbing triple zero T there exists $f, h \in R, d \in D$,then $[T:_R D][T:_R D]T = (0)$.

Proof:

Suppose that $[T:_R D][T:_R D]T \neq (0)$, then $mkt \neq 0$ there exists $m, k \in [T:_R D]$ and some $t \in T$. Then it follows by (Proposition 2 and Proposition 8) $0 \neq mkt = (m + f)(k + h)(t + d) \in T$ and T is WNP2- Absorbing submodule of D, either $(m + f)(t + d) = md + mt + ft + fd \in rad_D(T) + J(D)$, then $fd \in rad_D(T) + J(D)$ a contradiction or $(m + f)(k + h) = (mk + mh + fk + fh)D \subseteq T + J(D)$, then $fhD \subseteq T + J(D)$ a contradiction. Hence $[T:_R D][T:_R D]T = (0)$.

Proposition 10

Let T be a WNP2-Absorbing of D and suppose $fhA \subseteq T$ there exists $f, h \in R$, and some submodule A of $D \ni (f, h, m)$ is not a WNP2-Absorbing triple zero of T for every $m \in A$. If $fhD \nsubseteq T + J(D)$ then $fA \subseteq rad_D(T) + J(D)$ or $hA \subseteq rad_D(T) + J(D)$.

Proof:

Suppose that (f, h, m) is not a WNP2-Absorbing triple zero of T for every $m \in A$ and suppose that $fA \nsubseteq rad_D(T) + J(D)$ and $hA \nsubseteq rad_D(T) + J(D)$. Then $fa_1 \notin rad_D(T) + J(D)$ or $ha_2 \notin rad_D(T) + J(D)$ there exists $a_1, a_2 \in A$. If $0 \neq fha_1 \in T$ with $fhD \nsubseteq T + J(D)$ and $fa_1 \notin rad_D(T) + J(D)$ then $ha_2 \in rad_D(T) + J(D)$ (because T is WNP2-Absorbing). If $fha_1 = 0$ and $fa_1 \notin rad_D(T) + J(D)$, $fhD \nsubseteq T + J(D)$ and (f, h, a) is not WNP2-Absorbing triple zero of T, we get $ha_2 \in rad_D(T) + J(D)$. By simmilar argument, since (f, h, a_2) is not a WNP2-Absorbing triple zero of T and $fhD \nsubseteq T + J(D)$, $ha_2 \nsubseteq rad_D(T) + J(D)$ we get $fa_1 \notin rad_D(T) + J(D)$. Now, $fh(a_1 + a_2) \in T$ and $(f, h, a_1 + a_2)$ is not a WNP2-Absorbing triple zero of T and $fhD \nsubseteq T + J(D)$ we get $f(a_1 + a_2) \in rad_D(T) + J(D)$ or $f(a_1 + a_2) \in rad_D(T) + J(D)$. If $f(a_1 + a_2) = fa_1 + fa_2 \in rad_D(T) + J(D)$ and $fa_2 \in rad_D(T) + J(D)$, we have $fa_1 \in rad_D(T) + J(D)$ a contradiction. If $h(a_1 + a_2) = ha_1 + ha_2 \in rad_D(T) + J(D)$ and $ha_1 \in rad_D(T) + J(D)$ then $ha_2 \in rad_D(T) + J(D)$ a contradiction. Hence $fA \subseteq rad_D(T) + J(D)$ or $hA \subseteq rad_D(T) + J(D)$.

Proposition 11

Let B is a WNP2-Absorbing submodule of an R-module D with (s_1, s_2, d), is WNP2-Absorbing triple zero there exists $s_1, s_2 \in R$, $d \in D$. Then the following holds

1. $s_1 s_2 d = (0)$
2. $[B:_R D]s_1 d = [B:_R D]s_2 d = (0)$
3. $s_1[B:_R D]d = s_2[B:_R D]d = (0)$
4. $s_1[B:_R D]d = s_2[B:_R D]d = (0)$
5. $[B:_R D]s_1 B = [B:_R D]s_2 B = s_1[B:_R D]B = s_2[B:_R D]B = (0)$
6. $[B:_R D][B:_R D]B = [B:_R D]^2 B = (0)$, In particular $[B:_R D]^2 \subseteq Ann_R(D)$

Proof:

(1) Assume that $s_1 s_2 B \neq (0)$, then $s_1 s_2 b \neq 0$ there exists $b \in B$ since (s_1, s_2, d) is a WNP2-Absorbing Triple Zero of B then $s_1 s_2 b = 0$, $s_1 d \notin rad_D(B) + J(D)$ and $s_1 d \notin rad_D(B) + J(D)$ and $s_1 s_2 \notin [B + J(D):_R D]$. Since $0 \neq s_1 s_2 b \in B$ and B be aWNP-2AB submodule, and $s_1 s_2 \notin [B + J(D):_R D]$, then either $s_1 d \in rad_D(B) + J(D)$, or $s_2 d \in rad_D(B) + J(D)$. Now, $0 \neq s_1 s_2(d + b) = s_1 s_2 d + s_1 s_2 b = s_1 s_2 b \in B$ and $s_1 s_2 \notin [B + J(D):_R D]$, since B be a WNP2-Absorbing, then either $s_1(d + b) = s_1 d + s_1 b \in rad_D(B) + J(D)$ or $s_2(d + b) = s_2 d + s_2 b \in rad_D(B) + J(D)$. If $s_1(d + b) = s_1 d + s_1 b \in rad_D(B) + J(D)$ and $s_2(d + b) = s_2 d + s_2 b \in rad_D(B) + J(D)$. We get $s_1 d \in rad_D(B) + J(D)$, its contradiction. If $s_2(d + b) = s_2 d + s_2 b \in rad_D(B) + J(D)$ and $s_2 d \in rad_D(B) + J(D)$. We get $s_2 d \in rad_D(B) + J(D)$, its contradiction. Hence $s_1 s_2 B = (0)$.

(2) Assume that $[B:_R D]s_1 d \neq (0)$, then $m s_1 d \neq 0$ there exists $m \in [B:_R D]$ since (s_1, s_2, d) is a WNP2-Absorbing Triple Zero of B ,then $s_1 s_2 d = 0$, $s_1 d \notin rad_D(B) + J(D)$ and $s_2 d \notin rad_D(B) + J(D)$ and $s_1 s_2 \notin [B + J(D):_R D]$. Since $0 \neq m s_1 d \in B$ and B be WNP2-Absorbing submodule, then either $md \in rad_D(B) + J(D)$, or $s_1 d \in rad_D(B) + J(D)$ or $m s_1 \in [B + J(D):_R D]$ Now, $0 \neq (s_2 + m)s_1 b = s_1 s_2 d + m s_1 d = m s_1 d \in B$ since B be WNP2-Absorbing, then either $(s_2 + m)d = s_2 d + md \in rad_D(B) + J(D)$ or $s_1 d \in rad_D(B) + J(D)$ its contradiction or some $(s_2 + m)s_1 \in [B + J(D):_R D]$. If $(s_2 + m)d = s_2 d + md \in rad_D(B) + J(D)$ and $md \in rad_D(B) + J(D)$, We get $s_1 s_2 \in [B + J(D):_R D]$, its contradiction. Hence $[B:_R D]s_1 d = (0)$. Similarly, we can prove $[B:_R D]s_2 d = (0)$.

(3) Assume that $s_1[B:_R D]d \neq (0)$, then $s_1 md \neq 0$ there exists $m \in [B:_R D]$ since (s_1, s_2, d) is WNP2-Absorbing triple zero of B then $s_1 s_2 d = 0$, $s_1 d \notin rad_D(B) + J(D)$ and $s_2 d \notin rad_D(B) + J(D)$ and $s_1 s_2 \notin [B + J(D):_R D]$. Since $0 \neq s_1 md \in B$ and B be WNP2-Absorbing submodule, then either $s_1 d \in rad_D(B) + J(D)$, or $md \in rad_D(B) + J(D)$ or $s_1 m \in [B + J(D):_R D]$ Now, $0 \neq s_1(m + s_2)d = s_1 md + s_1 s_2 d = m s_1 d \in B$ since B be WNP2-Absorbing, then either $s_1 d \in rad_D(B) + J(D)$ its contradiction or $(m + s_2)d = s_2 d + md + s_2 d \in rad_D(B) + J(D)$ or $s_1(m + s_2)d = s_1 m + s_1 s_2 \in [B + J(D):_R D]$. If $(m + s_2)d = s_2 d + md + s_2 d \in rad_D(B) + J(D)$ and $md \in rad_D(B) + J(D)$ we get $s_2 d \in rad_D(B) + J(D)$, its contradiction. If $(m + s_2)d = s_1 m + s_1 s_2 \in [B + J(D):_R D]$. and $s_1 m \in [B + J(D):_R D]$, we get $s_1 s_2 \in [B + J(D):_R D]$, its conytradiction. Hence $s_1[B:_R D]d = (0)$ Similarly, we can prove $s_2[B:_R D]d = (0)$.

(4) Assume that $[B:_R D]s_1 B \neq (0)$, then $s_1 mb \neq 0$ there exists $m \in [B:_R D]$ and $b \in B$ since (s_1, s_2, d) is a WNP2-Absorbing triple zero of B then $s_1 s_2 d = 0$, $s_1 d \notin rad_D(B) + J(D)$ and $s_2 d \notin rad_D(B) + J(D)$ and $s_1 s_2 \notin [B + J(D):_R D]$ Since $0 \neq s_1 mb \in B$ and B be WNP2-Absorbing submodule, and either $s_1 d \in rad_D(B) + J(D)$, or $md \in rad_D(B) + J(D)$ or $s_1 m \in [B + J(D):_R D]$ Now, $0 \neq (s_2 + m)s_1(b + d) = s_1 s_2 b + s_1 s_2 d + m s_1 b + m s_1 b = m s_1 b \in B$, because $s_1 s_2 d = 0$ and $s_1 s_2 b = 0$, $m s_1 b = 0$ by 1 and $m s_1 d = 0$ by 2 since B WNP2-Absorbing then either $(s_2 + m)(b + d) = s_2 b + s_2 d + mb + mb \in rad_D(B) + J(D)$ its contradiction or $s_1(d + b) = s_1 d + s_1 b \in rad_D(B) + J(D)$ its contradiction or $(s_2 + m)s_1 = s_1 s_2 + m s_1 \in [B + J(D):_R D]$. Its contradiction. Hence $[B:_R D]s_1 B = (0)$. Similarly, we can prove $[B:_R D]s_2 B = s_1[B:_R D]B = s_2[B:_R D]B = (0)$.

(5) Assume that $[B:_R D] = [B:_R D]B \neq (0)$, then $m_1 m_2 b \neq 0$ there exists $m_1, m_2 \in [B:_R D]$ and $b \in B$ since (s_1, s_2, d) is a WNP2-Absorbing triple zero of B then $s_1 s_2 d = 0$, $s_1 d \notin rad_D(B) + J(D)$ and $s_2 d \notin rad_D(B) + J(D)$ and $s_1 s_2 \notin [B + J(D):_R D]$ Since $0 \neq m_1 m_2 b \in B$ and B be a WNP2-Absorbing submodule, then either $m_1 b \in$

$rad_D(B) + J(D)$, or $m_2b \in rad_D(B) + J(D)$ or $m_1m_2 \in [B + J(D):_R D]$ Now, $(m_1 + s_2)(m_2 + s_1)(b + d) = m_1m_2b + m_1m_2d + m_1s_1b + s_2m_2b + s_2m_2d + s_2s_1b + s_2s_1d = m_1m_2b \in B$, because $s_2s_1d = 0$ and $s_2s_1b = 0$ by 1, $m_1s_1b = 0$, $s_2m_2b = 0$ by 4, and $m_1s_1b = 0$ (2). Since B be WNP2-Absorbing then either $(m_1 + s_2)(b + d) = m_1b + m_1d + s_2b + s_2b \in rad_D(B) + J(D)$ its contradiction or $(m_2 + s_1)(b + d) = m_2b + m_2d + s_1b + s_1b \in rad_D(B) + J(D)$ its contradiction $(m_1 + s_2)(m_2 + s_1) = m_1m_2 + m_1s_1 + s_2m_2 + s_2s_1 \in [B + J(D):_R D]$. Its contradiction. Then we get $[B:_R D][B:_R D]B = [B:_R D]^2B = (0)$ we get $[B:_R D]^3 \subseteq [[B:_R D]^2B:_R D] = [(0):_R D] = Ann_R(D)$.

Definition 12

Let B be a WNP2-Absorbing submodule of an R-module D and let $(0) \neq N_1N_2T \subseteq B$ there exists N be an ideal of R and some submodule T of D. B is called WNP2-Absorbing Free Triple Zero in regard to N_1, N_2, T if (n_1, n_2, t) is not triple zero for every $n_1 \in N_1, n_2 \in N_2$ and $t \in T$.

Proposition 13

Let B be a WNP2-Absorbing submodule of an R-module D ,and $(0) \neq N_1N_2T \in B$ there exists ideal N_1, N_2 in R, some submodule T of D, where B is Free triple zero in regard to N_1, N_2, T. Then either $N_1T \subseteq rad_D(B) + J(D)$ or $N_2T \subseteq rad_D(B) + J(D)$ or $N_1N_2 \subseteq [B + J(D)_R D]$.

Proof:

Suppose that $N_1N_2 \nsubseteq [B + J(D):_R D]$. Now, we must show that either $N_1T \subseteq rad_D(B) + J(D)$ or $N_2T \subseteq rad_D(B) + J(D)$. Let $N_1T \nsubseteq rad_D(B) + J(D)$ and $N_2T \nsubseteq rad_D(B) + J(D)$. Then there exist $m \in N_1$ and $f \in N_2 \ni mT \nsubseteq rad_D(B) + J(D)$ and $fT \nsubseteq rad_D(B) + J(D)$. Where $m \in N_1$ and $f \in N_2$ since B be a WNP2-Absorbing submodule and by Proposition 10, we get $mf \in [B + J(D):_R D]$ since $mfD \subseteq B + J(D)$, $mT \nsubseteq rad_D(B) + J(D)$ and $fT \nsubseteq rad_D(B) + J(D)$. By our assumption $(0) \neq N_1N_2T \in B$, there exist $b \in N_1$ and $h \in N_2$, with $bh \notin [B + J(D):_R D]$ by Proposition 10, we get either $bT \subseteq rad_D(B) + J(D)$ or $bT \subseteq rad_D(B) + J(D)$, as $(0) \neq bhT \in B$ and $bh \notin [B + J(D):_R D]$. We investigate (3) cases.

Case 1: Assume that $bT \subseteq rad_D(B) + J(D)$ and $hT \nsubseteq rad_D(B) + J(D)$. Since $mhTB \subseteq B$ and $mT \nsubseteq rad_D(B) + J(D)$ and $hT \nsubseteq rad_D(B) + J(D)$. By Proposition 10, we get $mh \notin [B + J(D):_R D]$. We have $(m + b)T \notin rad_D(B) + J(D)$ as $(0) \neq (m + b)hT \in B$, $mT \nsubseteq rad_D(B) + J(D)$ and $bT \subseteq rad_D(T) + J(D)$. Since $hT \nsubseteq rad_D(B) + J(D)$ and $(m + b)T \notin rad_D(B) + J(D)$, then we obtain $h(m + b) \in [B + J(D):_R D]$ By Proposition 10 .Thus since $h(m + b) = hm + hb \in [B + J(D):_R D]$ and $mh \in [B + J(D):_R D]$, then $mh \in [B + J(D):_R D]$, which is a contradiction.

Case 2: Assume that $bT \nsubseteq rad_D(B) + J(D)$ and $hT \subseteq rad_D(B) + J(D)$. It is clearly shown similarly to case (1).

Case 3: Assume that $bT \subseteq rad_D(B) + J(D)$ and $hT \nsubseteq rad_D(B) + J(D)$. Then $(f + h) \nsubseteq rad_D(B) + J(D)$ as $fT \nsubseteq rad_D(B) + J(D)$ and $hT \subseteq rad_D(B) + J(D)$. By Proposition 10 we get $m(f + h) \in [B + J(D):_R D]$ since $(0) \neq m(f + h)T \in B$, $mT \nsubseteq rad_D(B) + J(D)$ and $(f + h)T \nsubseteq rad_D(B) + J(D)$. Then $mh \in [B + J(D):_R D]$ since $m(f + h) \in [B + J(D):_R D]$ and $mf \in [B + J(D):_R D]$. As $bT \nsubseteq rad_D(B) + J(D)$ and $mT \nsubseteq rad_D(B) + J(D)$, then $(m + b)T \notin rad_D(B) + J(D)$. As $(0) \neq (m + b)fT \subseteq B$, and $mT \nsubseteq rad_D(B) + J(D)$, then $(m + b)T \nsubseteq rad_D(B) + J(D)$. Since $(0) \neq (m + b)fT \subseteq B$, $fT \nsubseteq rad_D(B) + J(D)$ then $(m + b)T \nsubseteq rad_D(B) + J(D)$ By Proposition 10 we get $(m + b)f = mf + bf \in [B + J(D):_R D]$. Since $mf \in [B + J(D):_R D]$ and $mf + bf \in [B + J(D):_R D]$ then $bf \in [B + J(D):_R D]$ By Proposition 10 we get $(m + b)(f + h)T \subseteq B$, $(m + b)T \nsubseteq rad_D(B) + J(D)$ and $(f + h)T \nsubseteq rad_D(B) + J(D)$. As $mf, mh, bf \in [B + J(D):_R D]$, then $mf + mh + bf + bh \in [B + J(D):_R D]$. Thus $bh \in [B + J(D):_R D]$, since $mf + mh + bf + bh \in [B + J(D):_R D]$ a contradiction. Hence either $N_1T \subseteq rad_D(B) + J(D)$ or $N_2T \subseteq rad_D(B) + J(D)$.

Proposition 14

Let B is WNP2-Absorbing submodule of an R-module D with D is semi simple, and T is submodule of D with $T \subseteq B$, then $\forall s \in R$ and $d \in D$, (s_1, s_2, d) be a triple zero of $B \Leftrightarrow (s_1, s_2, d + T)$ is a triple zero of $\frac{B}{T}$.

Proof:

Assume that (s_1, s_2, d) be a triple zero of B there exists $s \in R$ and $d \in D$. Then $s_1 s_2 d = 0, s_1 d \notin rad_D(B) + J(D)$ and $s_2 d \notin rad_D(B) + J(D)$ and $s_1 s_2 \notin [B + J(D):_R D]$.By Propositio13 we get $\frac{B}{T}$ is a WNP2-Absorbing submodule of an R-module $\frac{D}{T}$. Thus $s_1 s_2(d + T) = T$, $s_1 d \notin rad_D\left(\frac{B}{T}\right) + J(D)$ and $s_2 d \notin rad_D\left(\frac{B}{T}\right) + J(D)$ and $s_1 s_2 \notin \left[\frac{B}{T} + J\left(\frac{D}{T}\right):_R \frac{D}{T}\right]$. Hence $(s_1, s_2, d + T)$ be a triple zero of $\frac{B}{T}$. Conversely, Assume, that $(s_1, s_2, d + T)$ be a Triple Zero of $\frac{B}{T}$. Suppose that $s_1 s_2 d \neq 0$. Then $s_1 s_2 d \in B$. Since $(s_1, s_2, d + T) = T$. Thus $s_1 d \in rad_D(B) + J(D)$ or $s_2 d \in rad_D(B) + J(D)$ or $s_1 s_2 \in [B + J(D):_R D]$ as B be a WNP2-Absorbing, a contradiction. So, it must be $s_1 s_2 d = 0$ consequently, (s_1, s_2, d) is a triple zero of B.

3. Conclusion

In this paper, we have introduced and explored the concept of weakly nearly primary 2-Absorbing triple zero submodule as a new generalization in the study of submodule. We provided a formal definition and proved several foundational theorems that clarify the behavior and structure of such submodule within different mod types. The results obtained enrich the theory of submodule and open new directions for future research in related algebraic structures. We hope that this work will serve as a small but meaningful step towards deeper understanding in module theory and will inspire further mathematical investigations in this area.

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