Combinatorial Realizability Of The New Exact Sequence

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ABSTRACT

In this work we introduce and study the generalization of combinatorial realizability of J.H.C.Whitehead and a new notion analogous of J_m-complex , which we call $J_{m,q}$-complex .

Consider the following sequence :

$$\begin{array}{cccc} & \sigma_{p+1,q} & & \sigma_{p,q} \\ \downarrow & & \downarrow\ \downarrow & & \downarrow \end{array}$$

$$\rightarrow \pi_{p+q+1}(k^{p+1},k^{p}) \rightarrow \pi_{p+q}(k^{p}) \xrightarrow{j_{p,q}} \pi_{p+q}(k^{p},k^{p-1}) \xrightarrow{\beta_{p,q}} \pi_{p+q+1}(k^{p-!}) \rightarrow \pi_{p+q+1}(k^{p-!},k^{p}) \rightarrow$$

Denote $\pi_{p+q}(k^{p},k^{p-1})$ *by* $C_{p,q}$ *and* $\pi_{p+q}(k^{p})$ *by* $A_{p,q}$

$\{In \ case \ q<-p, \ then \ C_{p,q}=A_{p,q}=0, \ by \ the \ convevtion \ a \ mong \ algbraic \ topologists \ to \ assume \ \pi_n(X,A)=0 \ and \ \pi_n(X)=0 \ if \ n\leq 0\}$

The above sequence { T_q } is called a composite chain system , if the following tow conditions are satisfied ;

(1) $C_{p,q} = A_{p,q} = 0$ if $_p < 2$, and

(2) each $C_{p,q}$ is a free abelain group .

Denote

$$\begin{array}{ll} j_{p-1,q}\circ\beta_{p,q} & by \quad \sigma_{p,q} \\ \ker\sigma_{p,q}/\mathrm{Im}\,\sigma_{p+1,q} & by \quad \mathrm{H}_{p,q} \\ \ker j_{p,q} & by \quad \Gamma_{p,q} \\ \pi_{p+q}(k^{p})/\mathrm{Im}\,\beta_{p+1,q} & by \quad \Sigma_{p,q} \end{array}$$

and

$$E_q \ : \ \cdots \ \rightarrow \ \mathrm{H}_{p+1,q} \xrightarrow{\nu_{p+1,q}} \Gamma_{p,q} \xrightarrow{\varphi_{p,q}} \Sigma_{p,q} \xrightarrow{\mu_{p,q}} \mathrm{H}_{p,q} \ \rightarrow \ \cdots$$

Let

$$\mathrm{E}_{\mathrm{q}}^{-} : \cdots \rightarrow \mathrm{H}_{\mathrm{p+1,q}}^{-} \rightarrow \Gamma_{\mathrm{p,q}}^{-} \rightarrow \Sigma_{\mathrm{p,q}}^{-} \rightarrow \mathrm{H}_{\mathrm{p,q}}^{-} \rightarrow \cdots \rightarrow H_{2,q}^{-} \rightarrow 0$$

be an exact sequence in which the groups are abelain .
A composite chain system T_q will be called a combinatorial realization of E_q^- if and only if $E_q(K)$ is isomorphic to E_q^- .
We obtain some results , which are ;
Under certain condition the "a new exact sequence" has a combinatorial realization .
If K is a complex and $A_{p,q} = \pi_{p+q}(K^p)$, then
$i_{p,q} : A_{p-1,q+1} \rightarrow A_{p,q}$, maps $A_{p-1,q+1}$ onto 0 ,
$\forall_{p=2,3,4,\cdots,m} \Leftrightarrow K$ *is a* $J_{m,q}$-complex .
If K is a $J_{m,q}$-complex , then
$\sum_{p,q} \cong H_{p,q} \quad \forall_{p=1,2,3,\cdots,m}$ *and* $\mu_{m+1,q}$ *is onto.*

INTRODUCTION

This work is inspired by the papers of " combinatorial homotopy I " , and " combinatorial homotopy II " of J.H.C.Whitehead ,[W1] and [W2],respectively . We introduce in it a generalization of the main concept of that papers .

This work contains two sections;In first section,we introduce the generalization of the combinatorial realizability and $J_{m,q}$-complex . In second section , we establish some results about combinatorial realizability and the $J_{m,q}$-complex .

<u>Section(1) "Definitions"</u>

We mean by the term "complex" in the sequel "connected cw-complex". Let k be a cw-complex ,
where $\pi_{p+q}(k^p, k^{p-1})$ by $C_{p,q}$ and $\pi_{p+q}(k^p)$ by $A_{p,q}$
{ In case q < -p , then $C_{p,q} = A_{p,q} = 0$, by the convention a mong algebraic topologists to assume
$\pi_n(X,A) = 0$ *and* $\pi_n(X) = 0$ *if* $n \leq 0$ }

Now , consider the following sequence ;

$$T_q : \cdots \rightarrow C_{p+1,q} \xrightarrow{\beta_{p+1,q}} A_{p,q} \xrightarrow{j_{p,q}} C_{p,q} \xrightarrow{\beta_{p,q}} A_{p-1,q} \rightarrow \cdots$$

where $\beta_{p,q}$, $j_{p,q}$ *are the homotopy boundary and the realativizing operators ,respectively .*

The system T_q is called a composite chain system , if the following tow conditions are satisfied ;

(1) $C_{p,q} = A_{p,q} = 0$ *if* $p < 2$, *and*

(2) $each \quad C_{p,q} \quad is\ a\ free\ abelain\ group$.

The a new exact sequence $E_q(k)\ induced\ by\ T_q$ as follows;

(1) $E_q(k) : \cdots \to H_{p+1,q} \xrightarrow{\gamma_{p+1,q}} \Gamma_{p,q} \xrightarrow{\varphi_{p,q}} \sum_{p,q} \xrightarrow{\mu_{p,q}} H_{p,q} \to \cdots$

Where

$j_{p-1,q} \circ \beta_{p,q}$ by $\sigma_{p,q}$

$\ker \sigma_{p,q} / im\ \sigma_{p+1,q}$ by $H_{p,q}$

$\ker j_{p,q}$ by $\Gamma_{p,q}$

$\pi_{p+q}(k^p)/im\ \beta_{p+1,q}$ by $\sum_{p,q}$

Let

$E_q^- : \cdots \to H_{p+1,q}^- \to \Gamma_{p,q}^- \to \sum_{p,q}^- \to H_{p,q}^- \to \cdots \to H_{2,q}^- \to 0$

be an exact sequence in which the groups are abelain .

A composite chain system T_q will be called a combinatorial realization of E_q^- iff $E_q(k)$ is isomorphic to E_q^- .

(1) $Let\ E_q = E_q(k)\ ,\ E_q^- = E_q\ (k^-)\ ,\ where\ T_q\ and\ T_q^-\ are\ composite\ chain\ system\ . If\ a\ homomorphism\ \varepsilon(q) : E_q \to E_q^-\ is\ induced\ by\ a\ homomorphism\ \delta(q) = (h_q, f_q) : T_q \to T_q^-\ , we\ shall\ call\ \delta(q)\ a\ combinatorial\ realization\ of\ \varepsilon(q)$.

(2) $We\ say\ that\ a\ complex\ k\ is\ a\ J_{m,q} - complex\ if$ $\ker j_{p,q} = 0 \quad \forall \quad p = 2,3,4,\cdots,m$.

Section (2) "results and conclusion"

We obtain some results . some of these results purely algebraic and others depend on topology of space . we use the " a new exact sequence " mention in [Dh] . we will write some of these results without proof .

Theorem 1

$E_q^- \quad has\ a\ combinatorial\ realization$.

Proof

$the\ proof\ consists of\ two\ stage\ . the\ first\ is\ to\ construct$

(1) Borrowing from [Dh] the our symbols .
(2) In an analogous terminology of J.H.C.Whitehead ,see[W1] .

a *convenient composite chain system, the second is to prove that auch a construction is a realization of* E_q^- .

We start with the second stage, that is assuming the constructionis already done. Thus we are assuming that we have at our disposal.

Assume that T_q *is a composite chain system which is*

constructed and a homomorphisms

$$l^*_{p+1,q} : Z_{p+1,q} \to H^-_{p+1,q} ,$$
$$\gamma_{p,q} : \Gamma_{p,q} \approx \Gamma^-_{p,q} \quad \text{and}$$
$$t^*_{p,q} : A_{p,q} \to \textstyle\sum^-_{p,q} .$$

are given, for every $p = 1,2,3,\cdots$ *such that the following condition are satisfied;*

(*)

$(c1)$ $\left(\gamma^-_{p+1,q} \circ l^*_{p+1,q}\right)(z) = \left(\vartheta_{p,q} \circ \beta_{p+1,q}\right)(z) \quad \forall z \in Z_{p+1,q}$.

$(c2)$ $\left(\varphi^-_{p,q} \circ \vartheta_{p,q}\right)(x) = t^*_{p,q}(x) \quad \forall x \in \Gamma_{p,q}$ *and*

$(c3)$ $\left(\mu^-_{p,q} \circ t^*_{p,q}\right)(x) = \left(l^*_{p,q} \circ j_{p,q}\right)(x) \quad \forall x \in A_{p,q}$.

(**)

$(r1)$ $l^*_{p,q}\left(Z_{p+1,q}\right) = H^-_{p+1,q}$.

$(r2)$ $\sigma_{p+1,q}\left(C_{p+1,q}\right) = \ker l^*_{p,q}$ *and*

$(r3)$ $\beta_{p+1,q}\left(C_{p+1,q}\right) \subset \ker t^*_{p,q}$.

Then it follows from $\{**\}$ *that one can determine homomorphisms*

$\tau_{p+1,q} : H_{p+1,q} \to H^-_{p+1,q}$ *for* $p = 1,2,3,\ldots$

$\rho_{p,q} : \sum_{p,q} \to \sum^-_{p,q}$ *for* $p = 1,2,3,\ldots$

by $\tau \circ l = l^*$ *and* $\rho \circ t = t^*$,

where $H_{p+1,q}, \sum_{p,q}$ *are in* $E_q = E_q(k)$, *and*

$l : Z_{p+1,q} \to H_{p+1,q}$ *is defined by* $l(z) = [z]$,

$t : A_{p,q} \to \sum_{p,q}$ *is defined by* $t(x) = [x]$.

Clearly $\tau_{p+1,q}$ *by* $\{**\}$ *is an isomorphism.*

Consider the following diagram;

$$\begin{array}{ccccccccccc} E_q & : \cdots \to & H_{p+1,q} & \overset{\gamma_{p+1,q}}{\longrightarrow} & \Gamma_{p,q} & \overset{\varphi_{p,q}}{\longrightarrow} & \sum_{p,q} & \overset{\mu_{p,q}}{\longrightarrow} & H_{p,q} & \to \cdots \\ \downarrow_{\varepsilon(q)} & & \downarrow & & \downarrow_{\vartheta_{p,q}} & & \downarrow_{\rho_{p,q}} & & \downarrow_{\tau_{p,q}} & \\ E_q^- & : \cdots \to & H_{p+1,q}^- & \overset{\gamma_{p+1,q}^-}{\longrightarrow} & \Gamma_{p,q}^- & \overset{\varphi_{p,q}^-}{\longrightarrow} & \sum_{p,q}^- & \overset{\mu_{p,q}^-}{\longrightarrow} & H_{p,q}^- & \to \cdots \end{array}$$

It is easy to show that this diagram commutes in

each square by simply chasing the diagram and utilizing $\{*\}$ *and* $\{**\}$. *Therefore*
$\varepsilon(q) = (\vartheta_q, \rho_q, \tau_q) : E_q \to E_q^-$ *is a homomorphism*.

Since $\tau_{p+1,q}$, $\vartheta_{p,q}$, $\tau_{p,q}$ *and* $\vartheta_{p,q}$ *are isomorphism for every* p , *so is* $\rho_{p,q}$, *(by the five lemma theorem, see [H]).*

Now we turn to the first stage namely the construction stage. The proof is by induction on p .
Starting with $C_{1,q} = A_{0,q} = 0$.

Let $p \geq 1$, *and assume that we have constructed the groups and homomorphisms ;*

$$C_{p,q} \overset{\beta_{p,q}}{\longrightarrow} A_{p-1,q} \overset{j_{p-1,q}}{\longrightarrow} C_{p-1,q} \to \cdots \to C_{1,q} \overset{\beta_{1,q}}{\longrightarrow} A_{0,q} .$$

Likewise $l_{n+1,q}^*$ *and* $t_{n,q}^*$ *are also constrected satisfying* $\{*\}$ *and* $\{**\}$, *for* $n = 0,1,2,3, \ldots , p-1$.

We need to construct $A_{p,q}$ *and* $j_{p,q}$. *this task is carried as follows ;*

We define $A_{p,q}$ *and* $j_{p,q}$ *as follows ;*

$$A_{p,q} = \Gamma_{p,q}^- \otimes \ker \beta_{p,q} \quad \text{and} \quad j_{p,q} : A_{p,q} \to C_{p,q}$$

by $j_{p,q}(e,b) = b$ *for* $e \in \Gamma_{p,q}^-$ *and* $b \in \ker \beta_{p,q}$.

It is easy to show that ; $j_{p,q}$ *is well – defined , and since* $C_{p,q}$ *is a free abelian group so is* $\ker \beta_{p,q}$. *Let* $\{b_i\}$

be a family of generators of $\ker \beta_{p,q}$ *indexed by a set* I , *it follows from* $\{*\}$ *when* $n = p-1$, *that*

$$\begin{aligned} (\gamma_{p,q}^- \circ l_{p,q}^*)(j_{p,q}(b_i)) &= (\vartheta_{p-1,q} \circ \beta_{p,q})(j_{p,q}(b_i)) \\ &= \vartheta_{p-1,q}\left(\beta_{p,q}(j_{p,q}(b_i))\right) \end{aligned}$$

$= \quad 0 \quad .$

$\Rightarrow \quad l^*_{p,q}(\, j_{p,q}(b_i)\,) \in \ker \gamma^-_{p,q} = Im\, \mu^-_{p,q} \quad .$

Therefore, there is for each b_i *an* $x_i^- \in \sum^-_{p,q}$,
such theat $\quad l^*_{p,q}(\, j_{p,q}(b_i)\,) = \mu^-_{p,q}(x_i^-) \quad .$
By axiom of choice we choose an x_i^- *for each* b_i
and will denote such x_i^- *by* $c(b_i)$.

We define $\quad t^*_{p,q} : A_{p,q} \longrightarrow \sum^-_{p,q}$ *by*
$t^*_{p,q}\,(e, b_i) = (\varphi^-_{p,q} \circ \vartheta_{p,q})(e) + x_i^-$ *, where* x_i^- *is* $c(b_i)$
Then

$$\mu^-_{p,q}(\, t^*_{p,q}(e,b_i)\,) = \mu^-_{p,q}(\,(\varphi^-_{p,q}\circ\vartheta_{p,q})(e) + x_i^-\,)$$
$$= \mu^-_{p,q}(\,(\varphi^-_{p,q}\circ\vartheta_{p,q})(e)\,) + \mu^-_{p,q}(x_i^-)$$
$$= \quad 0 + \mu^-_{p,q}(x_i^-)$$
$$= l^*_{p,q}(\, j_{p,q}(b_i)\,)$$
$$= l^*_{p,q}(\, j_{p,q}(e)\,) + l^*_{p,q}(\, j_{p,q}(b_i)\,)$$
$$= (l^*_{p,q}\circ j_{p,q})(e, b_i) \quad .$$

Therefore
$(1) \quad \cdots \quad (\mu^-_{p,q}\circ t^*_{p,q})(x) = (l^*_{p,q}\circ j_{p,q})(x) \quad \forall x \in A_{p,q}$
$(2) \quad \cdots \quad t^*_{p,q}(e) = (\varphi^-_{p,q}\circ\vartheta_{p,q})(e) \quad \forall e \in \Gamma^-_{p,q}$.

Therefore conditions $(c2)$ *and* $(c3)$ *in* $\{*\}$ *are satisfied.*

Now it remains to construct $C_{p+1,q}$ *and* $\beta_{p+1,q}$ *, thus will be done, as follows;*

Let $\{y_m\}$ *be a family of generators of* $H^-_{p+1,q}$ *indexed by a set* M .

Let $Z_{p+1,q}$ *be the free abelian group over a family* $\{z_m\}$ *which is in one – to – one correspondence with* $\{y_m\}$.

Let $l^*_{p+1,q} : Z_{p+1,q} \rightarrow H^-_{p+1,q}$, *be defined by*
$l^*_{p,q}(z_m) = y_m$ *then* $l^*_{p+1,q}(Z_{p+1,q}) = H^-_{p+1,q}$.
We define $\quad C_{p+1,q} = Z_{p+1,q} \otimes \ker l^*_{p,q}$.
Since $C_{p,q}$ *is a free abelian group, so is* $\ker l^*_{p,q}$,
hence $C_{p+1,q}$ *is a free abelian group* .

Let $\{r_d\}$ *be a family of generators of* $\ker l^*_{p,q}$ *indexed by a set* D, *it follows from* $\{*\}$, *when* $n = p-1$, *that*

$$\begin{aligned}\beta_{p,q}(r_d) &= \vartheta^{-1}_{p-1,q}\left(\left(\gamma^-_{p,q}\circ l^*_{p,q}\right)(r_d)\right)\\ &= \vartheta^{-1}_{p-1,q}\left(\gamma^-_{p,q}\left(l^*_{p,q}(r_d)\right)\right)\\ &= 0\end{aligned}$$

$$\Longrightarrow \quad r_d \in \ker\beta_{p,q} = j_{p,q}(A_{p,q}) .$$

Therefore, there is for each r_d *an* $a_d \in A_{p,q}$, *such that* $j_{p,q}(a_d) = r_d$.

By axiom of choice we choose an a_d *for each* r_d *and we will denote such* a_d *by* $c(r_d)$.

From (1), *we have*

$$\begin{aligned}\left(\mu^-_{p,q}\circ t^*_{p,q}\right)(a_d) &= \left(l^*_{p,q}\circ j_{p,q}\right)(a_d)\\ &= l^*_{p,q}\left(J_{p,q}(a_d)\right)\\ &= l^*_{p,q}(r_d)\\ &= 0\end{aligned}$$

$$\Longrightarrow \quad t^*_{p,q}(a_d) \in \ker\mu^-_{p,q} = Im\,\varphi^-_{p,q}$$

Therefore, there is for each a_d *an* $e^-_d \in \Gamma^-_{p,q}$, *such that* $t^*_{p,q}(a_d) = \varphi^-_{p,q}(e^-_d)$.

By axiom of choice we choose an e^-_d *for each* a_d *and we will denote such* e^-_d *by* $c(a_d)$.

From (2), *we have* $t^*_{p,q}(a_d) = t^*_{p,q}(e_d)$, *where* $e_d = \vartheta^{-1}_{p,q}(e^-_d)$.

$$\begin{aligned}&t^*_{p,q}\left(\left(\vartheta^{-1}_{p,q}\circ\gamma^-_{p+1,q}\right)\left(l^*_{p+1,q}(z)\right)\right)\\ &= \left(\varphi^-_{p,q}\circ\vartheta_{p,q}\right)\left(\left(\vartheta^{-1}_{p,q}\circ\gamma^-_{p+1,q}\right)\left(l^*_{p+1,q}(z)\right)\right)\\ &= \left(\varphi^-_{p,q}\circ\gamma^-_{p+1,q}\right)\left(l^*_{p+1,q}(z)\right)\\ &= 0 .\end{aligned}$$

Now, we define $\beta_{p+1,q} : C_{p+1,q} \to A_{p,q}$, *by*

$$\beta_{p+1,q}(z, r_d) = \left(\vartheta^{-1}_{p,q}\circ\gamma^-_{p+1,q}\right)\left(l^*_{p+1,q}(z)\right) + (a_d - e_d).$$

where a_d *is* $c(r_d)$ *and* e_d *dependent on choose of* a_d .

Then $\varphi_{p+1,q}(z,r_d) = (j_{p,q}\circ\beta_{p+1,q})(z,r_d)$

$$= j_{p,q}\left(\beta_{p+1,q}(z,r_d)\right)$$

$$= j_{p,q}\left(\left(\vartheta_{p,q}^{-1}\circ\gamma_{p+1,q}^{-}\right)\left(l_{p+1,q}^{*}(z)\right)\right) + j_{p,q}(a_d - e_d)$$

$$= 0 + j_{p,q}(a_d)$$

$$= r_d .$$

Therefore

$\ker\varphi_{p,q} = Z_{p+1,q}$ *and* $\varphi_{p+1,q}(C_{p+1,q}) = \ker l_{p,q}^{*}$.

Also

$$t_{p,q}^{*}\left(\beta_{p+1,q}(z,r_d)\right)$$

$$= t_{p,q}^{*}\left(\left(\vartheta_{p,q}^{-1}\circ\gamma_{p+1,q}^{-}\right)\left(l_{p+1,q}^{*}(z)\right)\right) + t_{p,q}^{*}(a_d - e_d)$$

$$= 0 + t_{p,q}^{*}(a_d - e_d)$$

$$= 0$$

Then $\beta_{p+1,q}(C_{p+1,q}) \subset \ker t_{p,q}^{*}$ *and*

$$\vartheta_{p,q}\left(\beta_{p+1,q}(z,0)\right) = \vartheta_{p,q}\left(\left(\vartheta_{p,q}^{-1}\circ\gamma_{p+1,q}^{-}\right)\left(l_{p+1,q}^{*}(z)\right)\right)$$

$$= \gamma_{p+1,q}^{-}\left(l_{p+1,q}^{*}(z)\right)$$

Therefore

$(\gamma_{p+1,q}^{-}\circ l_{p+1,q}^{*})(z) = (\vartheta_{p,q}\circ\beta_{p+1,q})(z) \quad \forall z \in Z_{p+1,q}$.

Therefore $\{*\}$ *and* $\{**\}$ *are satisfied when* $n = p$ *, and the prove is complet .*

<u>Theorem 2</u>

Any homomorphism $\varepsilon(q): E_q \to E_q^{-}$ *, has a combinatorial realization* $\delta(q) : T_q \to T_q^{-}$.

<u>Proof</u>

Let $\varepsilon(q) = (\vartheta_q, \rho_q, \tau_q)$ *, as in proof of theorem 1 , we have* $C_{p+1,q} = Z_{p+1,q} \otimes \ker l_{p,q}^{*}$ *and* $A_{p,q} = \Gamma_{p,q} \otimes \ker\beta_{p,q}$ *, where*

$\ker l^*_{p,q} = \varphi_{p+1,q}(C_{p+1,q})$ *and* $\ker \beta_{p,q} = j_{p,q}(A_{p,q})$.

Let

$i_{p,q} : \Gamma_{p,q} \to A_{p,q}$ *, be the inclusion homomorphism ,*
define by $i_{p,q} = [x]$ *,*

$t^*_{p,q} : A_{p,q} \to \sum_{p,q}$ *, be the natural homomorphism ,*
defin by $t^*_{p,q}(x) = [x]$ *,*

$l_{p,q} : Z_{p,q} \to H_{p,q}$ *, be the natural homomorphism ,*
defin by $l_{p,q}(z) = [z]$.

Let $\{b_i\}_{i\in I}$, $\{z_m\}_{m\in M}$ *and* $\{r_d\}_{d\in D}$ *be a family of generators of* $\ker \beta_{p,q}$, $Z_{p+1,q}$ *and* $\ker l_{p,q}$ *, respectively .*

Then $\tau_{p+1,q}\left(l_{p+1,q}(z_m)\right) \in H^-_{p+1,q}$

$\Longrightarrow \quad \dot{l}^{-1}_{p+1,q}\left(\tau_{p+1,q}\left(l_{p+1,q}(z_m)\right)\right) \in Z^-_{p+1,q}$ *, where*

$\dot{l}_{p+1,q}: Z^-_{p+1,q} \to H^-_{p+1,q}$ *be the natural homomorphis.*

Let $z^-_m \in \dot{l}^{-1}_{p+1,q}\left(\tau_{p+1,q}\left(l_{p+1,q}(z_m)\right)\right)$ *implies that*

(1) $\cdots$ $\dot{l}_{p+1,q}(z^-_m) = \tau_{p+1,q}\left(l_{p+1,q}(z_m)\right)$.

And let $h^*_{p+1,q}: Z_{p+1,q} \to Z^-_{p+1,q}$ *be define by*
$h^*_{p+1,q}(z_m) = z^-_m \quad \forall\ p \geq 1$. *Then*

(2) $\cdots$ $\dot{l}_{p+1,q}\left(h^*_{p+1,q}(z_m)\right) = \tau_{p+1,q}\left(l_{p+1,q}(z_m)\right)$.

Also $\rho_{p,q}\left(t_{p,q}(b_i)\right) \in \sum^-_{p,q} \quad \Longrightarrow$

$\dot{t}^{-1}_{p,q}\left(\rho_{p,q}\left(t_{p,q}(b_i)\right)\right) \subset A^-_{p,q}$ *, where*

$\dot{t}_{p,q}: A^-_{p,q} \to \sum^-_{p,q}$ *, be the natural homomorphism .*

Let $a^-_i \in \dot{t}^{-1}_{p,q}\left(\rho_{p,q}\left(t_{p,q}(b_i)\right)\right)$ *implies that*

(3) $\cdots$ $\dot{t}_{p,q}(a^-_i) = \rho_{p,q}\left(t_{p,q}(b_i)\right)$.

$\Longrightarrow \quad \dot{l}_{p,q}\left(j^-_{p,q}(a^-_i)\right) = \mu^-_{p,q}\left(k^-_{p,q}(a^-_i)\right)$

$$= \mu^{-}_{p,q}\big((\rho_{p,q} \circ t_{p,q})(b_i) \big)$$
$$= \left(\mu^{-}_{p,q} \circ \rho_{p,q}\right)\left(t_{p,q}(b_i)\right)$$
$$= \left(\tau_{p,q} \circ \mu_{p,q}\right)\left(t_{p,q}(b_i)\right)$$
$$= \tau_{p,q}\left(\mu_{p,q}\left(t_{p,q}(b_i)\right)\right)$$
$$= \tau_{p,q}\left(l_{p,q}\left(j_{p,q}(b_i)\right)\right)$$
$$= \dot{l}_{p,q}\left(h^{*}_{p,q}\left(j_{p,q}(b_i)\right)\right)$$

$$\Longrightarrow \quad j^{-}_{p,q}(a^{-}_i) - h^{*}_{p,q}(j_{p,q}(b_i)) \in \ker \dot{l}_{p,q} = \varphi^{-}_{p+1,q}(C^{-}_{p+1,q})$$

Therefore

$$(4) \quad \cdots \quad j^{-}_{p,q}(a^{-}_i) = h^{*}_{p,q}\left(j_{p,q}(b_i)\right) + \varphi^{-}_{p+1,q}(c^{-}_i)$$

for some $c^{-}_i \in C^{-}_{p+1,q}$

By axiom of choice we choose an c^{-}_i *for each* b_i *and we will denote such* c^{-}_i *by* $c(b_i)$.

Now we define $f_{p,q}$ *, as follows ;*

Let $f_{p,q} : A_{p,q} \to A^{-}_{p,q}$ *be define by*

$f_{p,q}(x) = \vartheta_{p,q}(x) \;\; \forall x \in \Gamma_{p,q}$ *and* $f_{p,q}(b_i) = a^{-}_i - \beta^{-}_{p+1,q}(c^{-}_i)$.

Let $x \in \Gamma_{p,q}$ *we have*

$$\dot{t}_{p,q}\big(f_{p,q}(x) \big) = \dot{t}_{p,q}\big(\vartheta_{p,q}(x) \big)$$
$$= \varphi^{-}_{p,q}\big(\vartheta_{p,q}(x) \big)$$
$$= \big(\varphi^{-}_{p,q} \circ \vartheta_{p,q} \big)(x)$$
$$= \big(\rho_{p,q} \circ \varphi_{p,q} \big)(x)$$
$$= \rho_{p,q}\big(\varphi_{p,q}(x) \big)$$
$$= \rho_{p,q}\big(t_{p,q}(x) \big)$$

and $\quad \dot{t}_{p,q}\left(f_{p,q}(b_i)\right) = \dot{t}_{p,q}(a^{-}_i) - \dot{t}_{p,q}\left(\beta^{-}_{p+1,q}(c^{-}_i)\right)$

$$= \dot{t}_{p,q}(a^{-}_i)$$
$$= \rho_{p,q}\left(t_{p,q}(b_i)\right) \quad .$$

Therefore , we obtain

$$(5)\quad \cdots \quad \begin{cases} \tau(l(z_m) = \dot{l}(h^*(z_m) \\ \vartheta(x) = f(x) \quad \forall x \in \Gamma_{p,q} \\ \rho\big(t(x)\big) = \dot{t}\big(f(x)\big) \quad \forall x \in A_{p,q} \end{cases}$$

From (5), we have

$$\dot{t}_{p,q}\left(f_{p,q}\left(\beta_{p+1,q}(r_d)\right)\right) = \rho_{p,q}\left(t_{p,q}\left(\beta_{p+1,q}(r_d)\right)\right)$$
$$= \rho_{p,q}(0)$$
$$= 0$$

$$\Longrightarrow \quad f_{p,q}\left(\beta_{p+1,q}(r_d)\right) \in \ker \dot{t}_{p,q} = Im\, \beta^{-}_{p+1,q}$$

Therefore $\beta^{-}_{p+1,q}(\dot{c}_d) = f_{p,q}\left(\beta_{p+1,q}(r_d)\right)$

for some $\dot{c}_d \in C^{-}_{p+1,q}$.

By axiom of choice we choose an $\dot{c}_d$ *for each* r_d *and we will denote such* $\dot{c}_d$ *by* $c(r_d)$.

Now we will define $h_{p+1,q}$ *, as follows ;*

Let $h_{p+1,q} : C_{p+1,q} \rightarrow C^{-}_{p+1,q}$ *, be define by* $h_{p+1,q}(z) = h^*_{p+1,q}(z) \quad \forall z \in Z_{p+1,q}$ *and* $h_{p+1,q}(r_d) = \dot{c}_d$.

Then $\beta^{-}_{p+1,q}\left(h_{p+1,q}(r_d)\right) = \beta^{-}_{p+1,q}(\dot{c}_d)$

$$= f_{p,q}\left(\beta_{p+1,q}(r_d)\right) \ .$$

Also , it follows from (5), *that*

$$f_{p,q}\left(\beta_{p+1,q}(z)\right) = f_{p,q}\left(\gamma_{p+1,q}\left(l_{p+1,q}(z)\right)\right)$$
$$= \vartheta_{p,q}\left(\gamma_{p+1,q}\left(l_{p+1,q}(z)\right)\right)$$
$$= \left(\vartheta_{p,q} \circ \gamma_{p+1,q}\right)\left(l_{p+1,q}(z)\right)$$
$$= \left(\gamma^{-}_{p+1,q} \circ \tau_{p+1,q}\right)\left(l_{p+1,q}(z)\right)$$
$$= \gamma^{-}_{p+1,q}\left(\tau_{p+1,q}\left(l_{p+1,q}(z)\right)\right)$$
$$= \gamma^{-}_{p+1,q}\left(\dot{l}_{p+1,q}\left(h^*_{p+1,q}(z)\right)\right)$$

$$= \gamma^{-}_{p+1,q}\left(i_{p+1,q}\left(h_{p+1,q}(z)\right)\right)$$

$$= \beta^{-}_{p+1,q}\left(h_{p+1,q}(z)\right)$$

$$\Rightarrow \quad \beta^{-}_{p+1,q}\left(h_{p+1,q}(z)\right) = f_{p,q}\left(\beta_{p+1,q}(z)\right) \qquad \forall\, z \in Z_{p+1,q}$$

Therefore $(\beta^{-}_{p+1,q} \circ h_{p+1,q})(x) = (f_{p,q} \circ \beta_{p+1,q})(x)\ \forall x \in C_{p+1,q}$

Let $x \in \Gamma_{p,q} \quad \Rightarrow \quad \vartheta_{p,q}(x) \in \Gamma^{-}_{p,q}$

$$\Rightarrow j^{-}_{p,q}\left(f_{p,q}(x)\right) = j^{-}_{p,q}\left(\vartheta_{p,q}(x)\right) = 0$$

and $h_{p,q}\left(j_{p,q}(x)\right) = h_{p,q}(0) = 0$.

Then $(j^{-}_{p,q} \circ f_{p,q})(x) = (h_{p,q} \circ j_{p,q})(x) \quad \forall x \in \Gamma_{p,q}$.

On the other hand , we have

$$j^{-}_{p,q}\left(f_{p,q}(b_i)\right) = j^{-}_{p,q}\left(a^{-}_i - \beta^{-}_{p+1,q}(c^{-}_i)\right)$$

$$= j^{-}_{p,q}(a^{-}_i) - (j^{-}_{p,q} \circ \beta^{-}_{p+1,q})(c^{-}_i)$$

$$= j^{-}_{p,q}(a^{-}_i) - \varphi^{-}_{p+1,q}(c^{-}_i)$$

$$= j^{-}_{p,q}(a^{-}_i) - \left(j^{-}_{p,q} - h^{*}_{p,q}\left(j^{-}_{p,q}(b_i)\right)\right)$$

$$= h^{*}_{p,q}\left(j_{p,q}(b_i)\right)$$

$$= h_{p,q}\left(j_{p,q}(b_i)\right)$$

Therefore $(j^{-}_{p,q} \circ f_{p,q}) = (h_{p,q} \circ j_{p,q})(x) \quad \forall x \in A_{p,q}$

Therefore (h, f) *is a homomorphism. Since* $h(z) = h^{*}(z)$, *it follows from* (5) *that* (h, f) *is a combinatorial realization of* (τ, ϑ, ρ).

<u>*Theorem 3*</u>

Let k *be a complex and* $A_{p,q} = \pi_{p+q}(k^{p})$. *Then* $i_{p,q}: A_{p-1,q+1} \to A_{p,q}$ *maps* $A_{p-1,q+1}$ *onto* 0 , $\forall\, p = 2,3,4, \ldots , m$. $\Leftrightarrow$ k *is a* $J_{m,q}$ − *complex* .

<u>*proof*</u>

We know that $Im\, i_{p,q} = \ker j_{p,q}$, *then*

$i_{p,q}(A_{p-1,q+1}) = 0 \quad \forall\, p = 2,3,\dots,m.$

$\Leftrightarrow \quad \ker j_{p,q} = 0 \ \ \forall p = 2,3,4,\dots,m.$

$\Leftrightarrow \quad k\ is\ a\ J_{m,q} - complex\ .$

Theorem 4

(1) $If\ \ K \equiv_m L\ ,\quad then$

$K\ \ is\ a\ J_{m,q} - complex \ \Leftrightarrow\ L\ \ is\ a\ J_{m,q} - complex.$

Corollary 5

$If\ \ K \equiv_m K^-\ ,then \quad \Gamma_{p,q} = 0 \ \Leftrightarrow\ \Gamma^-_{p,q} = 0 \ \ \forall p = 2,3,\dots,m.$

Theorem 6

$If\ \ K\ \ is\ a\ J_{m,q} - complex\ ,then \quad \sum_{p,q} \cong H_{p,q}$

$\forall\ p = 1,2,3,\dots,m. \quad and \quad \mu_{m+1,q} \ \ is\ onto\ .$

Proof

$Consider\ the\ following\ a\ new\ exact\ sequence\ ;$

$$\cdots \to \sum_{m+1,q} \xrightarrow{\mu_{m+1,q}} H_{m+1,q} \xrightarrow{\gamma_{m+1,q}} \Gamma_{m,q} \xrightarrow{\varphi_{m,q}} \sum_{m,q} \xrightarrow{\mu_{m,q}} H_{m,q} \to \cdots$$

$Since\ \ \Gamma_{p,q} = \ker j_{p,q} = 0 \ \ \forall p = 1,2,3,\dots,m. It\ follows\ from$ $exactness\ ,that \quad \sum_{p,q} \cong H_{p,q} \quad \forall p = 1,2,3,\dots,m. \ \ and$ $Im\ \mu_{m+1,q} = \ker \gamma_{m+1,q} = H_{m+1,q}\ \ .Therefore\ \ \mu_{m+1,q} is\ onto$

Corollary 7

$If\ \ K\ \ is\ a\ J_{m,q} - complex\ ,and\ \ K \equiv_m K^-\ .Then$ $\sum^-_{p,q} \cong H^-_{p,q} \quad \forall p = 1,2,\dots,m. \ \ and \ \ \mu^-_{m+1,q} \ \ is\ onto\ ,$ $where \quad \sum^-_{p,q}\ ,H^-_{p,q} \quad are\ in \quad E_q(K^-)\ .$

corollary 8 **(Hurewicz's theorem for cw − complexes),[H]**

$If\ \ K\ \ is\ a\ cw - complex\ such\ that\ \ \pi_r(K) = 0,$ $for\ \ 1 \le r \le n-1. Then\ \ j_n : \pi_n(K) \cong H_n(K)$ $and \quad j_{n+1} \quad maps \quad \pi_{n+1}(K)\ onto\ H_{n+1}(K)\ .$

Theroem 9

$Let \quad K \quad be\ an \quad (m-1) - connected\ .$ $Then \quad K \quad is\ a\ J_{m,q} - complex\ \ .$

Theorem 10

$$\Gamma_{p,q} \ \cong\ \sum_{p-1,q+1}\ .$$

---

(1) K and L are n-equivalent,see[W1]

Proof

From the exact homotopy sequence of the pair (k^p, k^{p-1})

$$\cdots \xrightarrow{i_{p,q+1}} \pi_{p+q+1}(k^p) \xrightarrow{j_{p,q+1}} \pi_{p+q+1}(k^p, k^{p-1}) \xrightarrow{\beta_{p,q+1}} \pi_{p+q}(k^{p-1}) \xrightarrow{i_{p,q}} \pi_{p+q}(k^p) \xrightarrow{j_{p,q}} \cdots$$

we have $Im\ i_{p,q} = ker\ j_{p,q} \quad \Rightarrow \quad i_{p,q}(A_{p-1,q+1}) = \Gamma_{p,q}$,

So from the fundamental homomorphism theorem ,

we have $\Gamma_{p,q} \cong \dfrac{A_{p-1,q+1}}{ker\ i_{p,q}}$

But from exactness, it follows $Im\ \beta_{p,q+1} = ker\ i_{p,q}$.

Therefore $\Gamma_{p,q} \cong \sum_{p-1,q+1}$.

Theorem 11

$$\Gamma_{p,q} \cong \pi_{p+q}(k) \qquad where \ -p \leq q < 0 \ .$$

Proof

Consider the following sequence ;

$$\cdots \to A_{p-1,q} \xrightarrow{i_{p,q-1}} A_{p,q-1} \xrightarrow{j_{p,q-1}} C_{p,q-1} \xrightarrow{\beta_{p,q-1}} A_{p-1,q-1} \xrightarrow{i_{p,q-2}} A_{p,q-2} \to$$

when $q = 1$ *, remembering that* $C_{p,q} = \pi_{p+q}(k^p, k^{p-1})$

which is zero in case $-p \leq q < 0$.

Also $A_{p,q} = \pi_{p+q}(k^p)$*, and by a well known theorem ,*

$\pi_{p+q}(k^p)$ *is isomorphic to* $\pi_{p+q}(k)$ *when* $-p \leq q < 0$.

Then we get

$ker\ j_{p,q} = A_{p,q} = \pi_{p,q}(k^p) \cong \pi_{p+q}(k)$ *where* $-p \leq q < 0$.

Therefore $\Gamma_{p,q} \cong \pi_{p+q}(k)$ *where* $-p \leq q < 0$.

Theorem 12

If k *is a cw − complex such that* k^{n-1} *consists of a single* $0 -$ *cell* e^0 *. Then*

(i) $\Gamma_{p,q} = 0$ *if* $p \leq o$,

(ii) $\sum_{p,q} = 0$ *if* $p \leq n-1$,

(iii) $\sum_{n,q} \cong H_{n,q}$.

Proof

Since $k^0 = k^1 = k^2 = \cdots = k^{n-1} = \{e^0\}$,

it follows that $C_{p,q} = \pi_{p+q}(k^p, k^{p-1}) = 0$ *if* $p \leq n-1$

and $A_{p,q} = \pi_{p+q}(k^p) = 0$ *if* $p \leq n-1$

$and\ hence\quad \sum_{p,q} = \frac{A_{p,q}}{D_{p,q}} = 0\ \ if\ \ p \leq n-1\ .$

$Consider\ the\ following\ sequence\ ;$

$$\cdots \to C_{n+1,q} \xrightarrow{\beta_{n+1,q}} A_{n,q} \xrightarrow{j_{n,q}} C_{n,q} \to 0$$

$Since\quad \Gamma_{p,q} = ker\ j_{p,q}\ \ ,it\ followsthat\quad \Gamma_{p,q} = 0\quad \forall\, p \leq n-1$

$Notice\ that\quad C_{n,q} = \pi_{n+q}(k^n, k^{n-1}) \cong \pi_{n+q}(k^n) = A_{n,q}$

$then\ \ j_{n,q}\ is\ an\ isomorphism. Therefore\ \ \Gamma_{n,q} = ker j_{n,q} = 0.$

$Now\ , consider\ the\ following\ exact\ sequence\ ;$

$$\to H_{n+1,q} \to \Gamma_{n,q} \to \sum_{n,q} \to H_{n,q} \to \Gamma_{n-1,q} \to \sum_{n-1,q} \to$$

$it\ follows\ from\ exactness\ that\quad \sum_{n,q} \cong H_{n,q}\ .$

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