Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.17.(2) 2025,pp.Math 37–45

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

Certain Sub-Classes of Harmonic Univalent Functions Associated With the Differential Operator

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ARTICLE INFO

Article history:
Received: 21/03/2025
Rrevised form: 11/04/2025
Accepted : 17/04/2025
Available online: 30/06/2025

Keywords:

Harmonic Functions,
univalent Functions,
Sense-Preserving,
extreme Points,
distortion Theorem.

ABSTRACT

In the present study, a subclass of harmonic univalent functions defined by a differential operator acting on complex harmonic functions is tackled. A sufficient condition and a representation theorem for the subclass are derived. Some geometric properties associated with it are also investigated, including coefficient bound, extreme points, distortion and convex combinations in connection to the subclass $\mathcal{S}_{\Upsilon}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$.

MSC..

https://doi.org/10.29304/jqcsm.2025.17.22223

1. Introduction

The Υ denotes the family of continuous complex-valued harmonic functions that are harmonic in open unit disk $\Delta=\{z:|z|<1\}$. The $\mathcal{H}\subset\Upsilon$ contains analytic functions in Δ. A harmonic function in Δ is represented by the form $f=h+\bar{g}$, where $h\in\mathcal{H}$ and $g\in\mathcal{H}$. Here , h is referred to as the analytic part, and g is known as the co-analytic part of f, A condition that is both necessary and sufficient for f to be locally univalent and sense-preserving in Δ is that $|h'(z)|>|g'(z)|$(refer to [1]). Hence, without loss of generality, this can be expressed as:

$$h(z)=z+\sum_{s=2}^{\infty}a_s z^s \quad , \quad g(z)=\sum_{s=1}^{\infty}b_s z^s \quad ,z\in\Delta. \tag{1}$$

The $\mathcal{S}_{\Upsilon}$ denotes the family of sense-preserving, harmonic, and univalent functions $f(z)=h(z)+\overline{g(z)}$ within Δ, satisfying the condition $f_z(0)-1=f(0)=0$. It can be demonstrated that the sense-preserving characteristic implies $|b_1|<1$. The $\mathcal{S}_{\Upsilon}^0\subset\mathcal{S}_{\Upsilon}$ encompasses all functions of $\mathcal{S}_{\Upsilon}$ that satisfy the condition $f_z(0)=0$ (refer to [1]).

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Communicated by 'sub editor'

The geometric subclass and certain coefficient bounds in the class ϒ were examined in [1]. For more foundational outcomes, one may consult the standard introductory textbook [2] and explore additional insights from [3], [4]. Various researchers have unveiled a multitude of compelling findings in several articles, as indicated by references [5], [6], [7] and [8]. Moreover, researchers have extensively examined the related class and its subclasses in [9], [10] and [11].

For a function f in $\mathcal{S}$, the differential operator $\mathcal{D}^n (n \in \mathbb{N}_0)$ of f was defined in [11]. For $f = h + \bar{g}$ given by (1), in [10] the modified Sãlãgean operator of f was introduced as :

Drawing inspiration from the preceding studies of [12] and [9], the present study focuses on examining coefficient condition, convex combination, distortion, and extreme points.

$$\mathcal{D}^n f(z) = \mathcal{D}^n h(z) + (-1)\overline{\mathcal{D}^n g(z)},$$

where

$$\mathcal{D}^n h(z) = z + \sum_{s=2}^{\infty} s^n a_s z^s \quad \text{and} \quad \mathcal{D}^n g(z) = \sum_{s=1}^{\infty} s^n b_s z^s .$$

Next, for the functions $f = h + \bar{g} \in \mathcal{H}$ defined in (1), in [9] multiplier transformations were introduced, as the modified multiplier transformation of f is denoted as:

$$I_{\mu,\tau}^0 f(z) = \mathcal{D}^0 f(z) = h(z) + \overline{g(z)},$$

$$I_{\mu,\tau}^1 f(z) = \frac{\mu\, \mathcal{D}^0 f(z) + \tau\, \mathcal{D}^1 f(z)}{\mu + \tau} = \frac{\mu\left(h(z) + \overline{g(z)}\right) + \tau\left(h'(z) + \overline{g'(z)}\right)}{\mu + \tau}, \tag{2}$$

$$I_{\mu,\tau}^n f(z) = I_{\mu,\tau}^1 \left(I_{\mu,\tau}^{n-1} f(z)\right), (n \in \mathbb{N}_0). \tag{3}$$

For $0 \le \mu \le \tau$.If f is given by (1), then from (2) and (3) we see that

$$I_{\mu,\tau}^n f(z) = z + \sum_{s=2}^{\infty} (\frac{\tau s + \mu}{\mu + \tau})^n a_s z^s + (-1)^n \sum_{s=1}^{\infty} (\frac{\tau s - \mu}{\mu + \tau})^n \overline{b_s z^s} \tag{4}$$

Also, *as* f is given by (1):

$$I_{\mu,\tau}^n f(z) = f * \underbrace{\left(\Theta_1(z) + \overline{\Theta_2(z)}\right) * \dots * \left(\Theta_1(z) + \overline{\Theta_2(z)}\right)}_{n-times}$$

$$= h * \underbrace{\left(\Theta_1(z) * \dots * \Theta_1(z)\right)}_{n-times} + \overline{g * \underbrace{\Theta_2(z) * \dots * \Theta_2(z)}_{n-times}},$$

here " $*$ " represents power series convolution or the Hadamard product and

$$\Theta_1(z) = \frac{(\mu + \tau)z - \mu z^2}{(\mu + \tau)(1 - z)^2} \quad , \quad \Theta_2(z) = \frac{(\mu - \tau)z - \mu z^2}{(\mu + \tau)(1 - z)^2}.$$

The operators examined by various researchers are obtained through the parameter specialization for all $f \in \mathcal{H}$:

(i) $I_{0,1}^n f(z) = \mathcal{D}^n f(z)$ ([11]);
(ii) $I_{\lambda}^n f(z)$ ([9]);
(iii) $I_{1,1}^n f(z) = I^n f(z)$ ([13]) for $f \in \Upsilon$;
(iv) $I_{\mu,1}^n f(z) = I_{\mu}^n f(z)$ ([10]).

Definition 1.1. For $\rho \neq 0$ and $\rho \in \mathbb{C}$, with $|\rho| \le 1, \sigma \in \mathbb{R}, 0 \le \gamma \le 1$and $0 \le \beta < 1$, let $\mathcal{S}_\Upsilon(\mu, \tau, \sigma, n, \rho, \gamma, \beta)$ denoted the class of harmonic functions f given by (1) satisfying the condition:

$$Re\left\{1+\frac{1}{\rho}\left(\frac{(1+e^{i\sigma})\, I_{\mu,\tau}^{n+1} f(z)}{(1-\gamma)z+\gamma\, I_{\mu,\tau}^{n} f(z)}-e^{\sigma i}-1\right)\right\}\geq \beta \tag{5}$$

As $I_{\mu,\tau}^{n} f(z)$ is defined by (4).Further, by $\overline{\mathcal{S}_{\Upsilon}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$ the subclass of $\mathcal{S}_{\Upsilon}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$ harmonic functions $f_n(z)=h(z)+\overline{g_n(z)}$ so that $h(z)$ and $g_n(z)$ are of the form:

$$h(z)=z-\sum_{s=2}^{\infty} a_s z^s \quad , \quad g_n(z)=(-1)^n\sum_{s=1}^{\infty} b_s z^s \;, \quad a_s, b_s\geq 0 \tag{6}$$

Through the selection of appropriate parameter values, the class $\mathcal{S}_{\Upsilon}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$ is transformed into various subclasses of harmonic univalent functions.

(i) $\mathcal{S}_{\Upsilon}(0,1,\sigma,0,1,\gamma,\beta)=\mathcal{G}_{\Upsilon}(\sigma,\gamma,\beta)$ in ([14]).
(ii) $\mathcal{S}_{\Upsilon}(0,1,\sigma,0,1,1,\beta)=G_{\Upsilon}(\beta)$ in ([15]).
(i) $\mathcal{S}_{\Upsilon}(0,1,0,0,1,1,0)=\mathcal{S}H_{\Upsilon}^{*}(0)$in ([16], [17], [18]).
(ii) $\mathcal{S}_{\Upsilon}(0,1,0,0,1,1,\beta)=\mathcal{S}H_{\Upsilon}^{*}(\beta)$in ([19]).
(iii) $\mathcal{S}_{\Upsilon}(\mu,1,0,n,1,1,\beta)=\mathcal{S}H_{\Upsilon}(\mu,n,\beta)$ in([20]).
(iv) $\mathcal{S}_{\Upsilon}(0,1,0,n,1,1,0)=HK_{\Upsilon}(0)$ in ([16], [17], [18]).
(i) $\mathcal{S}_{\Upsilon}(0,1,0,1,1,1,\beta)=HK_{\Upsilon}(\beta)$in ([19]).
(ii) $\mathcal{S}_{\Upsilon}(0,1,0,n,1,1,\beta)=H_{\Upsilon}(n,\beta)$in ([4])
(iii) $\mathcal{S}_{\Upsilon}(0,1,\sigma,n,1,1,\beta)=R\mathcal{S}_{\Upsilon}(n,\beta)$ in ([21]).
(iv) $\mathcal{S}_{\Upsilon}(\mu,1,\sigma,n,1,1,\beta)=R\mathcal{S}_{\Upsilon}(\mu,n,\beta)$in ([22]).

$\mathcal{S}_{\Upsilon}^{0}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)=\mathcal{S}_{\Upsilon}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)\cap\mathcal{S}_{\Upsilon}^{0}$ and $\overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)=\overline{\mathcal{S}_{\Upsilon}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)\cap\mathcal{S}_{\Upsilon}^{0}$ is defined.

2. The Coefficient Condition

In this section, we demonstrate the sufficient condition for $f\in\mathcal{S}_{\Upsilon}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$ as indicated by the following result.

Theorem 2.1. If $f=h+\bar{g}$ such that h and g are defined by (1) with $b_1=0$. Moreover:

$$\sum_{s=2}^{\infty}\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n}|a_s| + \sum_{s=2}^{\infty}\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n}|b_s|\leq(1-\beta) \tag{7}$$

As: $\rho\neq 0$ and $\rho\in\mathbb{C}$, with $|\rho|\leq 1,\sigma\in\mathbb{R},0\leq\gamma\leq 1,\ 0\leq\mu\leq {}^{\tau}/_{2}, n\in\mathbb{N}_0$ and $\frac{\mu}{\mu+\tau}\leq\beta\leq\frac{\tau}{\mu+\tau}$. Then f is sense-preserving, harmonic univalent in unit disk Δ and $f\in\mathcal{S}_{\Upsilon}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$.

Proof. If z_1 and $z_2$are two distinct points then:

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right|\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|=1-\left|\frac{\sum_{s=1}^{\infty} b_s(z_1^s-z_2^s)}{(z_1-z_2)-\sum_{s=2}^{\infty} a_s(z_1^s-z_2^s)}\right|>1-\frac{\sum_{s=1}^{\infty} s|b_s|}{1-\sum_{s=2}^{\infty} s|a_s|}$$

$$\geq 1-\frac{\sum_{s=1}^{\infty}\frac{\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n}}{1-\beta}|b_s|}{1-\sum_{s=2}^{\infty}\frac{\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n}}{1-\beta}|a_s|}\geq 0$$

This shows univalent. f is sense-preserving in unit disk Δ because,

$$
\begin{aligned}
|h'(z)| &\geq 1-\sum_{s=2}^{\infty} s|a_s|\,|z|^{s-1} \\
&> 1-\sum_{s=2}^{\infty}\frac{1}{1-\beta}\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n |a_s| \\
&\geq \sum_{s=2}^{\infty}\frac{1}{1-\beta}\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n |b_s| \\
&> \sum_{s=2}^{\infty} s|b_s|\,|z|^{s-1} \\
&\geq |g'(z)|.
\end{aligned}
$$

Using the given condition that $|1-\beta+\omega| \geq |1+\beta+\omega|$ if and only if $Re(\omega)\geq\beta$, it is sufficient to demonstrate that $|1-\beta+\omega|-|1+\beta+\omega|\geq 0$ gives,

$$
\begin{aligned}
&\left|\gamma(2\rho-\rho\beta-1-e^{i\sigma})I_{\mu,\tau}^{n}f(z)+(1+e^{i\sigma})I_{\mu,\tau}^{n+1}f(z)+(1-\gamma)(2\rho-\rho\beta-1-e^{i\sigma})z\right| \\
&\quad -\left|\gamma(1+\rho\beta+e^{i\sigma})I_{\mu,\tau}^{n}f(z)-(1+e^{i\sigma})I_{\mu,\tau}^{n+1}f(z)+(1-\gamma)(1+\rho\beta+e^{i\sigma})z\right|\geq 0
\end{aligned} \tag{8}
$$

Substituting for $I_{\mu,\tau}^{n}f(z)$ and $I_{\mu,\tau}^{n+1}f(z)$ in (8) , we get

$$
\begin{aligned}
&\left|\begin{array}{l}\gamma\left(2\rho-\rho\beta-(1+e^{i\sigma})\right)\left(z+\sum_{s=2}^{\infty}\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n a_s z^s+(-1)^n\sum_{s=1}^{\infty}\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n\overline{b_s z^s}\right) \\ +(1+e^{i\sigma})\left(z+\sum_{s=2}^{\infty}\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n+1} a_s z^s+(-1)^{n+1}\sum_{k=1}^{\infty}\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n+1}\overline{b_s z^s}\right)+(1-\gamma)(2\rho-\rho\beta-1-e^{i\sigma})z\end{array}\right| \\
&-\left|\begin{array}{l}\gamma(1+\rho\beta+e^{i\sigma})\left(z+\sum_{s=2}^{\infty}\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n a_s z^s+(-1)^n\sum_{s=1}^{\infty}\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n\overline{b_s z^s}\right) \\ -(1+e^{i\sigma})\left(z+\sum_{s=2}^{\infty}\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n+1} a_s z^s+(-1)^{n+1}\sum_{s=1}^{\infty}\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n+1}\overline{b_s z^s}\right)+(1-\gamma)(1+\rho\beta+e^{i\sigma})z\end{array}\right| \\
&\geq 2(1-\beta)|z|-\sum_{s=2}^{\infty}\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(2-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n|a_s||z|^s \\
&\quad-\sum_{s=2}^{\infty}\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(2-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n|b_s||z|^s \\
&\quad-\sum_{s=2}^{\infty}\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho\beta\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n|a_s||z|^s \\
&\quad-\sum_{s=2}^{\infty}\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho\beta\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n|b_s||z|^s \\
&\geq 2(1-\beta)|z|\left\{\begin{array}{l}1-\sum_{s=2}^{\infty}\frac{1}{1-\beta}\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n|a_s| \\ \sum_{s=2}^{\infty}\frac{1}{1-\beta}\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n|b_s|\end{array}\right\}.
\end{aligned}
$$

The final result is non-negative by (7), thus concluding the demonstration.

Theorem 2.2. If $f_n = h+\overline{g_n}$ defined by (6) with $b_1=0$. Then $f\in\overline{S_Y}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$ if and only if

$$
\begin{aligned}
&\sum_{s=2}^{\infty}\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n a_s \\
&+\sum_{s=2}^{\infty}\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n b_s \leq (1-\beta)
\end{aligned} \tag{9}
$$

As: $\rho\neq 0$ and $\rho\in\mathbb{C}$, with $|\rho|\leq 1, \sigma\in\mathbb{R}, 0\leq\gamma\leq 1, 0\leq\mu\leq {}^{\tau}/_{2}, n\in\mathbb{N}_0$ and $\frac{\mu}{\mu+\tau}\leq\beta\leq\frac{\tau}{\mu+\tau}$.

Proof. since $\overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta) \subset \mathcal{S}_{\Upsilon}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$. then the " if " part follows from Theorem 2.1 not that if the functions h and g in $f = h + \bar{g} \in \mathcal{S}_{\Upsilon}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$ are given in (6) then $f \in \overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$. For the " only if " part, by contradiction, $f \notin \overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$ if the condition (5) dose not hold.
Thus:

$$Re\left\{\frac{(1-\beta)z - \left(\begin{array}{c}\sum_{s=2}^{\infty}\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n} a_s z^s + \\ \sum_{s=2}^{\infty}\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n} b_s \overline{z^s}\end{array}\right)}{1-\sum_{s=2}^{\infty}\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n} a_s z^s + \sum_{s=2}^{\infty}\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n} b_s \overline{z^s}}\right\} \geq 0.$$

The above condition satisfies all values of , $|z| = r < 1$. By choosing z on the positive real axis $(0 \leq z = r < 1)$,:

$$Re\left\{\frac{(1-\beta) - \left(\begin{array}{c}\sum_{s=2}^{\infty}\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n} a_s r^{s-1} + \\ \sum_{s=2}^{\infty}\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n} b_s r^{s-1}\end{array}\right)}{1-\sum_{s=2}^{\infty}\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n} a_s r^{s-1} + \sum_{s=2}^{\infty}\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n} b_s r^{s-1}}\right\} \geq 0. \quad (10)$$

If the condition (9) is not satisfied, the numerator in (10) is negative. This contradicts with$f \in \overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$. Here, the proof is complete.

3. Extreme points

To examine the extreme points of the function $f_n \in \overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$, we utilize the coefficient condition obtained in Section 2.
Theorem 3.1. Let $f_n$by given by (2) then $f_n \in \overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$ if and only if

$$f_n(z) = \sum_{s=1}^{\infty}\left(\mathcal{X}_s h_s(z) + \mathcal{Y}_s g_{n_s}(z)\right),$$

$$where, h_1(z) = z \quad , \quad h_s(z) = z - \frac{1-\beta}{\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n}} z^s,$$

$$and\ g_{n_1}(z) = z \quad , \quad g_{n_s}(z) = z + (-1)^n \frac{1-\beta}{\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n}} \overline{z^s},$$

$\mathcal{X}_s \geq 0, \mathcal{Y}_s \geq 0, \sum_{s=1}^{\infty}(\mathcal{X}_s + \mathcal{Y}_s) = 1, \rho \neq 0$ and $\rho \in \mathbb{C}$, with $|\rho| \leq 1, \sigma \in \mathbb{R}, 0 \leq \gamma \leq 1,$ $0 \leq \mu \leq \tau/2, n \in \mathbb{N}_0,$ $\frac{\mu}{\mu+\tau} \leq \beta \leq \frac{\tau}{\mu+\tau}$ and $(s = 2,3,\ldots)$.

Specially, the extreme points of $f_n \in \overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$ are $\{h_s\}$and $\{g_{n_s}\}$.

Proof. From (6), for functions $f_n$as:

$$f_n(z)=\sum_{k=1}^{\infty}\left(\mathcal{X}_s h_s(z)+\mathcal{Y}_s g_{n_s}(z)\right)=\sum_{s=1}^{\infty}(\mathcal{X}_s+\mathcal{Y}_s)z-\sum_{s=2}^{\infty}\frac{1-\beta}{\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n}\mathcal{X}_s z^s$$
$$+(-1)\sum_{s=2}^{\infty}\frac{1-\beta}{\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n}\mathcal{Y}_s\overline{z^s}.$$

Then:

$$\sum_{s=2}^{\infty}\left(\frac{\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n}{1-\beta}\right)\left(\frac{1-\beta}{\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n}\mathcal{X}_s\right)$$
$$+\sum_{s=2}^{\infty}\left(\frac{\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n}{1-\beta}\right)\left(\frac{1-\beta}{\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n}\mathcal{Y}_s\right)$$
$$=\sum_{s=2}^{\infty}\mathcal{X}_s+\sum_{s=2}^{\infty}\mathcal{Y}_s=1-\mathcal{X}_1-\mathcal{Y}_1\le 1$$

and so $f_n\in\overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$. Conversely, if $f_n\in\overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$, then:

$$a_s\le\frac{1-\beta}{\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n}$$
$$and\quad b_s\le\frac{1-\beta}{\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n}.$$

$$\mathcal{X}_s=\frac{\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n}{1-\beta}a_s,\quad(s=2,3,\dots),$$
$$\mathcal{Y}_s=\frac{\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n}{1-\beta}b_k,\quad(s=2,3,\dots),$$
$$and\ \ \mathcal{X}_1+\mathcal{Y}_1=1-\sum_{s=1}^{\infty}(\mathcal{X}_s+\mathcal{Y}_s),$$

As: $\mathcal{X}_s,\mathcal{Y}_s\ge 0$.Then, as necessary, we obtain

$$f_n(z)=(\mathcal{X}_1+\mathcal{Y}_1)z+\sum_{s=1}^{\infty}\mathcal{X}_s h_s(z)+\mathcal{Y}_s g_{n_s}(z)$$
$$=\sum_{s=1}^{\infty}\left(\mathcal{X}_s h_s(z)+\mathcal{Y}_s g_{n_s}(z)\right).$$

4. Distortion and Convex Combination

The Theorem outlined below demonstrates that the $\overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$ remains invariant under distortion and convex combinations of its numbers.

Theorem 4.1. Let $f_n \in \overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$. Then for $|z| = r < 1$ and $\rho \neq 0$, $\rho \in \mathbb{C}$, with $|\rho| \leq 1, \sigma \in \mathbb{R}, 0 \leq \gamma \leq 1$, $0 \leq \mu \leq \tau/2$, $n \in \mathbb{N}_0$ *and* $\frac{\mu}{\mu+\tau} \leq \beta \leq \frac{\tau}{\mu+\tau}$ we have

$$|f_n(z)| \leq r + \frac{1-\beta}{\left[\frac{2[(2\tau+\mu)-\gamma(\mu+\tau)]}{\mu+\tau} + \gamma\rho(1-\beta)\right]\left(\frac{2\tau+\mu}{\mu+\tau}\right)^n} r^2$$

and

$$|f_n(z)| \geq r - \frac{1-\beta}{\left[\frac{2[(2\tau+\mu)-\gamma(\mu+\tau)]}{\mu+\tau} + \gamma\rho(1-\beta)\right]\left(\frac{2\tau+\mu}{\mu+\tau}\right)^n} r^2$$

Proof. To establish the validity of the left-hand side, we assume that $f_n \in \overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$ then:

$$|f_n(z)| \leq r + \sum_{s=2}^{\infty}(a_s + b_s)r^2$$

$$\leq r + \frac{(1-\beta)r^2}{\left[\frac{2[(2\tau+\mu)-\gamma(\mu+\tau)]}{\mu+\tau} + \gamma\rho(1-\beta)\right]\left(\frac{2\tau+\mu}{\mu+\tau}\right)^n} \sum_{s=2}^{\infty}\left\{\begin{array}{c}\frac{\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau} + \gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^n}{1-\beta} a_s \\ \frac{\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau} - \gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^n}{1-\beta} b_s\end{array}\right\}$$

$$\leq r + \frac{1-\beta}{\left[\frac{2[(2\tau+\mu)-\gamma(\mu+\tau)]}{\mu+\tau} + \gamma\rho(1-\beta)\right]\left(\frac{2\tau+\mu}{\mu+\tau}\right)^n} r^2$$

Similarly, the validity of the right-hand is demonstrated.

Corollary 4.2. Let f_n of type (6)be so that, $f_n \in \overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$, where $0 \leq \gamma \leq 1$, $0 \leq \mu \leq \tau/2$, $n \in \mathbb{N}_0$ *and* $\frac{\mu}{\mu+\tau} \leq \beta \leq \frac{\tau}{\mu+\tau}$. then

$$\left\{w : |w| < 1 - \frac{1-\beta}{\left[\frac{2[(2\tau+\mu)-\gamma(\mu+\tau)]}{\mu+\tau} + \gamma\rho(1-\beta)\right]\left(\frac{2\tau+\mu}{\mu+\tau}\right)^n}\right\} \subset f_n(\Delta).$$

Theorem 4.3. The class $\overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$is closed under convex combinations.

Proof . Let $f_{n_i} \in \overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$for $(t = 1,2,\dots)$ is given by:

$$f_{n_t}(z) = z - \sum_{s=2}^{\infty} a_{s_t} z^s + (-1)\sum_{s=2}^{\infty} b_{s_t} \bar{z}^s.$$

Then by (9),

$$\sum_{s=2}^{\infty}\frac{\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n}}{1-\beta}a_{s_t}\quad+\sum_{s=2}^{\infty}\frac{\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n}}{1-\beta}b_{s_t}\leq 1. \tag{11}$$

For $\sum_{t=1}^{\infty} q_t = 1, 0 < q_t < 1$,we express the convex combination of f_{n_t} as follows

$$\sum_{t=1}^{\infty} q_t\, f_{n_t}(z) = z-\sum_{s=2}^{\infty}\left(\sum_{t=1}^{\infty} q_t\, a_{s_t}\right)z^s+(-1)\sum_{s=2}^{\infty}\left(\sum_{t=1}^{\infty} q_t\, b_{s_t}\right)\overline{z^s}.$$

Then by (11),

$$\begin{aligned}
&\sum_{s=2}^{\infty}\frac{\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n}}{1-\beta}\left(\sum_{t=1}^{\infty} q_t\, a_{s_t}\right)\\
&+\sum_{s=2}^{\infty}\frac{\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n}}{1-\beta}\left(\sum_{t=1}^{\infty} q_t\, b_{s_t}\right)\\
&=\sum_{t=1}^{\infty} q_t\left\{\sum_{s=2}^{\infty}\frac{\left[\frac{2[(\tau s+\mu)-\gamma(\mu+\tau)]}{\mu+\tau}+\gamma\rho(1-\beta)\right]\left(\frac{\tau s+\mu}{\mu+\tau}\right)^{n}}{1-\beta}a_{s_t}\right.\\
&\left.+\sum_{s=2}^{\infty}\frac{\left[\frac{2[(\tau s-\mu)+\gamma(\mu+\tau)]}{\mu+\tau}-\gamma\rho(1-\beta)\right]\left(\frac{\tau s-\mu}{\mu+\tau}\right)^{n}}{1-\beta}b_{s_t}\right\}
\end{aligned}$$

$$\leq \sum_{t=1}^{\infty} q_t = 1.$$

This is the condition required by (9) and so $\sum_{i=1}^{\infty} q_i\, f_{n_i}(z) \in \overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$.

5. Conclusion

This article introduces a subclass of harmonic univalent functions defined by linear operator $I_{\mu,\tau}^{n} f(z)$.Furthermore,various intriguing outcomes are examined, such as the coefficient bound for the subclass $\overline{\mathcal{S}_{\Upsilon}^{0}}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$.Subsequently, geometric properties pertaining to the considered function, including coefficient bound, extreme points, distortion and convex combinations in connection to the subclass $\mathcal{S}_{\Upsilon}(\mu,\tau,\sigma,n,\rho,\gamma,\beta)$.

Acknowledgements

The authors express their gratitude to the reviewers and editors for their valuable feedback and suggestions.

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