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Bounds on coefficients for a class of Analytic functions Defined by Quasi-Subordination

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ABSTRACT

This study delineates particular subclasses of analytic univalent functions linked to quasi-subordination and establishes results including coefficient bounds and Fekete-Szego problem for functions inside these subclasses.

MSC..

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1. Introduction

Letting $\mathcal{M}$ denote the class of analytic functions defined on the open unit disc $\mathfrak{U} = \{\mathfrak{z} : |\mathfrak{z}| < 1\}$, normalized by the conditions $\mathcal{F}(0) = 0$ and $\mathcal{F}'(0) = 1$. An analytic function $\mathcal{F} \in \mathcal{M}$ possesses a Taylor series expansion represented as:

$$\mathcal{F}(\mathfrak{z}) = \mathfrak{z} + \sum_{j=2}^{\infty} a_j \mathfrak{z}^j , \qquad (\mathfrak{z} \in \mathfrak{U}) \tag{1.1}$$

Let $\mathcal{F}$and g be two analytic functions in $\mathfrak{U}$. Then $\mathcal{F}$ is said to be subordinate to g, which is written as

$$\mathcal{F} \prec g \text{ or } \mathcal{F}(\mathfrak{z}) \prec g(\mathfrak{z}) \qquad (\mathfrak{z} \in \mathfrak{U}). \tag{1.2}$$

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If there exists a Schwarz function w analytic in $\mathfrak{A}$, such that $w(0)=0$ and $|w(z)|<1$, with $\mathcal{F}(z)=g(w(z))$. Moreover, if the function g is univalent in A, then $\mathcal{F}(z)\prec g(z)$ is equivalent to $\mathcal{F}(0)=g(0)$ and $\mathcal{F}(\mathfrak{A})$ is a subset of $g(\mathfrak{A})$. An analytic function f quasi-subordinate to an analytic function g in the open unit disk $\mathfrak{A}$, is shown by

$$\mathcal{F}(z)\prec_q g(z).$$

With the existence of analytic functions φ along with w, where $|\varphi(z)|<1, w(0)=0$, and $|w(z)|<1$, resulting in $\mathcal{F}(z)=\varphi(z)g\big(w(z)\big)$,

Note, when $\varphi(z)=1$, then $\mathcal{F}(z)=g(w(z))$ so that $\mathcal{F}(z)\prec g(z)$ in $\mathfrak{A}$. Furthermore, if $w(z)=z$, then $\mathcal{F}(z)=\varphi(z)g(z)$, and in this case, $\mathcal{F}$ is majorized by g. Written $\mathcal{F}(z)\ll g(z)$ in $\mathfrak{A}$. Therefore, it is clear that quasi-subordination serves as a generalisation of both subordination and majorization; refer to ([1],[2],[5],[6],[7],[8],[12],[13],[14],[15] and [16]). The term quasi-subordination was initially introduced in 1970 by [12]. Ma and Minda [11] established a category of starlike functions by the method of subordination and examined the classes $S^*(\varphi)$ and $G^*(\varphi)$, defined as follows:

$$S^*(\varphi)=\left\{\mathcal{F}\in\mathcal{H}:\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\prec\varphi(z), z\in\mathfrak{A}\right\}$$

and

$$G^*(\varphi)=\left\{\mathcal{F}\in\mathcal{H}:1+\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}\prec\varphi(z), z\in\mathfrak{A}\right\},$$

where every coefficient is real. Additionally, let θ be a univalent, analytic function with a positive real portion in $\mathfrak{A}$, such that $\theta(0)=1, \theta'(0)>0$, and φ maps the open unit disk $\mathfrak{A}$ onto an area that is symmetric concerning the real axis and starlike concerning 1. The form of Taylor's series expansion for such a function is:

$$\theta(z)=1+B_1z+B_2z^2+B_3z^3+\cdots, B_1>0$$

and

$$\varphi(z)=C_0+C_1z+C_2z^2+\cdots.$$

Geometric function theory has a long history with the Fekete-Szego functional $|a_3-\mu a_2^2|$ for normalized univalent functions of the kind provided by (1.1).

Since it has drawn a lot of attention, especially in relation to numerous subclasses of the class of normalized analytic and univalent functions [4].

Definition(1.1): A function $\mathcal{F}\in\mathcal{M}$ is said to be in the class $S^q_{\varrho,\tau}(\theta)$, $(\tau>0, \varrho\geq 1)$, satisfies the following quasi-subordination:

$$\left(\frac{\mathcal{F}(z)}{z}+\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1\right)^{\tau}+\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{\varrho}\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1\right)^{\tau}\right)-2\prec_q\theta(z)-1.$$

Definition(1.2): A function $\mathcal{F}\in\mathcal{M}$ is said to be in the class $R^q_{\sigma,r}(\delta,\varepsilon,\theta)$, $(\varrho\geq 1, \tau\geq 1, 0\leq\delta<1)$ and $\varepsilon>1$ satisfies the following quasi-subordination:

$$(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)})^{\varrho}(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1)^{\tau}+\left(\frac{z\mathcal{F}'(z)+\delta z^2\mathcal{F}''(z)}{\mathcal{F}(z)}\right)+(1-\varepsilon)(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1)\prec_q\theta(z)-1.$$

Lemma (1.1)[3]: Consider w be an analytic function within $\mathfrak{A}$, such that $w(0)=0, |w(z)|<1$ with

$$w(z)=w_1z+w_2z^2+w_3z^3+\cdots,$$

then

$$|w_2+tw_1^2|\leq\max\{1,|t|\}, t\in\mathbb{C}$$

The result is sharp for the function $w(\mathfrak{z}) = \mathfrak{z}^2$ or $w(\mathfrak{z}) = \mathfrak{z}$.

Lemma(1.2) [9]: Consider $\varphi(\mathfrak{z})$ be an analytic function within $\mathfrak{A}$, such that $|\varphi(\mathfrak{z})| < 1$ with

$$\varphi(\mathfrak{z}) = C_0 + C_1\mathfrak{z} + C_2\mathfrak{z}^2 + C_3\mathfrak{z}^3 + \cdots.$$

Then $|C_0| < 1$ and $|C_n| < 1 - |C_0|$ for $n > 0$.

2. Main Results

Theorem(2.1): If $\mathcal{F}$ *is* given by (1.1) and belongs to the class $S^q_{\varrho,\tau}(\theta)$, then

$$|a_2| \le \frac{|B_1|}{1 + \varrho + 3\tau}, \tag{2.1}$$

and

$$|a_3| \le \frac{|B_1|}{1 + 18\tau + 4\varrho}[1 + \max\left\{1, \frac{\left[4(\tau^2 - 3\tau) + \frac{1}{2}((\varrho + 2\tau)^2 - 3(\varrho + 4\tau))\right]}{(1 + \varrho + 3\tau)^2}|B_1| + \left|\frac{B_2}{B_1}\right|\right\}. \tag{2.2}$$

and for some $\mu \in \mathbb{C}$, we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{|B_1|}{1+18\tau+4\varrho}[1 + \max\left\{1, \frac{\left[4(\tau^2-3\tau)+\frac{1}{2}((\varrho+2\tau)^2-3(\varrho+4\tau))\right]}{(1+\varrho+3\tau)^2}|B_1| - \frac{1+18\tau+4\varrho}{(1+\varrho+3\tau)^2}\mu|B_1| + \left|\frac{B_2}{B_1}\right|\right\}. \tag{2.3}$$

Proof: If $\mathcal{F} \in S^q_{\varrho,\tau}(\theta)$, then there exists an analytic function θ in $\mathfrak{A}$ such that $|\theta(\mathfrak{z})| \le 1$ and $w: \mathfrak{A} \to \mathfrak{A}$, with $w(0) = 0$ and $|w(\mathfrak{z})| < 1$ such that :

$$\left(\frac{\mathcal{F}(\mathfrak{z})}{\mathfrak{z}} + \left(\frac{\mathfrak{z}\mathcal{F}''(\mathfrak{z})}{\mathcal{F}(\mathfrak{z})} + 1\right)^\tau + \left(\frac{\mathfrak{z}\mathcal{F}'(\mathfrak{z})}{\mathcal{F}(\mathfrak{z})}\right)^\varrho \left(\frac{\mathfrak{z}\mathcal{F}''(\mathfrak{z})}{\mathcal{F}(\mathfrak{z})} + 1\right)^\tau\right) - 2 = \emptyset(\mathfrak{z})\big(\theta\big(w(\mathfrak{z})\big) - 1\big) \tag{2.4}$$

since

$$\frac{\mathcal{F}(\mathfrak{z})}{\mathfrak{z}} + \left(\frac{\mathfrak{z}\mathcal{F}''(\mathfrak{z})}{\mathcal{F}(\mathfrak{z})} + 1\right)^\tau = 2 + (1 + 2\tau)a_2\mathfrak{z} + (2(\tau^2 - 3\tau)a_2^2 + (1 + 6\tau)a_3)\mathfrak{z}^2 + \cdots, \tag{2.5}$$

and

$$\left(\frac{\mathfrak{z}\mathcal{F}'(\mathfrak{z})}{\mathcal{F}(\mathfrak{z})}\right)^\varrho \left(\frac{\mathfrak{z}\mathcal{F}''(\mathfrak{z})}{\mathcal{F}(\mathfrak{z})} + 1\right)^\tau = 1 + (\varrho + \tau)a_2\mathfrak{z} + \left(\frac{1}{2}((\varrho + 2\tau)^2 - 3(\varrho + 4\tau))a_2^2 + 4(\varrho + 3\tau)a_3\right)\mathfrak{z}^2 + \cdots, \tag{2.6}$$

put (2.6) and (2.5) in (2.4) as equal coefficients on both sides, and we get

$$\left(\frac{\mathcal{F}(\mathfrak{z})}{\mathfrak{z}} + \left(\frac{\mathfrak{z}\mathcal{F}''(\mathfrak{z})}{\mathcal{F}(\mathfrak{z})} + 1\right)^\tau + \left(\frac{\mathfrak{z}\mathcal{F}'(\mathfrak{z})}{\mathcal{F}(\mathfrak{z})}\right)^\varrho \left(\frac{\mathfrak{z}\mathcal{F}''(\mathfrak{z})}{\mathcal{F}(\mathfrak{z})} + 1\right)^\tau\right) - 2 = 1 +$$

$$(1 + \varrho + 3\tau)a_2\mathfrak{z} + \left[2(\tau^2 - 3\tau) + \frac{1}{2}((\varrho + 2\tau)^2 - 3(\varrho + 4\tau))\right]a_2^2\mathfrak{z}^2 + (1 + 6\tau) + (4(\varrho + 3\tau))a_3\mathfrak{z}^2 + \cdots, \tag{2.7}$$

$$\emptyset(\mathfrak{z})\big(\theta\big(w(\mathfrak{z})\big) - 1\big) = C_0w_1B_1 + C_1w_1B_1 + \big(C_0(w_2B_1 + w_1^2B_2)\big) \tag{2.8}$$

$$a_2 = \frac{1}{1 + \varrho + 3\tau}C_0w_1B_1, \tag{2.9}$$

and

$$a_3 = \frac{1}{1+18\tau+4\varrho}\left[C_1 w_1 B_1 + \left(C_0(w_2 B_1 + w_1^2 B_2)\right) - \frac{\left[4(\tau^2-3\tau)+\frac{1}{2}((\varrho+2\tau)^2-3(\varrho+4\tau))\right]}{(1+\varrho+3\tau)^2} C_0^2 w_1^2 B_1^2\right], \qquad (2.10)$$

also

$$a_3 - \mu a_2^2 = \frac{1}{1+18\tau+4\varrho}[C_1 w_1 B_1 + \left(C_0(w_2 B_1 + w_1^2 B_2)\right)$$

$$-\frac{\left[4(\tau^2-3\tau)+\frac{1}{2}((\varrho+2\tau)^2-3(\varrho+4\tau))\right]}{(1+\varrho+3\tau)^2} C_0^2 w_1^2 B_1^2] - \frac{1+18\tau+4\varrho}{(\varrho+3\tau)^2}\mu C_0^2 w_1^2 B_1^2, \qquad (2.11)$$

Utilizing this fact along with the established inequalities $|C_0| \le 1$ and $|C_1| \le 1$, $|w_1| \le 1$, as well as Lemma (1.1), we derive,

$$|a_2| \le \frac{|\mathcal{B}_1|}{1+\varrho+3\tau},$$

and

$$|a_3| \le \frac{|\mathcal{B}_1|}{1+18\tau+4\varrho}[1 + \max\left\{1, \frac{\left[4(\tau^2-3\tau)+\frac{1}{2}((\varrho+2\tau)^2-3(\varrho+4\tau))\right]}{(1+\varrho+3\tau)^2}|\mathcal{B}_1| + \left|\frac{\mathcal{B}_2}{\mathcal{B}_1}\right|\right\}.$$

and for some $\mu \in \mathbb{C}$, we have

$$|a_3 - \mu a_2^2| = \le \frac{|\mathcal{B}_1|}{1+18\tau+4\varrho}[1 + \max\left\{1, \frac{\left[4(\tau^2-3\tau)+\frac{1}{2}((\varrho+2\tau)^2-3(\varrho+4\tau))\right]}{(1+\varrho+3\tau)^2}|\mathcal{B}_1| + \left|\frac{\mathcal{B}_2}{\mathcal{B}_1}\right| - \frac{1+18\tau+4\varrho}{(1+\varrho+3\tau)^2}\mu|\mathcal{B}_1|\right\}.$$

Theorem(2.2): If $\mathcal{F} \in \mathcal{M}$ satisfies

$$\left(\frac{\mathcal{F}(z)}{z} + \left(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1\right)^{\tau} + \left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{\varrho}\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1\right)^{\tau}\right) - 2 \ll \theta(z) - 1.$$

then

$$|a_2| \le \frac{|\mathcal{B}_1|}{1+\varrho+3\tau},$$

$$|a_3| \le \frac{1}{1+18\tau+4\varrho}\left[|\mathcal{B}_1| + \frac{\left[4(\tau^2-3\tau)+\frac{1}{2}((\varrho+2\tau)^2-3(\varrho+4\tau))\right]}{(1+\varrho+3\tau)^2}|\mathcal{B}_1| + |\mathcal{B}_2|\right],$$

and

$$|a_3 - \mu a_2^2| \le \frac{1}{1+18\tau+4\varrho}\left[|\mathcal{B}_1| + \frac{\left[4(\tau^2-3\tau)+\frac{1}{2}((\varrho+2\tau)^2-3(\varrho+4\tau))\right]}{(1+\varrho+3\tau)^2}|\mathcal{B}_1| + |\mathcal{B}_2| - \frac{1+18\tau+4\varrho}{(1+\varrho+3\tau)^2}\mu|\mathcal{B}_1|^2\right].$$

Proof: The results are derived by substituting $w(z) = z$ in the evidence of Theorem (2.1).

Substituting $\varrho = 1$ into Theorem (2.1) yields the subsequent consequence.

Corollary(2.1): If $\mathcal{F}$ is defined using (1.1) in the $S_{1,\tau}^{q}(\theta)$, subsequently

$$|a_2| \leq \frac{|\mathcal{B}_1|}{2+3\tau},$$

$$|a_3| \leq \frac{|\mathcal{B}_1|}{5+18\tau}[1+\max\left\{1, \frac{\left[4(\tau^2-3\tau)+\frac{1}{2}((1+2\tau)^2-3(1+4\tau))\right]}{(2+3\tau)^2}|\mathcal{B}_1|+\left|\frac{\mathcal{B}_2}{\mathcal{B}_1}\right|\right\}.$$

and for some $\mu \in \mathbb{C}$, we have

$$|a_3-\mu a_2^2| \leq \frac{|\mathcal{B}_1|}{5+18\tau}[1 + \max\left\{1, \frac{\left[4(\tau^2-3\tau)+\frac{1}{2}((1+2\tau)^2-3(1+4\tau))\right]}{(2+3\tau)^2}|\mathcal{B}_1|+\left|\frac{\mathcal{B}_2}{\mathcal{B}_1}\right|-\frac{1+6\tau+4(1+3\tau)}{(2+3\tau)^2}\mu|\mathcal{B}_1|\right\}.$$

Corollary(2.2): If $\mathcal{F}$ is defined using (1.1) in the $S_{\varrho,1}^{q}(\theta)$, subsequently

$$|a_2| \leq \frac{|\mathcal{B}_1|}{4+\varrho},$$

and

$$|a_3| \leq \frac{|\mathcal{B}_1|}{4\varrho+19}[1+\max\left\{1, \frac{\left[-8+\frac{1}{2}((\varrho+2)^2-3(\varrho+4))\right]}{(4+\varrho)^2}|\mathcal{B}_1|+\left|\frac{\mathcal{B}_2}{\mathcal{B}_1}\right|\right\}.$$

and for some $\mu \in \mathbb{C}$, we have

$$|a_3-\mu a_2^2| \leq \frac{|\mathcal{B}_1|}{4\varrho+19}[1+\max\left\{1, \frac{\left[-8+\frac{1}{2}((\varrho+2)^2-3(\varrho+4))\right]}{(4+\varrho)^2}|\mathcal{B}_1|+\left|\frac{\mathcal{B}_2}{\mathcal{B}_1}\right|-\frac{4\varrho+19}{(4+\varrho)^2}\mu|\mathcal{B}_1|\right\}.$$

Remark(2.1): For $\tau = 0, \varrho \geq 1$, and $(0 \leq \delta < 1)$, it is classified within the subclass $S_{\varrho}^{q}(\theta)$ if the subsequent quasi-subordination criteria is fulfilled:

$$\left(\frac{\mathcal{F}(z)}{z}+\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{\varrho}\right)-1 \prec_q \theta(z)-1.$$

The subsequent section presents estimates for the coefficients $|a_2|$ and $|a_3|$, also $|a_3-\mu a_2^2|$ for some $\mu \in \mathbb{C}$, related to the function within the class $S_{\varrho}^{q}(\theta)$.

Theorem(2.3): If $\mathcal{F}$ is defined by (1.1), it is categorized under the subclass $\mathcal{R}_{\sigma,r}^{q}(\delta,\varepsilon,\theta)$. Subsequently

$$|a_2| \leq \frac{|\mathcal{B}_1|}{((\sigma+r)+3(2\delta-\varepsilon))}, \tag{2.12}$$

and

$$|a_3| \leq \frac{1}{((\sigma+3r)+8+3\delta+6\varepsilon)}[|\mathcal{B}_1|+\max\left\{|\mathcal{B}_1|, \frac{\left(\frac{1}{2}\left((\sigma+2r)^2-3(\sigma+4r)\right)-23-2\delta-\varepsilon\right)}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)^2}|\mathcal{B}_1^2|+|\mathcal{B}_2|\right\}. \tag{2.13}$$

and for some $\mu \in \mathbb{C}$, we have

$$|a_3-\mu a_2^2| \leq \frac{1}{((\sigma+3r)+8+3\delta+6\varepsilon)}[|\mathcal{B}_1|$$

$$+\max\left\{|\mathcal{B}_1|,\frac{\left(\frac{1}{2}\left((\sigma+2r)^2-3(\sigma+4r)\right)-23-2\delta-\varepsilon\right)}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)^2}|\mathcal{B}_1^2|+\frac{\left((\sigma+3r)+8+3\delta+6\varepsilon\right)}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)^2}\mu|\mathcal{B}_1^2|+|\mathcal{B}_2|\right\}. \quad (2.14)$$

Proof: If $\mathcal{F}\in\mathcal{R}_{\sigma,r}^{q}(\delta,\varepsilon,\theta)$, then there exists an analytic function θ in $\mathfrak{A}$ with $|\theta(z)|\le 1$ and a mapping $w:\mathfrak{A}\to\mathfrak{A}$, such that $w(0)=0$ and $|w(z)|<1$, satisfying the following conditions:

$$\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{\varrho}\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1\right)^{\tau}+\left(\frac{z\mathcal{F}'(z)+\delta z^2\mathcal{F}''(z)}{\mathcal{F}(z)}\right)+(1-\varepsilon)\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1\right)=\emptyset(z)\big(\theta(w(z))-1\big), \quad (2.15)$$

since

$$\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{\sigma}\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1\right)^{r}+\left(\frac{z\mathcal{F}'(z)+\delta z^2\mathcal{F}''(z)}{\mathcal{F}(z)}\right)+(1-\varepsilon)\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1\right)=(1-\varepsilon)+\left((\sigma+r)+3(2\delta-\varepsilon)\right)a_2 z+$$

$$\left((\sigma+3r)+8+3\delta+6\varepsilon\right)a_3 z^2+\left(\frac{1}{2}\left((\sigma+2r)^2-3(\sigma+4r)\right)-23-2\delta-\varepsilon\right)a_2^2 z^2+\cdots. \quad (2.16)$$

By substituting (2.8) and (2.13) and equal the coefficients on both sides, we obtain

$$a_2=\frac{1}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)}C_0 w_1\mathcal{B}_1, \quad (2.17)$$

and

$$a_3=\frac{1}{\left((\sigma+3r)+8+3\delta+6\varepsilon\right)}\left[C_1 w_1\mathcal{B}_1+\left(C_0(w_2\mathcal{B}_1+w_1^2\mathcal{B}_2)\right)-\frac{\left(\frac{1}{2}\left((\sigma+2r)^2-3(\sigma+4r)\right)-23-2\delta-\varepsilon\right)}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)^2}C_0^2 w_1^2\mathcal{B}_1^2\right], \quad (2.18)$$

also

$$a_3-\mu a_2^2=$$
$$\frac{1}{\left((\sigma+3r)+8+3\delta+6\varepsilon\right)}\left[C_1 w_1\mathcal{B}_1+\left(C_0(w_2\mathcal{B}_1+w_1^2\mathcal{B}_2)\right)-\frac{\left(\frac{1}{2}\left((\sigma+2r)^2-3(\sigma+4r)\right)-23-2\delta-\varepsilon\right)}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)^2}C_0^2 w_1^2\mathcal{B}_1^2-\frac{\left((\sigma+3r)+8+3\delta+6\varepsilon\right)}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)^2}\mu C_0^2 w_1^2\mathcal{B}_1^2\right], \quad (2.19)$$

Utilizing this knowledge alongside the established inequalities, $|\mathbb{C}_0|\le 1$ and $|\mathbb{C}_1|\le 1$, $|w_1|\le 1$, and Lemma (1.1), we derive

$$|a_2|\le\frac{|\mathcal{B}_1|}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)},$$

and

$$|a_3|\le\frac{1}{\left((\sigma+3r)+8+3\delta+6\varepsilon\right)}\left[|\mathcal{B}_1|+\max\left\{|\mathcal{B}_1|,\frac{\left(\frac{1}{2}\left((\sigma+2r)^2-3(\sigma+4r)\right)-23-2\delta-\varepsilon\right)}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)^2}|\mathcal{B}_1^2|+|\mathcal{B}_2|\right\}.\right.$$

Further

$$|a_3-\mu a_2^2|\le\frac{1}{\left((\sigma+3r)+8+3\delta+6\varepsilon\right)}\left[|\mathcal{B}_1|+\max\left\{|\mathcal{B}_1|,\frac{\left(\frac{1}{2}\left((\sigma+2r)^2-3(\sigma+4r)\right)-23-2\delta-\varepsilon\right)}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)^2}|\mathcal{B}_1^2|+\frac{\left((\sigma+3r)+8+3\delta+6\varepsilon\right)}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)^2}\mu|\mathcal{B}_1^2|+|\mathcal{B}_2|\right\}.\right.$$

Theorem(2.4): If $\mathcal{F}\in\mathcal{H}$ satisfies

$$\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{\varrho}\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1\right)^{\tau}+\left(\frac{z\mathcal{F}'(z)+\delta z^2\mathcal{F}''(z)}{\mathcal{F}(z)}\right)+(1-\varepsilon)\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}(z)}+1\right)\ll\theta(z)-1,$$

then

$$|a_2| \leq \frac{|\mathcal{B}_1|}{((\sigma+r)+3(2\delta-\varepsilon))}, \text{ and } |a_3| \leq \frac{1}{((\sigma+3r)+8+3\delta+6\varepsilon)}\left[|\mathcal{B}_1|+|\mathcal{B}_2|-\frac{(\frac{1}{2}\left((\sigma+2r)^2-3(\sigma+4r)\right)-23-2\delta-\varepsilon)}{((\sigma+r)+3(2\delta-\varepsilon))^2}|\mathcal{B}_1^2|\right],$$

and for some $\mu \in \mathbb{C}$, we have

$$|a_3-\mu a_2^2| \leq \frac{1}{((\sigma+3r)+8+3\delta+6\varepsilon)}\left[|\mathcal{B}_1|+|\mathcal{B}_2|-\frac{\left(\frac{1}{2}\left((\sigma+2r)^2-3(\sigma+4r)\right)-23-2\delta-\varepsilon\right)}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)^2}|\mathcal{B}_1^2| - \frac{\left((\sigma+3r)+8+3\delta+6\varepsilon\right)}{\left((\sigma+r)+3(2\delta-\varepsilon)\right)^2}\mu|\mathcal{B}_1^2|\right].$$

Proof: The findings are derived by substituting $w(\mathfrak{z}) = \mathfrak{z}$ in the proving equation of Theorem (2.3).

Substituting $\delta = 1$ into Theorem (2.3) yields the subsequent consequence.

Corollary(2.3): If $\mathcal{F}$ is defined by (1.1) within the class $R_{\sigma,r}^{q}(\delta,\varepsilon,\theta)$ and $\delta = 1$, then

$$|a_2| \leq \frac{|\mathcal{B}_1|}{((\sigma+r)+3(2-\varepsilon))},$$

and

$$|a_3| \leq \frac{1}{\left((\sigma+3r)+11+6\varepsilon\right)}[|\mathcal{B}_1| + \max\left\{|\mathcal{B}_1|, \frac{(\frac{1}{2}\left((\sigma+2r)^2-3(\sigma+4r)\right)-21-\varepsilon)}{((\sigma+r)+3(2-\varepsilon))^2}|\mathcal{B}_1^2|+|\mathcal{B}_2|\right\}.$$

Further

$$|a_3-\mu a_2^2| \leq \frac{1}{\left((\sigma+3r)+11+6\varepsilon\right)}[|\mathcal{B}_1| + \max\left\{|\mathcal{B}_1|, \frac{(\frac{1}{2}\left((\sigma+2r)^2-3(\sigma+4r)\right)-21-\varepsilon)}{((\sigma+r)+3(2\delta-\varepsilon))^2}|\mathcal{B}_1^2| + \frac{\left((\sigma+3r)+11+6\varepsilon\right)}{((\sigma+r)+3(2-\varepsilon))^2}\mu|\mathcal{B}_1^2|+|\mathcal{B}_2|\right\}.$$

Corollary(2.4): If $\mathcal{F}$ is defined by (1.1) in the class $\mathcal{R}_{\sigma,1}^{q}(\delta,\varepsilon,\theta)$ with $r = 1$, then

$$|a_2| \leq \frac{|\mathcal{B}_1|}{\left((\sigma+1)+3(2\delta-\varepsilon)\right)},$$

and

$$|a_3| \leq \frac{1}{\left((\sigma+3)+8+3\delta+6\varepsilon\right)}[|\mathcal{B}_1| + \max\left\{|\mathcal{B}_1|, \frac{(\frac{1}{2}\left((\sigma+2)^2-3(\varrho+4)\right)-23-2\delta-\varepsilon)}{((\sigma+1)+3(2\delta-\varepsilon))^2}|\mathcal{B}_1^2|+|\mathcal{B}_2|\right\}.$$

Further

$$|a_3-\mu a_2^2| \leq \frac{1}{((\sigma+3)+8+3\delta+6\varepsilon)}[|\mathcal{B}_1| + \max\left\{|\mathcal{B}_1|, \frac{(\frac{1}{2}\left((\sigma+2)^2-3(\varrho+4)\right)-23-2\delta-\varepsilon)}{((\sigma+1)+3(2\delta-\varepsilon))^2}|\mathcal{B}_1^2| + \frac{((\sigma+3)+8+3\delta+6\varepsilon)}{((\sigma+1)+3(2\delta-\varepsilon))^2}\mu|\mathcal{B}_1^2|+|\mathcal{B}_2|\right\}.$$

Remark(2.3): For $\tau = \varrho = 0, (0 \leq \delta < 1)$, it is classified into the subclass $\mathcal{R}_{0,0}^{q}(\delta,\varepsilon,\theta)$ if the subsequent quasi-subordination requirement is fulfilled:

$$1+\left(\frac{\mathfrak{z}\mathcal{F}'(\mathfrak{z})+\delta\mathfrak{z}^2\mathcal{F}''(\mathfrak{z})}{\mathcal{F}(\mathfrak{z})}\right)+(1-\varepsilon)(\frac{\mathfrak{z}\mathcal{F}''(\mathfrak{z})}{\mathcal{F}(\mathfrak{z})}+1) \prec_q \theta(\mathfrak{z})-1.$$

Subsequently, we present estimates for the coefficients $|a_2|$ and $|a_3|$, as well as $|a_3 - \mu a_2^2|$ for some $\mu \in \mathbb{C}$, pertaining to the function within the class $\mathcal{R}_{0,0}^{q}(\delta, \varepsilon, \theta)$.

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