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# New Subclasses of Bi-univalent Functions Associated with Quasi-subordination

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ABSTRACT

In this paper, we obtain some new subclasses of bi-univalent functions by using quasi-subordination. Also, we obtain the bounds for the modulus of the initial coefficients of the function inside these classes.

MSC..

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## 1. Introduction

Assume $\Xi$ exist the class of all normalized analytic functions $f$ in an open unit disk $\Delta = \{z: z \in \mathbb{C}, |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \Delta). \tag{1.1}$$

A function $f$ has an inverse $f^{-1}$ has become satisfying $f^{-1}\big(f(z)\big) = z, (z \in \Delta)$, and $f\big(f^{-1}(w)\big) = w$, $\left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$,

location

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$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots, (w \in \Delta). \quad (1.2)$$

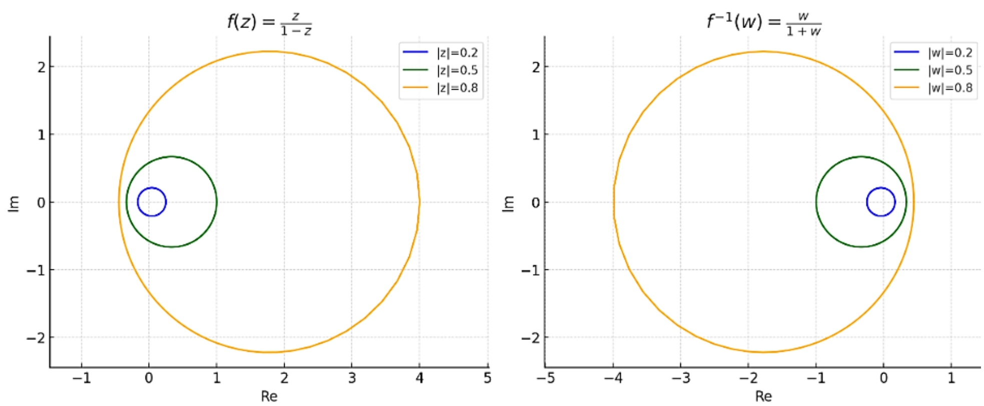


Figure : plots of the real and imaginary parts of the bi-univalent function and its inverse

If $f$ and $f^{-1}$ are univalent functions in $\Delta$, then $f$ is described as bi-univalent in $\Delta$ and the class of bi-univalent functions defined in $\Delta$ is denoted by $\sum$. (see [14]).

Let $f$ and $g$ are analytic functions in $\Xi$. Then $f$ is said to be quasi-subordinated to $g$ in $\Delta$ and articulated as follows:

$$f(z) \prec_q g(z), \qquad (z \in \Delta),$$

if there exists $\theta(z)$ also $w(z)$ exist two analytic functions in $\Delta$, accompanied by $w(0) = 0$ such that $|\theta(z)| < 1$, $|w(z)| < 1$ also $f(z) = \theta(z)g(w(z))$. If $\theta(z) = 1$, then $f(z) = g(w(z))$, in order to $f(z) \prec g(z)$ in $\Delta$. If $w(z) = z$, then $f(z) = \theta(z)g(z)$, It is asserted that $f$ is majorized by $g$ also written $f(z) \ll g(z)$ in $\Delta$. (see [1], [16])

Ma and Manda [15] established a category of starlike also convex functions by the use of subordination also the examination of classes $S^*(\phi)$ and $G^*(\phi)$ that is characterized by

$$S^*(\phi) = \left\{ f \in H: \frac{zf'(z)}{f(z)} \prec \phi(z), z \in \Delta \right\},$$

and

$$G^*(\phi) = \left\{ f \in H: \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in \Delta \right\}.$$

By $S^*_{\Sigma}(\phi)$ and $G^*_{\Sigma}(\phi)$, we denote to bi-starlike also bi-convex functions $f$ is bi-starlike also bi-convex of Ma-Minda designate accordingly [15].

In the sequel, it is assumed that $\phi$ of the form

$$\phi(z) = 1 + c_1z + c_2z^2 + \cdots, \quad (1.3)$$

where $\phi(0) = 1$ and $\phi'(0) > 0$, also

$$\theta(z) = k_0 + \sum_{i=1}^{\infty} k_i z^i, \quad (1.4)$$

which are analytic and constrained in $\Delta$. Nonetheless, there are only a few works determining the overarching coefficient limits $|a_2|$ also $|a_3|$ ([2,3,4,6,10,11,12,13,20,21,22] and [23]) for the analytic bi-univalent functions inside the scholarly discourse. ([2,5,7,8,9,17,18,19])

**Lemma (1.1) [10].** Let $h(z) = 1 + h_1 z + h_2 z^2 + \cdots \in P$, in which location $P$ represents the set of all functions $h$, analytic in $\Delta$, That is why $Re\{h(z)\} > 0, (z \in \Delta)$, then $|h_i| \leq 2$ for $i = 1,2,3,\cdots$.

## 2. Main Results

**Definition (2.1).** A function $f \in \sum$ defined by (1.1) is said to be in the class $\mathfrak{G}^{\alpha}_{q,\Sigma}(\varepsilon,\beta,\phi)$ if the following quasi-subordination conditions:

$$\frac{1}{\beta}\left[(1-\varepsilon)\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}+\frac{zf''(z)}{f'(z)}\right] \prec_q (\phi(z)-1) \tag{2.1}$$

and

$$\frac{1}{\beta}\left[(1-\varepsilon)\left(1+\frac{wg''(w)}{g'(w)}\right)+\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}+\frac{wg''(w)}{g'(w)}\right] \prec_q (\phi(w)-1) \tag{2.2}$$

where ($\varepsilon \geq 1, \beta \in \mathbb{C}|\{0\}$ and $0 \leq \alpha \leq 1,$), $z, w \in \Delta$ and $g = f^{-1}$ and the function $g, \phi$ are given by (1.2) and (1.3) respectively.

If we put $\alpha = 0$ in Definition(2.1), we obtain the following Remark such that $\mathfrak{G}^{0}_{q,\Sigma}(\varepsilon,\beta,\phi) = \mathfrak{G}_{q,\Sigma}(\varepsilon,\beta,\phi)$

**Remark (2.1).** A function $f \in \sum$ defined is classified as belonging to the class $\mathfrak{G}_{q,\Sigma}(\varepsilon,\beta,\phi)$ if the subsequent quasi-subordination conditions:

$$\frac{1}{\beta}\left[(1-\varepsilon)\left(1+\frac{zf''(z)}{f'(z)}\right)+\frac{zf''(z)}{f'(z)}\right] \prec_q (\phi(z)-1) \tag{2.3}$$

also

$$\frac{1}{\beta}\left[(1-\varepsilon)\left(1+\frac{wg''(w)}{g'(w)}\right)+\frac{wg''(w)}{g'(w)}\right] \prec_q (\phi(w)-1) \tag{2.4}$$

where ($\varepsilon \geq 1, \beta \in \mathbb{C}|\{0\}$) and $g = f^{-1}$ is given by (1.2).

**Theorem (2.1).** If $f$ is given by The Taylor-Maclaurin series expansion (1.1) is classified under $\mathfrak{G}^{\alpha}_{q,\Sigma}(\varepsilon,\beta,\phi)$, subsequently

$$|a_2| \leq \min\left\{\frac{\beta A_0 B_1}{(4-2\varepsilon+\alpha)}, \sqrt{\frac{\beta A_0 B_2}{6(3\varepsilon-1)+\frac{1}{2}\alpha(\alpha-1)}}\right\} \tag{2.5}$$

and

$$|a_3| \leq \min\left\{\frac{\beta A_0 B_1}{4(7-3\varepsilon+\alpha)}+\frac{\beta A_0 B_2}{6(3\varepsilon-1)+\frac{1}{2}\alpha(\alpha-1)}, \frac{\beta A_0 B_1}{4(7-3\varepsilon+\alpha)}+\frac{\beta^2 A_0^2 B_1^2}{(4-2\varepsilon+\alpha)^2}\right\} \tag{2.6}$$

**Proof**. Let $f \in \mathfrak{G}^{\alpha}_{q,\Sigma}(\varepsilon,\beta,\phi)$ and $g = f^{-1}$. Subsequently, there exist two analytic functions $u, v: \Delta \to \Delta$ accompanied by $v(0) = 0$ also $u(0) = 0, |u(z)| < 1, |v(w)| < 1$, fulfilling

$$\frac{1}{\beta}\left[(1-\varepsilon)\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}+\frac{zf''(z)}{f'(z)}\right] = \theta(z)\big(\phi(u(z)-1)\big) \tag{2.7}$$

also

$$\frac{1}{\beta}\left[(1-\varepsilon)\left(1+\frac{wg''(w)}{g'(w)}\right)+\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}+\frac{wg''(w)}{g'(w)}\right]=\theta(w)\big(\phi(v(w)-1)\big). \quad (2.8)$$

We define the function $p_1(z)$ and $p_2(w)$ by:

$$p_1(z)=\frac{1+u(z)}{1-u(z)}=1+r_1z+r_2z^2+\cdots, \quad (2.9)$$

and

$$p_2(w)=\frac{1+v(w)}{1-v(w)}=1+k_1z+k_2z^2+\cdots. \quad (2.10)$$

Or equivalent,

$$u(z)=\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}\left[c_1z+\left(c_1-\frac{c_1^2}{2}\right)z^2+\cdots\right], \quad (2.11)$$

and

$$v(w)=\frac{p_2(w)-1}{p_2(w)+1}=\frac{1}{2}\left[b_1w+\left(b_1-\frac{b_1^2}{2}\right)w^2+\cdots\right], \quad (2.12)$$

then $p_1(z)$ and $p_2(w)$ are analytic functions in $\Delta$, with $p_1(0)=p_2(0)=1$. due to, $v,u:\Delta\to\Delta$, possess a positive real component in $\Delta$, also $|b_i|\le 2$

also $|c_i|\le 2$, for $i=1,2$. Using (2.11) also (2.12) in (2.7) also (2.8), thus, we obtain

$$\frac{1}{\beta}\left[(1-\varepsilon)\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}+\frac{zf''(z)}{f'(z)}\right]=\theta(z)\left(\phi\left[\frac{p_1(z)-1}{p_1(z)+1}\right]-1\right) \quad (2.13)$$

also

$$\frac{1}{\beta}\left[(1-\varepsilon)\left(1+\frac{wg''(w)}{g'(w)}\right)+\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}+\frac{wg''(w)}{g'(w)}\right]=\theta(w)\left(\phi\left[\frac{p_2(w)-1}{p_2(w)+1}\right]-1\right). \quad (2.14)$$

Since $f\in\sum$ possesses the Maclurian series defined by (1.1), a calculation indicates that its inverse $g=f^{-1}$ according to the expansion in (1.2), we obtain

$$\frac{1}{\beta}\left[(1-\varepsilon)\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}+\frac{zf''(z)}{f'(z)}\right]=\frac{1}{\beta}\left[(2-\varepsilon)+(4-2\varepsilon+\alpha)a_2z+\left[2(7-3\varepsilon+\alpha)a_3+\left(4(6\varepsilon-5)+\frac{1}{2}\alpha(\alpha-3)\right)a_2^2\right]z^2+\cdots\right], \quad (2.15)$$

and

$$\frac{1}{\beta}\left[(1-\varepsilon)\left(1+\frac{wg''(w)}{g'(w)}\right)+\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}+\frac{wg''(w)}{g'(w)}\right]=\frac{1}{\beta}\left[(2-\varepsilon)-(4-2\varepsilon+\alpha)a_2w+\left[2(7-3\varepsilon+\alpha)(2a_2^2-a_3)+\left(4(6\varepsilon-5)+\frac{1}{2}\alpha(\alpha-3)\right)a_2^2\right]w^2+\cdots\right]. \quad (2.16)$$

By employing equations (2.11) as well as (2.12) in conjunction with (1.3) as well as (1.4), it becomes apparent that

$$\theta(z)\left(\phi\left[\frac{p_1(z)-1}{p_1(z)+1}\right]-1\right)=\frac{1}{2}A_0B_1c_1z+\left(\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\times\left(c_1-\frac{c_1^2}{2}\right)+\frac{1}{4}A_0B_2c_1^2\right)z^2+\cdots, \quad (2.17)$$

and

$$\theta(w)\left(\phi\left[\frac{p_2(w)-1}{p_2(w)+1}\right]-1\right)=\frac{1}{2}A_0B_1b_1w+\left(\frac{1}{2}A_1B_1b_1+\frac{1}{2}A_0B_1\times\left(b_1-\frac{b_1^2}{2}\right)+\frac{1}{4}A_0B_2b_1^2\right)w^2+\cdots. \quad (2.18)$$

Utilizing equations (2.17) and (2.15) to compare the coefficients of$z$ and $z^2$, we obtain

$$\frac{1}{\beta}(4-2\varepsilon+\alpha)a_2=\frac{1}{2}A_0B_1c_1, \quad (2.19)$$

and

$$\frac{1}{\beta}\left[2(7-3\varepsilon+\alpha)a_3+\left(4(6\varepsilon-5)+\frac{1}{2}\alpha(\alpha-3)\right)a_2^2\right]=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\times\left(c_1-\frac{c_1^2}{2}\right)+\frac{1}{4}A_0B_2c_1^2. \quad (2.20)$$

Also utilizing (2.18) also (2.16) to compare the coefficients of$w$ also $w^2$, we obatin

$$-\frac{1}{\beta}(4-2\varepsilon+\alpha)a_2=\frac{1}{2}A_0B_1b_1, \quad (2.21)$$

and

$$\frac{1}{\beta}\left[2(7-3\varepsilon+\alpha)(2a_2^2-a_3)+\left(4(6\varepsilon-5)+\frac{1}{2}\alpha(\alpha-3)\right)a_2^2\right]=$$

$$\frac{1}{2}A_1B_1b_1+\frac{1}{2}A_0B_1\times\left(b_1-\frac{b_1^2}{2}\right)+\frac{1}{4}A_0B_2b_1^2. \quad (2.22)$$

From equations (2.19) with (2.21), it follows that its $c_1=-b_1$, also

$$8(4-2\varepsilon+\alpha)^2a_2^2=\beta^2A_0^2B_1^2(c_1^2+b_1^2). \quad (2.23)$$

By summing equations (2.20) and (2.22), we derive

$$a_2^2=\frac{2\beta A_0B_1(c_1+b_1)+\beta A_0(c_1^2+b_1^2)(B_2-B_1)}{8\left[\left(4(6\varepsilon-5)+\frac{1}{2}\alpha(\alpha-3)\right)+2(7-3\varepsilon+\alpha)\right]}, \quad (2.24)$$

or equivalently,

$$a_2^2=\frac{\beta A_0[2B_1(c_1+b_1)+(c_1^2+b_1^2)(B_2-B_1)]}{8\left[6(3\varepsilon-1)+\frac{1}{2}\alpha(\alpha-1)\right]}. \quad (2.25)$$

Applying $|c_i|\le 2$ also $|b_i|\le 2$ for the occasion coefficients $c_2$ also $b_2$, we possess immediately

$$|a_2|\le\sqrt{\frac{\beta A_0B_2}{6(3\varepsilon-1)+\frac{1}{2}\alpha(\alpha-1)}}$$

and

$$|a_2|\le\frac{\beta A_0B_1}{(4-2\varepsilon+\alpha)}.$$

Furthermore, to ascertain the limit on$|a_3|$ through the subtraction of (2.20) and (2.22), we derive

$$\frac{4}{\beta}[4(7-3\varepsilon+\alpha)a_3-4(7-3\varepsilon+\alpha)a_2^2]=2A_0B_1c_1+A_0B_1(c_2-b_2). \quad (2.26)$$

Utilizing (2.26) also (2.25), we obtain

$$|a_3| \leq \frac{\beta A_0 B_1}{4(7-3\varepsilon+\alpha)} + \frac{\beta A_0 B_2}{6(3\varepsilon-1)+\frac{1}{2}\alpha(\alpha-1)}$$

and

$$|a_3| \leq \frac{\beta A_0 B_1}{4(7-3\varepsilon+\alpha)} + \frac{\beta^2 A_0^2 B_1^2}{(4-2\varepsilon+\alpha)^2}.$$

This completes the proof Theorem (2.1).

By setting $\alpha = 0$ in Theorem (2.1), we get the next Corollary:

**Corollary (2.1).** Let $f$ defined by (1.1) belongs to the class $\mathfrak{G}_{q,\Sigma}(\varepsilon,\beta,\phi)$. Then

$$|a_2| \leq \min\left\{\frac{\beta A_0 B_1}{(4-2\varepsilon)}, \sqrt{\frac{\beta A_0 B_2}{6(3\varepsilon-1)}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{\beta A_0 B_1}{4(7-3\varepsilon)} + \frac{\beta A_0 B_2}{6(3\varepsilon-1)}, \frac{\beta A_0 B_1}{4(7-3\varepsilon)} + \frac{\beta^2 A_0^2 B_1^2}{(4-2\varepsilon)^2}\right\}.$$

By setting $\alpha = 1$ in Theorem (2.1), we get the next Corollary:

**Corollary (2.2).** Assume $f$ defined by (1.1) belongs to the class $\mathfrak{G}^1_{q,\Sigma}(\varepsilon,\beta,\phi)$. Then

$$|a_2| \leq \min\left\{\frac{A_0 B_1}{3}, \sqrt{\frac{A_0 B_2}{12}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{A_0 B_1}{20} + \frac{A_0 B_2}{12}, \frac{A_0 B_1}{20} + \frac{A_0^2 B_1^2}{9}\right\}.$$

By putting $\alpha = \varepsilon = 1$ and $\beta = 1$ in Theorem(2.1), we get the next Corollary:

**Corollary (2.3).** Let $f$ defined by (1.1) belongs to the class $\mathfrak{G}^0_{q,\Sigma}(1,1,\phi)$. Then

$$|a_2| \leq \min\left\{\frac{A_0 B_1}{2}, \sqrt{\frac{A_0 B_2}{12}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{A_0 B_1}{16} + \frac{A_0 B_2}{12}, \frac{A_0 B_1}{16} + \frac{A_0^2 B_1^2}{4}\right\}.$$

**Definition (2.2).** A function $f \in \Sigma$ is called in the class $\mathfrak{X}^\delta_{q,\Sigma}(\tau,\lambda,\phi)$ if the subsequent quasi-subordination conditions are satisfied:

$$\frac{1}{\tau}\left[\left(\frac{zf'(z)}{f(z)}\right)^\delta \left[\lambda\frac{f(z)}{z} + (1-\lambda)f'(z) - 1\right]\right] \prec_q (\phi(z)-1), \tag{2.27}$$

also

$$\frac{1}{\tau}\left[\left(\frac{wg'(w)}{g(w)}\right)^{\delta}\left[\lambda\frac{g(w)}{w}+(1-\lambda)g'(w)-1\right]\right]\prec_q(\phi(w)-1), \quad (2.28)$$

where ($\lambda \geq 1$ and $0 \leq \delta \leq 1$) and $\tau \in \mathbb{C}|\{0\}$, $z, w \in \Delta$ and $g = f^{-1}$ is given by (1.2).

If we put $\delta = 0$ in Definition(2.2), we obtain the following Remark such that $\mathfrak{X}^{0}_{q,\Sigma}(\tau,\lambda,\phi) = \mathfrak{X}_{q,\Sigma}(\tau,\lambda,\phi)$

**Remark (2.2).** A function $f \in \Sigma$ is called in the class $\mathfrak{X}_{q,\Sigma}(\tau,\lambda,\phi)$ if the following quasi-subordination conditions satisfying:

$$\frac{1}{\tau}\left[\left[\lambda\frac{f(z)}{z}+(1-\lambda)f'(z)-1\right]\right]\prec_q(\phi(z)-1), \quad (2.29)$$

and

$$\frac{1}{\tau}\left[\left[\lambda\frac{g(w)}{w}+(1-\lambda)g'(w)-1\right]\right]\prec_q(\phi(w)-1), \quad (2.30)$$

location ($\lambda \geq 1$ also $\tau \in \mathbb{C}|\{0\}$), $z, w \in \Delta$ and $g = f^{-1}$ is given by (1.2).

**Theorem (2.2).** If $f$ is given by (1.1) is owned to the class $\mathfrak{X}^{\delta}_{q,\Sigma}(\tau,\lambda,\phi)$, then

$$|a_2| \leq \min\left\{\frac{A_0B_1}{(2-\lambda)}, \sqrt{\frac{\tau A_0 B_2}{[\delta(2-\lambda)+(3-2\lambda)]}}\right\} \quad (2.31)$$

and

$$|a_3| \leq \min\left\{\frac{A_0B_1}{2\tau(3-2\lambda)}+\frac{\tau A_0B_2}{\delta(2-\lambda)+(3-2\lambda)}, \frac{A_0B_1}{2\tau(3-2\lambda)}+\frac{\tau^2A_0^2B_1^2}{(2-\lambda)^2}\right\}. \quad (2.32)$$

**Proof.** Let $f \in \mathfrak{X}^{\delta}_{q,\Sigma}(\tau,\lambda,\phi)$ also $g = f^{-1}$. Subsequently, there exist two analytic functions $u, v: \Delta \to \Delta$ with $v(0) = 0$ and $u(0) = 0$, $|u(z)| < 1$ and $|v(w)| < 1$, satisfying

$$\frac{1}{\tau}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left[\lambda\frac{f(z)}{z}+(1-\lambda)f'(z)-1\right]\right]=\theta(z)(\phi(u(z)-1)) \quad (2.33)$$

and

$$\frac{1}{\tau}\left[\left(\frac{wg'(w)}{g(w)}\right)^{\delta}\left[\lambda\frac{g(w)}{w}+(1-\lambda)g'(w)-1\right]\right]=\theta(w)(\phi(v(w)-1)). \quad (2,34)$$

The series expansions for $f$ and $g$ as given in (1.1) and (1.2) respectively, we get

$$\frac{1}{\tau}\left[\left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left[\lambda\frac{f(z)}{z}+(1-\lambda)f'(z)-1\right]\right]=\frac{1}{\tau}[(2-\lambda)a_2z+[(3-2\lambda)a_3+\delta(2-\lambda)a_2^2]z^2+\cdots] \quad (2.35)$$

and

$$\frac{1}{\tau}\left[\left(\frac{wg'(w)}{g(w)}\right)^{\delta}\left[\lambda\frac{g(w)}{w}+(1-\lambda)g'(w)-1\right]\right]=\frac{1}{\tau}[-(2-\lambda)a_2w+[(3-2\lambda)(2a_2^2-a_3)+\delta(2-\lambda)a_2^2]w^2+\cdots]. \quad (2.36)$$

By utilizing equations (2.17) and (2.35) and comparing the coefficients of $z$ and $z^2$, we derive

$$\frac{1}{\tau}(2-\lambda)a_2=\frac{1}{2}A_0B_1c_1, \quad (2.37)$$

also

$$\frac{1}{\tau}[(3-2\lambda)a_3+\delta(2-\lambda)a_2^2]=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\times\left(c_1-\frac{c_1^2}{2}\right)+\frac{1}{4}A_0B_2c_1^2. \quad (2.38)$$

Likewise, by comparing the coefficients of w and $w^2$ in equations (2.18) and (2.36), we obtain

$$-\frac{1}{\tau}(2-\lambda)a_2=\frac{1}{2}A_0B_1b_1, \quad (2.39)$$

also

$$\frac{1}{\tau}[(3-2\lambda)(2a_2^2-a_3)+\delta(2-\lambda)a_2^2]=\frac{1}{2}A_1B_1b_1+\frac{1}{2}A_0B_1\times\left(b_1-\frac{b_1^2}{2}\right)+\frac{1}{4}A_0B_2b_1^2. \quad (2.40)$$

From equations (2.37) and (2.39), we derive

$$c_1=-b_1,$$

and

$$8(2-\lambda)^2a_2^2=\tau^2A_0^2B_1^2(c_1^2+b_1^2). \quad (2.41)$$

Currently, incorporating (2.38) also (2.40), we derive

$$a_2^2=\frac{\tau[2A_0B_1(c_2+b_2)+A_0(c_1^2+b_1^2)(B_2-B_1)]}{8[\delta(2-\lambda)+(3-2\lambda)]}.$$

Applying Lemma (1.1) for the coefficients $c_2$ and $b_2$, we have

$$|a_2|\le\sqrt{\frac{\tau A_0B_2}{\delta(2-\lambda)+(3-2\lambda)}}$$

and

$$|a_2|\le\frac{A_0B_1}{(2-\lambda)}.$$

Now, to find $|a_3|$, by subtracting (2.38) and (2.40), we get

$$4\tau[2(3-2\lambda)a_3-2(3-2\lambda)a_2^2]=2A_0B_1c_1+A_0B_1(c_2-b_2). \quad (2.42)$$

By using (2.41) and (2.42), we have

$$a_3=\frac{A_0B_1c_1}{4\tau(3-2\lambda)}+\frac{A_0B_1(c_2-b_2)}{8\tau(3-2\lambda)}+\frac{\tau^2A_0^2B_1^2(c_1^2+b_1^2)}{8(2-\lambda)^2}.$$

Utilizing Lemma (1.1) for coefficients $c_2$ also $b_2$, we drive

$$|a_3|\le\frac{A_0B_1}{2\tau(3-2\lambda)}+\frac{\tau^2A_0^2B_1^2}{(2-\lambda)^2}.$$

This complete the proof of Theorem (2.2).

By setting $\delta=0$ in Theorem (2.2), we get the next Corollary:

**Corollary (2.4).** Assume $f$ defined by (1.1) belongs to the class $\mathfrak{X}_{q,\Sigma}(\tau,\lambda,\phi)$. Then

$$|a_2| \leq \min\left\{\frac{A_0B_1}{(2-\lambda)}, \sqrt{\frac{\tau A_0B_2}{(3-2\lambda)}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{A_0B_1}{2\tau(3-2\lambda)} + \frac{\tau A_0B_2}{(3-2\lambda)}, \frac{A_0B_1}{2\tau(3-2\lambda)} + \frac{\tau^2A_0^2B_1^2}{(2-\lambda)^2}\right\}.$$

By setting $\lambda = \delta = 0$ in Theorem (2.2), we get the next Corollary:

**Corollary (2.5).** Assume $f$ defined by (1.1) belongs to the class $\mathfrak{X}_{q,\Sigma}^{1}(\tau, 1, \phi)$. Then

$$|a_2| \leq \min\left\{\frac{A_0B_1}{2}, \sqrt{\frac{\tau A_0B_2}{3}}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{A_0B_1}{6\tau} + \frac{\tau A_0B_2}{3}, \frac{A_0B_1}{6\tau} + \frac{\tau^2A_0^2B_1^2}{4}\right\}.$$

By putting $\tau = \lambda = 1$ in Theorem (2.2), we get the next Corollary:

**Corollary (2.6).** Let $f$ defined by (1.1) belongs to the class $\mathfrak{X}_{q,\Sigma}^{\delta}(1,1,\phi)$. Then

$$|a_2| \leq \min\left\{A_0B_1, \sqrt{\frac{A_0B_2}{[\delta+1]}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{A_0B_1}{2} + \frac{A_0B_2}{[\delta+1]}, \frac{A_0B_1}{2} + A_0^2B_1^2\right\}.$$

## References:

[1] O. Altintas and S. Owa, Majorizations and quasi-subordinations for certain analytic functions, Proc. Jpn. Acad. Ser. A, 68(7) (1992), 181-185.
[2] S. Altinkaya, S. Yalcin, Faber Polynomial coefficient bounds for a subclass of bi-univalent functions, C. R. Acad. Sci. Paris, Ser. I. (2015) (353), 1075-1080.
[3] S. Altinkaya, S.Yalcin, Coefficient bounds for a subclass of bi- univalent functions, TWMS. Journal of Pure and Applied Mathematics, (6) (2015), 180-185.
[5] S. Altinkaya, S.Yalcin, On a new subclass of bi-univalent functions satisfying subordinate conditions, Acta Universitatis Sapientiae, Mathematica (7) (2015), 5-14.
[5] S. R. Bakheet and W.G. Atshan, Third-order sandwich results for analytic univalent functions defined by integral operator. Adv. Mech.2022,10, 1178–1197.
[6] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, Studia Univ. Babes Bolyai Math., 31 (1986), 70-77.
[7] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, J. Classical Ana. 2 (1), (2013), 49-60.
[8] P. Duren, Subordination, in Complex Analysis, Vol. 599 of Lecture Notes in Mathematics, pp.22-29, Springer, Berlin, Germany, (1977).
[9] P. L. Duren, Univalent Functions, In: Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Hidelberg and Tokyo, (1983).
[10] M. El-Ityan, Q. A. Shakir, T. Al-Hawary, R. Buti, D. Breaz, L.-I. Cotîrlă. "On the Third Hankel Determinant of a Certain Subclass of Bi-Univalent Functions Defined by (p,q)-Derivative Operator," Mathematics 2025, 13, 1269.
[11] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett., 24 (2011), 1569-1573.
[12] S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficient estimates for analytic bi-close to convex functions, C. R. Acad. Sci. Paris, Ser. I. (2014) (352), 17-20.
[13] S. Kant, Coefficients estimate for certain subclass of bi-univalent functions associated with quasi-subordinations, Journal of fractional calculus and Applied.9 (1) Jan. (2018), 195-203.
[14] M. Lewin, On a coefficient problem for bi-univalent function, Proceedings of the American Mathematical Society, Vol. 18 (1967), 63-68.
[15] W. Ma and D. Minda, Auified treatment of some special classes of univalent functions, Proceedings of the conference on complex analysis, Z. Li. F. Ren, L. Yang and S. Zhang, eds., Int. Press (1994), 157-169.
[16] M. S. Robertson, Quasi-subordination and coefficient conjecture, Bull. Amer. Math. Soc., 76 (1970), 1-9.
[17] M. S . Muhammed and W. G. Atshan, Applications of Quasi-Subordination on Subclasses of bi-univalent Function Associated with Generalized Differential Operator, Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.16(14) 2024.
[18] M. S . Muhammed and W. G. Atshan, Coefficient Estimates and Fekete-Szegö Inequality for a Certain New Subclass of Bi-Univalent Functions by Using Generalized Operator with Bernoulli Polynomials , Advances in Nonlinear Variational Inequalities, 28(4s) (2025),pp.534-546.

[19] M. A. Sabri, W. G. Atshan, E. El-Seidy, New Differential Subordination and Superordination Results for a subclass of Meromorphic Univalent Functions Defined by a New Operator , Iraqi Journal of Science, 2025.
[20] Q. A. Shakir, A. S. Tayyah, D. Breaz, L.-I. Cotîrlă, E. Rapeanu, and F. M. Sakar, "Upper bounds of the third Hankel determinant for bi-univalent functions in crescent-shaped domains," Symmetry, vol. 16, p. 1281, 2024.
[21] Q. A. Shakir and W. G. Atshan, "On third Hankel determinant for certain subclass of bi-univalent functions," Symmetry, vol. 16, p. 239, 2024.
[22] A. S. Tayyah, W. G. Atshan, A class of bi-bazilevič and bi-pseudo-starlike functions involving tremblay fractional derivative operator. Probl. Anal. Issues Anal., 14(32)(2)(2025).
[23] A. S. Tayyah, W. G. Atshan, Starlikeness and bi-starlikeness associated with a new carathéodory function. J. Math. Sci. (2025). https://doi.org/10.1007/s10958-025-07604-8.