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Cartesian Product of Ultrasemiprime Algebras

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ABSTRACT

Normed finite-dimensional algebras isomorphic to the finite direct sum of ℝ or ℂ have been proven to be ultrasemiprime algebras using the max norm. Since all norms are equivalent in finite dimensions, normed finite-dimensional algebras will be ultrasemiprime under any norm. However, is this true for infinite-dimensional algebras? A study addressed the direct sum of ultrasemiprime algebras using the max norm and another considered the sum of norms. In this research, we proved that the direct sum of ultrasemiprime algebras is ultrasemiprime under different norms.

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1. Introduction

A critical connection in the mathematical inquiry has emerged which is Cartesian products, linking the pathways between abstract algebra, functional analysis, and topology. The principal work of Mathieu [1], who led the concept of ultra-prime algebras by introducing a norm on quotient algebras, has lighted up a series of extensive investigations to their properties done by many researchers. Our understanding of algebraic structures has been improved with this crossing of fields, not only that, but also encouraged further in-depth explorations of their applications across multiple mathematical domains [2][3][4].

In the past few years, the ultra-semiprime algebras have gained a significant attention especially in the field of direct product constructions. Their significance shines in their solid structure and wide-ranging utilities across operator theory, quantum algebra frameworks and non-Archimedean analysis.

This introduction intends to form the core results, explain the modern developments and point out the open problems regarding the direct product of ultra-semiprime algebras, hence confirming their deep relevance in mathematics.

Mathieu originally framed ultrasemiprime algebras as a specialized extension of prime algebras. These algebras represent an intermediate position relative to ultraprime algebras, which fulfill the requirement, and more general

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semiprime structures. A distinguishing feature of ultrasemiprime algebras is their preservation under finite direct product constructions. Unlike prime or ultraprime algebras, whose properties may not persist in finite direct products, ultrasemiprime algebras generally maintain their defining traits when subjected to Cartesian constructions [1].

All finite-dimensional normed algebras inherently possess ultrasemiprime properties. Specifically, such algebras exhibit categorical equivalence to finite direct products of the ultrasemiprime fields $\mathbb{R}$ or $\mathbb{C}$. This structural characterization naturally motivated the systematic investigation of Cartesian products of ultrasemiprime algebras.

Both the maximum and sum norms ensure sub-multiplicativity and completeness, essential properties for algebraic structures [8]. In [9], the researcher addressed distinctive non-associative almost prime algebras and also proved that the Cartesian product of these algebras is also a distinctive almost prime algebra, where the research addressed the criterion of summing norms.

A seminal contribution by Al-Neima et al. in 2022, [8] extended the earlier finite-dimensional results to arbitrary ultrasemiprime algebras. Their work demonstrated that the Cartesian product of ultrasemiprime algebras $A_1, A_2, \ldots, A_n$ remains ultrasemiprime under either the maximum or sum norm. This synthesis of algebraic rigidity and topological flexibility in Cartesian products of ultrasemiprime algebras enables critical decomposition techniques in functional analysis and operator theory. The foundational work of Mathieu and recent generalizations by Al-Neima et al. have solidified the theoretical framework, while cohomological advancements suggest deeper geometric classifications.

2. Ultrasemiprime Algebras

An algebra A over a normed field is termed ultrasemiprime if it fulfills the requirement that One can find a constant $k > 0$ such that there exists a positive constant $k > 0$ such that for all elements $a \in A$, the inequality $\| a^2 \|_A \geq k \| a \|_A$ is satisfied.

2.1. Theorem [8]

The following statements are equivalent for a normed algebra A.

1. For an arbitrary sequence $\{x_n\}, n \in N$ in A with $\|x_n\| = 1$, for all $n \in N$, a bounded sequence exists $\{y_n\}, n \in N$, in A such that the sequence $\{x_n y_n x_n\}, n \in N$ fails to converge to zero.
2. There exists a $c > 0$ such that $c\|x\|^2 \leq \|M_{x,x}\|$ for all $x \in A$.

In [5], Mohammed showed that finite dimensional normed algebras are ultrasemiprime behavior. Finite dimensional normed algebras isomorphism classes with $\mathbb{R}^n$ or $\mathbb{C}^n$ [7, Theorem 2.3.1]. This demonstrates that finite direct products of $\mathbb{R}$ or $\mathbb{C}$ are ultrasemiprime. The following theorems verify that finite direct products (with distinct definitions of the norm) of ultrasemiprime algebras are ultrasemiprime. We will use the q-norm defined as

$\|(a,b)\|_q = \sqrt[q]{\| a \|^q + \| b \|^q}$ where q is a positive integer number.

2.2. Theorem

Let A_1 and A_2 be arbitrary ultrasemiprime algebras. Then their direct product $A_1 \times A_2$ is an ultrasemiprime algebra under the q-norm.

Proof

Let A_1, A_2 be ultrasemiprime algebras, then there exist constants $c_{A_1} > 0$ for A_1 and $c_{A_2} > 0$ for A_2 such that

$$\|M_{a,a}\| \geq c_{A_1} \| a \|^2 \ \& \ \|M_{b,b}\| \geq c_{A_2} \| b \|^2 \quad \text{for } a \text{ in } A_1 \ \& \ b \text{ in } A_2$$

In $A_1 \times A_2$ we have $\|(a,b)\|_q = \sqrt[q]{\| a \|^q + \| b \|^q}$, for all $(a,b) \in A_1 \times A_2$, where $a \in A_1$ and $b \in A_2$

$$(\|(a,b)\|_2)^q = \| a \|^q + \| b \|^q$$

Let the sequence $(w_n), n \in \mathbb{N}$ be given in $A_1 \times A_2$, under the condition that $\|w_n\| = 1$ for any $n \in \mathbb{N}$. Consequently, there is a sequence $(a_n), n \in \mathbb{N}$ in A_1 & $(b_n), n \in \mathbb{N}$ in A_2 with the property that $(w_n) = (a_n, b_n), n \in \mathbb{N}$.

Assume that $A_1 \times A_2$ fails to satisfy the ultrasemiprime property. By Theorem 2.1(1), there does not exist any bounded sequence $(v_n), n \in \mathbb{N}$ in $A_1 \times A_2$ that makes the sequence $(w_n v_n w_n)$ not approach zero; but this implies for all bounded sequences $(v_n), n \in \mathbb{N}$, in $A_1 \times A_2$, the sequence $(w_n v_n w_n)$ has a limit of zero.

Three cases arise, without loss of generality, excluding zero elements.

The first case is when a_n equals zero, and $b_n$is non-zero, for all $n \in \mathbb{N}$. Then $w_n = (0, b_n)$, and now $\left(\frac{b_n}{\|b_n\|}\right)$, $n \in \mathbb{N}$ is a sequence in A_2. Since A_2 satisfies the ultrasemiprime property under Theorem 2.1(1), one can construct a bounded sequence $(y_n), n \in \mathbb{N}$ in A_2 with a bound c_y so that $\left(\frac{b_n}{\|b_n\|} y_n \frac{b_n}{\|b_n\|}\right), n \in \mathbb{N}$ does not approach zero. $\left(\frac{b_n}{\|b_n\|} y_n \frac{b_n}{\|b_n\|}\right) = \frac{1}{\|b_n\|^2}(b_n y_n b_n)$ does not converge to zero, which implies $(b_n y_n b_n)$ fails to converge to zero.

Define $(v_n), n \in \mathbb{N}$, by $v_n = (0, y_n)$ where $(y_n), n \in \mathbb{N}$ represents a sequence in A_2. (v_n) is a bounded sequence, due to $\|v_n\|_2{}^q = 0 + \|y_n\|^q \leq c_y{}^q$

Now $(w_n v_n w_n)$ converges to zero, so $(w_n v_n w_n) = (0, b_n)(0, y_n)(0, b_n) = (0, b_n y_n b_n)$, $(w_n v_n w_n)$ tends to zero, hence necessarily $(b_n y_n b_n)$ converges to zero, which leads to a contradiction, so $(w_n v_n w_n)$ does not converge to zero.

The other alternative is the case where $a_n$is non- zero, and b_n equals zero, for any $n \in \mathbb{N}$, then $(w_n) = (a_n, 0)$, now $\left(\frac{a_n}{\|a_n\|}\right)$, $n \in \mathbb{N}$ is a sequence in A_1. As A_1 is an ultrasemiprime algebra as shown in theorem 2.1(1), there exists a bounded sequence $(x_n), n \in \mathbb{N}$ in A_1, bounded by c_x, with the property that $\left(\frac{a_n}{\|a_n\|} x_n \frac{a_n}{\|a_n\|}\right), n \in \mathbb{N}$ does not converge to zero. Then $\left(\frac{a_n}{\|a_n\|} x_n \frac{a_n}{\|a_n\|}\right) = \frac{1}{\|a_n\|^2}(a_n x_n a_n)$ does not converge to zero, so $(a_n x_n a_n)$ does not approach zero.

Define $(k_n), n \in \mathbb{N}$, by $v_n = (x_n, 0)$ where $(x_n), n \in \mathbb{N}$ is a sequence in A_1. (v_n) is a bounded sequence, due to $\|v_n\|_2{}^q = \|x_n\|^q + 0 \leq c_x{}^q$.

Now, $(w_n v_n w_n)$ are convergent to zero, so $(w_n v_n w_n) = (a_n, 0)(x_n, 0)(a_n, 0) = (a_n x_n a_n, 0)$, $(w_n v_n w_n)$ are, so $(a_n x_n a_n)$ must as well, which results in a logical inconsistency, thus $(w_n v_n w_n)$ does not tend to zero.

The last alternative is the case where a_n equals zero, and b_n is non-zero, for all $n \in \mathbb{N}$. Then $(w_n) = (a_n, b_n)$, now $\left(\frac{a_n}{\|a_n\|}\right), \left(\frac{b_n}{\|b_n\|}\right)$, $n \in \mathbb{N}$, are sequences in A_1, A_2. Since A_1, A_2 are ultrasemiprime algebras, from Theorem 2.1(1), there exist bounded sequences $(x_n), (y_n)$ where $n \in \mathbb{N}$ in A_1, A_2 with bounds c_x, c_y such that $\left(\frac{a_n}{\|a_n\|} x_n \frac{a_n}{\|a_n\|}\right), \left(\frac{b_n}{\|b_n\|} y_n \frac{b_n}{\|b_n\|}\right), n \in \mathbb{N}$, do not converge to zero. Then $\left(\frac{a_n}{\|a_n\|} x_n \frac{a_n}{\|a_n\|}\right) = \frac{1}{\|a_n\|^2}(a_n x_n a_n)$ and $\left(\frac{b_n}{\|b_n\|} y_n \frac{b_n}{\|b_n\|}\right) = \frac{1}{\|b_n\|^2}(b_n y_n b_n)$ do not converge to zero, which means $(a_n x_n a_n)$ and $(b_n y_n b_n)$ do not either.

Put $v_n = (x_n, y_n)$ where $(x_n), (y_n), n \in \mathbb{N}$ are sequences in A_1, A_2. It is bounded, because $\|v_n\|_2{}^q = \|x_n\|^q + \|y_n\|^q \leq c_x{}^q + c_y{}^q$.

Now $(w_n v_n w_n)$ are convergent to zero, so $(w_n v_n w_n) = (a_n, b_n)(x_n, y_n)(a_n, b_n) = (a_n x_n a_n, b_n y_n b_n)$, $(w_n v_n w_n)$ tend to zero, then so must $(a_n x_n a_n), (b_n y_n b_n)$, which results in a logical inconsistency, so $(w_n v_n w_n)$ does not converge to zero. Thus, $A_1 \times A_2$ is an ultrasemiprime algebra.

The following theorem prove the ultrasemiprime algebra using the p-norm which is defined as $\|a\|_p = (\sum_{i=1}^{p} \|a_i\|^2)^{\frac{1}{2}}$, where $a = (a_1, a_2, \dots, a_p)$, $a_i \in A_i$, where $i = 1,2, \dots, p$.

2.3. Theorem

The direct product of any finite collection of ultrasemiprime algebras $A_1, A_2, \dots, A_p$ is ultrasemiprime with respect to the p-norm.

Proof:

Since $A_1, A_2, \dots, A_p$ are ultrasemiprime, there exist constants c_{A_i} in A_i, where $i = 1,2, \dots, p$, such that

$$\|M_{a_i,a_i}\| \geq c_{A_i} \| a_i \|^2 \text{ for } a_i \text{ in } A_i$$

In $A_1 \times A_2 \times \dots \times A_p$ we have $\|a\|_p = \|(a_1, a_2, \dots, a_p)\|_p = (\sum_{i=1}^{p} \|a_i\|^2)^{\frac{1}{2}}$, for all $(a_1, a_2, \dots, a_p) \in A_1 \times A_2 \times \dots \times A_p$.

$$\|a\|_p^2 = \sum_{i=1}^{p} \|a_i\|^2$$

Let the sequence $(w_n), n \in \mathbb{N}$ be given in $A_1 \times A_2 \times \ldots \times A_p$, under the condition that $\|w_n\| = 1$ for any $n \in \mathbb{N}$, Consequently, there are sequences $(a_{i_n}), n \in \mathbb{N}$ in A_i with the property that $(w_n) = (a_{1_n}, a_{2_n}, \ldots, a_{p_n}), n \in \mathbb{N}$.
Assume that $A_1 \times A_2 \times \ldots \times A_p$ fails to satisfy the ultrasemiprime property. By Theorem 2.1(1), there does not exist any bounded sequence $(v_n), n \in \mathbb{N}$ in $A_1 \times A_2 \times \ldots \times A_p$ that makes the sequence $(w_n v_n w_n)$ not approach zero, this implies for all bounded sequences $(v_n), n \in \mathbb{N}$, in $A_1 \times A_2 \times \ldots \times A_p$, the sequence $(w_n v_n w_n)$ has a limit of zero.
There are 2^p cases, without loss of generality, excluding zero elements.
The first possibility, when a_{1_n} equals zero, and a_{i_n} is non-zero, for all $n \in \mathbb{N}$, and $i = 2, \ldots, p$ then $w_n = (0, a_{2_n}, a_{3_n}, \ldots, a_{p_n})$, now $\left(\frac{a_{i_n}}{\|a_{i_n}\|}\right)$, $n \in \mathbb{N}$ is a sequence in A_i for $i = 2, \ldots, p$. Since A_i satisfies the ultrasemiprime property, Theorem 2.1(1) implies that one can construct a bounded sequence $(y_{i_n}), n \in \mathbb{N}$ in A_i with a bound c_{i_y} so that $\left(\frac{a_{i_n}}{\|a_{i_n}\|} y_{i_n} \frac{a_{i_n}}{\|a_{i_n}\|}\right), n \in \mathbb{N}$ does not approach zero. Then $\left(\frac{a_{i_n}}{\|a_{i_n}\|} y_{i_n} \frac{a_{i_n}}{\|a_{i_n}\|}\right) = \frac{1}{\|a_{i_n}\|^2}\left(a_{i_n} y_{i_n} a_{i_n}\right)$ does not converge to zero, which implies $\left(a_{i_n} y_{i_n} a_{i_n}\right)$ fails to converge to zero for $i = 2, \ldots, p$.
Define $(v_n), n \in \mathbb{N}$, by $v_n = (0, y_{2_n}, y_{3_n}, \ldots, y_{p_n})$ where $(y_{i_n}), n \in \mathbb{N}$ represents a sequence in A_i for $i = 2, \ldots, p$. (v_n) is a bounded sequence, due to $\|v_n\|_p^2 = 0 + \left\|y_{2_n}\right\|^2 + \left\|y_{3_n}\right\|^2 + \cdots + \left\|y_{p_n}\right\|^2 \leq c_{2_y}^2 + c_{3_y}^2 + \cdots + c_{p_y}^2$
Now $(w_n v_n w_n)$ are convergent to zero, so

$$\begin{aligned}(w_n v_n w_n) &= \left(0, a_{2_n}, a_{3_n}, \ldots, a_{p_n}\right)\left(0, y_{2_n}, y_{3_n}, \ldots, y_{p_n}\right)\left(0, a_{2_n}, a_{3_n}, \ldots, a_{p_n}\right) \\ &= (0, a_{2_n} y_{2_n} a_{2_n}, a_{3_n} y_{3_n} a_{3_n}, \ldots, a_{p_n} y_{p_n} a_{p_n}),\end{aligned}$$

$(w_n v_n w_n)$ tend to zero, hence necessarily $(a_{i_n} y_{i_n} a_{i_n})$ does for $i = 2, \ldots, p$, which leads to a contradiction, so $(w_n v_n w_n)$ does not converge to zero.
The rest of the cases for the proof follow the same method.
The following result is the special case of Theorem 2.2 when the value of q is equal to 2.

2.4. Corollary

For arbitrary ultrasemiprime algebras A_1 and A_2, their direct product $A_1 \times A_2$ is ultrasemiprime under the norm $\|(a,b)\|_2 = \sqrt{\| a \|^2 + \| b \|^2}$

3. Conclusions:

The aim of this paper is to enrich the scope of norms that ensure the Cartesian product of ultrasemiprime algebras remains ultrasemiprime, even in cases where these algebras are infinite-dimensional. In addition to the well-known max and sum norms, we introduce two new norms to this scope.

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