$b\text{–}open$ Sets In Topological Spaces

Raad Aziz Hussain Al-Abdulla

&

Sema Kadhim Gaber

Al-Qadisiya University

College of Science

Dept. of Mathematics

Abstract

This work consist of two sections. In section one we recall the definition of connected sets. We also introduce similar definition using $b-open$ sets and study the property of this definition. In [5] [1] $I-space$ and $MI-space$ are studied respectively. In section two we introduce similar definition $T_B-space$ using $b-open$ sets. In particular we will prove that in MI-space the $T_B-space$ and $T_D-space$ are equivalent.

Introduction

The concept of $b-open$ set in topological spaces was introduced in [2]. We recall the definition of connected spaces [4]. In first section of this paper we study similar definition using $b-open$ sets which is called b-connected space and we give several properties of this definition . In section two of this paper we study $I-space$ and some of their generalizations using $b-open$ sets and we study $MI-space$ and some of their generalization using $b-open$ sets.

Section one

b-connected spaces

Definition 1.1 [2]

Let X be a topological space $A \subseteq X$. A is called $b-open$ set in X iff $A \subseteq \overline{\overset{\circ}{A}} \cup \overset{\circ}{\overline{A}}$. A is called $b-closed$ iff A^{c} is $b-open$and it is easy to see that A is $b-closed$ set iff $\overset{\circ}{\overline{A}} \cap \overline{\overset{\circ}{A}} \subseteq A$

It is clear that every open set is $b-open$ and every closed set is $b-closed$. The intersection of $b-open$ set with open set is $b-open$ set. Also the union of any $b-open$ sets is $b-open$ set.

Definition 1.2:

Let X be topological space and $x \in X, A \subseteq X$. The point x is called a *b-limit* point of A, if each $b-open$ set containing x , contains a point of A distinct from x. We shall call the set of all *b-limit* points of A the $b-derived$ set of A and denoted it by $A^{/b}$. Therefore $x \in A^{/b}$ iff for every $b-open$ set V in X such that $x \in V$ implies that $(V \cap A) - \{x\} \neq \phi$.

Proposition 1.3

Let X be a topological space and $A \subseteq B \subseteq X$.then:

(i) $\overline{A}^{b} = A \cup A^{/b}$

(ii) A is $b-closed$ set iff $A^{/b} \subseteq A$.

(iii) $A^{/b} \subseteq B^{/b}$

Proof:

(i) If $x \notin \overline{A}^{b}$, then there exists a $b-closed$ set F in X such that $A \subseteq F$ and $x \notin F$. Hence $V = X - F$ is $b-open$ set such that $x \in V$ and $V \cap A = \phi$.

Therefore $x \notin A$ and $x \notin A^{/b}$, Then $x \notin A \cup A^{/b}$. Thus $A \cup A^{/b} \subseteq \overline{A}^{b}$. On the other hand, $x \notin A \cup A^{/b}$ implies that there exists a $b-open$ set V in X such that $x \in V$ and $V \cap A = \phi$. Hence $F = X - V$ is a $b-closed$ set in X such that $A \subseteq F$ and $x \notin F$.Hence $x \notin \overline{A}^{b}$. Thus $\overline{A}^{b} \subseteq A \cup \overline{A}^{b}$. Therefore $\overline{A}^{b} = A \cup A^{/b}$.

For (ii) and (iii) the proof is easy.

Definition 1.4 [2]

Let X be a topological space and $A \subseteq X$. The $b-closure$ of A is defined as the intersection of all $b-closed$ sets in X containing A, and is denoted by $\overline{A}^{b}$. It is clear that $\overline{A}^{b}$ is $b-closed$ set for any subset A of X , and $A \subseteq \overline{A}^{b}$.

Proposition 1.5

Let X be a topological space and $A \subset X$, then $\overline{A}^{b}$ is the smallest $b-closed$ set containing A .

Proof:

Suppose that B is $b-closed$ set such that $A \subseteq B$. Since $A^{-b} = A \cup A^{/b}$ by proposition 1.3 (i) , and $A \subset B$. Then $A^{-b} = A \cup A^{/b} \subseteq B \cup B^{/b} \subseteq B$. Thus $\overline{A}^{b} \subset B$. therefore $\overline{A}^{b}$ is the smallest $b-closed$ set containing A.

Proposition 1.6

Let X be a topological space and $A \subseteq B \subseteq X$. Then :

(i) $\overline{A}^{b} = \overline{A}^{s} \cap \overline{A}^{p}$

(ii) $\overline{A}^{b} \subseteq \overline{A}$

(iii) A is $b-closed$ set iff $A = \overline{A}^{b}$

(iv) $\overline{A}^{b} \subseteq \overline{B}^{b}$

(v) $\overline{\overline{A}^{b}}^{b} = \overline{A}^{b}$

Proof:

(i) See [2]

(ii) since $\overline{A}^{b} = \overline{A}^{s} \cap \overline{A}^{p} \subseteq \overline{A}^{s}$. Then $\overline{A}^{s} \subset \overline{A}$[3] therefore $\overline{A}^{b} \subset \overline{A}$.

(iii) Suppose that A is $b-closed$ set. Since $A \subset \overline{A}^{b}$ by definition $\overline{A}^{b}$, and $A \subseteq A$. Then $\overline{A}^{b} \subseteq A$. Therefore $\overline{A}^{b} = A$. Conversely suppose that $\overline{A}^{b} = A$, since $\overline{A}^{b}$ is $b-closed$ set by definition $\overline{A}^{b}$, and $\overline{A}^{b} = A$. Then A is $b-closed$

(iv) Since $B \subset \overline{B}^{b}$ by definition $\overline{B}^{b}$, and $A \subset B$. Then $A \subset \overline{B}^{b}$, since $\overline{B}^{b}$ is $b-closed$ and $A \subset \overline{B}^{b}$ by proposition 1.5, then $\overline{A}^{b} \subseteq \overline{B}^{b}$

For (v) the proof is easy.

Definition 1.7:

Let X be a topological space . Two subsets A and B of X are called $b-seperated$ iff $\overline{A}^{b} \cap B = A \cap \overline{B}^{b} = \phi$.

Definition 1.8 :

Let X be a topological space and $\phi \neq A \subseteq X$. Then A is called $b-connected$ set iff A is not the union of any two $b-seperated$ sets.

Remark 1.9

A set A is called $b-clopen$ iff it is $b-open$ and $b-closed$.

Proposition 1.10

Let X be a topological space, then the following statements are equivalent :

1- X is $b-connected$ space.

2- The only $b-clopen$ set in the space are X and ϕ.

3- The exist no two disjoint $b-open$ set and A and B such that $X = A \cup B$.

Proof:

(1) $\rightarrow$(2)

let X be $b-connected$ space. Suppose that D is $b-clopen$ set such that $D \neq \phi$ and $D \neq X$. Let $E = X - D$, then $E \neq \phi$ (since $D \neq X$). Since D is $b-open$ set, then E is $b-closed$ set. But $\overset{-b}{D} \cap E = D \cap E = \phi$ (since D is $b-clopen$set) and $D \cap \overset{-b}{E} = D \cap E = \phi$ (since E is $b-closed$ set), then D and E are two $b-seperated$ sets and $X = D \cup E$. Hence X is not $b-connected$ space which is a contradiction. Therefore the only $b-clopen$sets in the space are X and ϕ.

(2) $\rightarrow$ (3)

Suppose that the only $b-clopen$ sets in the space are X and ϕ. Assume that there exists two disjoint $b-open$ sets A and B such that $X = A \cup B$. Since $\overset{c}{B} = A$, then A is $b-clopen$ set. But $A \neq \phi$ and $A \neq X$, which is a contradiction. Hence there exist no two disjoint $b-open$ sets A and B such that $X = A \cup B$.

(3) $\rightarrow$ (1)

Suppose that X is not $b-connected$ space. Then there exist two $b-seperated$ sets A and B such that $X = A \cup B$. Since $\overset{-b}{A} \cap B = \phi$ and $A \cap B \subseteq \overset{-b}{A} \cap B$, thus $A \cap B = \phi$. Since $\overset{-b}{A} \subseteq \overset{c}{B} = A$, then A is $b-closed$ set. By the same way we can see that B is $b-closed$ set. Since $\overset{c}{A} = B$, then A and B are $b-open$ sets. Therefore A and B are two disjoint $b-open$ sets such that $X = A \cup B$ which is a contradiction. Hence X is $b-connected$ space.

Example 1.11

It is clear that each $b-connected$ space is connected. However, a connected space is not necessarily $b-connected$, as seen by the following example. Suppose X is any set with at least three points. Let $a \in X$, and $T = \{A : A \subset X, a \notin A\} \cup X$. It is clear that X is connected space. Let $A \neq \phi$, $A \neq X$ such that $a \notin A$. It is cleat that A is open set, then A is $b-open$ set (since each open set is $b-open$). Since $\bar{A} = A \cup \{a\}$, then $\overset{\circ}{\bar{A}} = A \subseteq A$. Therefore A is $b-closed$ set. Hence A is $b-clopen$ set and $A \neq \phi$, $A \neq X$, then X is not $b-connected$ space.

Example 1.12

In this example we show that $b-connectivity$ is not a hereditary property . Let $X = \{a,b,c,d\}$ and $T = \{\{a\},\{a,b\},\{a,c\},\{a,b,c\},X,\phi\}$ be a topological on X. The $b-open$ sets are:- $\{a\},\{a,b\}$, $\{a,c\},\{a,b,c\},\{a,d\},\{a,b,d\},\{a,c,d\},X$ and ϕ. It is clear that X is $b-connected$ space. Since the only $b-clopen$ sets are X and ϕ. Let $Y = \{b,c\}$, then

$Ty=\{\{b\},\{c\},Y,\phi\}$. It is clear that Y is not $b-connected$ space. Since $\{b\}\neq\phi,\{b\}\neq Y$ and $\{b\}$ is $b-clopen$ set in Y. Thus a $b-connectivity$ is not a hereditary property.

Proposition 1.13

Let A be a $b-connected$ set and D, E are $b-seperated$ sets. If $A\subseteq D\cup E$, then either $A\subseteq D$ or $A\subseteq E$.

Proof:

Suppose A be a $b-connected$ set and D, E are $b-seperated$ sets and $A\subseteq D\cup E$. Let $A\not\subset D$ and $A\not\subset E$. Suppose $A_1=D\cap A\neq\phi$ and $A_2=E\cap A\neq\phi$. Then $A=A_1\cup A_2$. Since $A_1\subseteq D$, hence $\overset{-b}{A_1}\subseteq\overset{-b}{D}$. Since $\overset{-b}{D}\cap E=\phi$, then $\overset{-b}{A_1}\cap A_2=\phi$. Since $A_2\subseteq E$, hence $\overset{-b}{A_2}\subseteq\overset{-b}{E}$. Since $\overset{-b}{E}\cap D=\phi$, then $\overset{-b}{A_2}\cap A_1=\phi$. But $A=A_1\cup A_2$, therefore A is not $b-connected$ space which is a contradiction. Then either $A\subseteq D$ or $A\subseteq E$.

Proposition 1.14

Let X be a topological space such that any two elements x and y of X are contained in some $b-connected$ subspace of X. Then X is $b-connected$.

Proof:

Suppose X is not $b-connected$ space. Then X is the union of two $b-seperated$ sets A,B. Since A,B are non empty sets, thus there exists a,b such that $a\in A,b\in B$. Let D be $b-connected$ subspace of X which contains a and b. Therefore either $D\subseteq A$ or $D\subseteq B$ which is a contradiction. (since $A\cap B=\phi$). Then X is $b-connected$ space.

Proposition 1.15

If A is $b-connected$ set then $\overset{-b}{A}$ is $b-connected$.

Proof :

Suppose A is $b-connected$ and $\overset{-b}{A}$ is not. Then there exist two $b-seperated$ sets D, E such that $\overset{-b}{A} = D \cup E$. But $A \subseteq \overset{-b}{A}$, then $A \subseteq D \cup E$ and since A is $b-connected$ set, then either $A \subseteq D$ or $A \subseteq E$.

(1) If $A \subseteq D$, then $\overset{-b}{A} \subseteq \overset{-b}{D}$. But $\overset{-b}{D} \cap E = \phi$, hence $\overset{-b}{A} \cap E = \phi$. Since $\overset{-b}{A} \subseteq D$, then $E = \phi$ which is a contradiction.

(2) If $A \subseteq E$, then $\overset{-b}{A} \subseteq \overset{-b}{E}$. But $E^{-b} \cap D = \phi$, hence $\overset{-b}{A} \cap D = \phi$. Since $\overset{-b}{A} \subseteq E$, then $D = \phi$ which is a contradiction Therefore $\overset{-b}{A}$ is $b-connected$ set.

Proposition 1.16

If D is $b-connected$ set and $D \subseteq E \subseteq \overset{-b}{D}$, then E is $b-connected$.

Proof:

If D is not $b-connected$, then there exist two sets A, B such that $\overset{-b}{A} \cap B = A \cap \overset{-b}{B} = \phi$. Since $D \subset E$, thus either $D \subseteq A$ or $D \subseteq B$. Suppose $D \subseteq A$, then $\overset{-b}{D} \subseteq \overset{-b}{B}$, thus $\overset{-b}{D} \cap B = \overset{-b}{A} \cap B = \phi$. But $B \subseteq E \subseteq \overset{-b}{D}$, thus $\overset{-b}{D} \cap B = B$. Therefore $B = \phi$ which is a contradiction. Thus E is $b-connected$ set.

If $D \subseteq B$, then $A = \phi$ which is a contradiction. Then E is $b-connected$ set.

Corollary 1.17

If space X contains a $b-connected$ subspace A such that $\overset{-b}{A}=X$, then X is $b-connected$.

Proof:

Suppose A is a $b-connected$ subspace of a space X such that $\overset{-b}{A}=X$, since $A\subseteq X=\overset{-b}{A}$, then by proposition 1.16, X is $b-connected$.

Proposition 1.18

If A and B are $b-connected$ subspaces of a space X such that $A\cap B\neq\phi$, then AUB is $b-connected$.

Proof:

Suppose that $A\cup B$ is not $b-connected$, then there are two $b-seperated$ sets D and E such that $A\cup B=D\cup E$. Since $A\subseteq A\cup B=D\cup E$ and A is a $b-connected$, then either $A\subseteq D$ or $A\subseteq E$. Since $B\subseteq A\cup B=D\cup E$ and B is $b-connected$, then either $B\subseteq D$ or $B\subseteq E$.

(1) If $A\subseteq D$ and $B\subseteq E$, then $A\cup B\subseteq D$. Hence $E=\phi$ which is a contradiction.

(2) f $A\subseteq D$ and $B\subseteq E$, then $A\cap B\subseteq D\cap E=\phi$. Therefore $A\cap B=\phi$ which is a contradiction.

By the same way we can get a contradiction if $A\subseteq D$ and $B\subseteq D$ or if $A\subseteq E$ and $B\subseteq E$. Therefore $A\cup B$ is $b-connected$ subspace of space X .

Proposition 1.19

If each $b-open$ subset of X is connected, then every pair of non empty open subsets of X has a non empty intersection.

Proof:

Let A,B be open subsets of X such that $A \cap B = \phi$. It is clear that $A \cup B$ is an open subset of X and A,B are open in $A \cup B$. Then $A \cup B$ is not connected set which is a contradiction since $A \cup B$ is $b-open$ subset of X. Therefore $A \cap B \neq \phi$.

Preposition 1.20

If each $b-open$ subset of X is $b-connected$, then every pair of non empty open set subset of X has a non empty intersection.

Proof:

Let each $b-open$ subset of X is $b-connected$. Since each $b-connected$ is connected, then by proposition 1.19 every pair of non empty subset of X has a non empty intersection.

Remark 1.21

The converse of preposition 1.20 is not true as shown by the following example.

Example 1.22

We consider the topological space (X,T) where $X = \{a,b,c\}$ and $T = \{\phi, X, \{a,b\}\}$. Let $A = \{a,b\}$ is $b-open$ set then $T_A = \{A, \phi\}$. Then $b-open$ sets is A are $\phi, A, \{a\}, \{b\}$. It is clear every pair of non-empty subsets of X has a non empty intersection but A is $b-open$ subset of X and it is not $b-connected$.

Preposition 1.23

If each $b-open$ subset of X is $b-connected$, then each $b-open$ subset of X is connected.

Proof:

It follows since every $b-connected$ set is connected.

Remark 1.24

The converse of preposition 1.23 is not true as shown by the following example.

Example 1.25

We consider the topological space given in example 1.22, then $A=\{a,b\}$ is connected but A is not $b-connected$.

Section Two

$T_B-Spaces$

Definition 2.1 [5]

A topological space (X,T) is called $T_D-space$ iff the set of limit points of any singeleton is closed . On the other hand ,a topological space (X,T) is called $I-space$ iff each open subset of X is connected.

Definition 2.2

Atopological space (X,T) is called $T_B-space$ iff the set of *b-limit* of any sington is $b-closed$.

We shall prove later that $T_B - Space$ and $T_D - Space$ are equivalent if the space is $MI - Space$. The following example show that the $T_D - Space$ and $T_B - Space$ are not equivalent in general.

Example 2.3

Let $X = \{a, b, c, d, e\}$ and $T_X = \{\{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}, \{d\}, \{a, d\}, \phi, X\}$ be a topological on X. The $b - open$ set $\{a\}, \{c, d\}$, $\{a, b, e\}$, $\{a, c, d\}, \{a, b, d, e\}, \{d\}, \{a, d\}, \{a, b, d\}, \{a, d, e\}, \{a, b, c, d\}, \{a, c, d, e\}, \{b, d, e\}, \{b, c, d, e\}$, $\{a, e\}, \{b, d\}, \{d, e\}, \{c, d, e\}, \{b, c, d\}, \{a, b\}, \phi, X$. It is clear that X is $T_B - Space$. But X is not $T_D - Space$. Since $\{b\}^{/} = \{e\}$ and $\{e\}$ is not closed .

Definition 2.4

Let (X, T) be a topological space, then (X, T) is called BT_o (resp.,B-T_1) iff for every $x, y \in X$ such that $x \neq y$, there exists a $b - open$ set containing x but not y or (resp- and) a $b - open$ set containing y but not x.

Proposition 2.5

Let (X, T) be a topological space. If for every $x \in X, \{x\}^{/b}$ is $b - closed$ set , then (X, T) is a BT_o-space.

Proof :

Let $x, y \in X$ such that $x \neq y$. Then either $y \notin \overline{\{x\}}^{b}$, in which case $Ny = \left(\overline{\{x\}}^{b}\right)^{c}$ is a $b - open$ set contain y which does not contain x or $y \in \overline{\{x\}}^{b}$. Then $y \in \{x\}^{/b}$. Hence $Nx = \left(\{x\}^{/b}\right)^{c}$ is a $b - open$ set which does not contain y. If $x \in Nx$, then $x \in \{x\}^{/b}$. Hence for each Vx is $b - open$ set contain

$x,(Vx \cap \{x\}) - \{x\} \neq \phi$ which is a contradiction . Then Nx contain x . Therefore (X,T) is $BTo-space$.

Proposition 2.6

Every $T_B-space$ is $BTo-space$**.**

Proof:

This follows immediately from proposition 2.5.

Proposition 2.7

A topological space (X,T) is $BT_1-space$ iff $\{x\} = \overline{\{x\}}^{b}$ for each $x \in X$.

Proof:

Let (X,T) be a $BT_1-space$ and $x \in X$. If $y \in X - \{x\}$, then there exist $b-open$ set such that $y \in G$ and $x \in X - G$. Hence $y \notin \overline{\{x\}}^{b}$ and $\overline{\{x\}}^{b} = \{x\}$. Conversely suppose that $\overline{\{x\}}^{b} = \{x\}$, for each $x \in X$. Let $y,z \in X$ with $y \neq z$. then $\overline{\{y\}}^{b} = \{y\}$ implies that $\overline{\{y\}}^{b\,c}$ is $b-open$ set contain z but not y . Also, $\overline{\{z\}}^{b} = \{y\}$ implies that $\overline{\{z\}}^{b\,c}$ is $b-open$ set contain y but not z . thus (X,T) is $TB-space$.

Proposition 2.8

Every $BT_1-space$ **is** $T_B-space$.

Proof:

In a $BT_1-space$ X. $\overline{\{x\}}^{b}=\{x\}$ for all $x\in X$. Hence $\{x\}^{/b}\subseteq\{x\}$. Therefore $\{x\}^{/b/b}\subseteq\{x\}^{/b}$. Then $\{x\}^{/b}$ is $b-closed$. Hence the space is $T_B-space$.

Definition 2.9 [1]

An $I-space$ (X,T) is called a maximal $I-space$ if for any topological U on X such that $T\subset U$, then (X,U) is not an $I-space$. We shall denote a maximal $I-space$ briefly by $MI-space$.

Proposition 2.10 [1]

Let (X,T) be a $MI-space$. If (X,T) is T_1-*space*. Then :

$(X,T)=\{A: x_\circ\in A, \textit{for}\ \text{some}\ x_\circ\in X\}\cup\{\phi\}$

$=\langle x,x_\circ\rangle$ for some $x_\circ\in X$.

Proposition 2.11

Let (X,T) be a $T_1-space$. Then (X,T) is $T_B-Space$ iff it is $T_D-Space$.

Proof:

Let (X,T) be a $T_1-space$. Then $\{x\}^{/}=\{x\}^{/b}=\phi$ for each $x\in X$. then for $x\in X,\{x\}^{/}$ and $\{x\}^{/b}$ are closed and hence they are $b-closed$ set. Therefore (X,T) is $T_B-Space$. Iff it is $T_D-Space$.

Proposition 2.12

Let (X,T) be a $MI-space$, and is not $T_1-space$. Then (X,T) is $T_B-Space$ iff it is $T_D-Space$.

Proof:

Let (X,T) be a $MI-space$ which is not $T_1-space$. Then $(X,T)=\langle x,x_\circ\rangle$ for some $x_\circ\in X$ (By proposition 2.10). thus $\{x\}^/=\phi$ for each $x\neq x_\circ$ and $\{x_\circ\}^/=X-\{x_\circ\}$therefore $\{x\}^/$ is closed for each $x\in X$. Since in this space the $b-open$ sets are the same as open sets, then $\{x\}^/$ is closed iff $\{x\}^{/b}$ is $b-closed$ for each $x\in X$. Therefore (X,T) is $T_B-space$ iff it is $T_D-space$.

Theorem 2.13

Let (X,T) be a $MI-space$. Then (X,T) is $T_B-space$ iff it is $T_D-space$.

Proof:

The theorem follows immediately from proposition 2.12, 2.11 .

References

[1] N.Al-Amili and A.G. Naoum, "On Maximal irreducible space". Bull. Coll. Sci.,Vol. 17, No.2, (1976).

[2] D. Andrijevic, "On $b-open$ sets" , Mat. Vesnik, 48 (1996),59-64.

[3] P. Das, "Note on some applications on semi-open sets", progress of Mathematics, 7 (1973).

[4] Michael C. Gemigani, "Elementary Topology" , Reading, Mass : Addison-Wesley publishing Co.,Inc. , (1967).

[5] J. Throne , "Topological structures" Holt Reinhart and Winston Book Co. , New York, (1966).