Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.17.(3) 2025,pp.Math 1–11

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

On ideal supra Z_open set in ideal supra topological space

Saja S. Faiad [a]**, *Rana B.Yaseen*** [b]

[a] *Department of Mathematics, College of Education for women , Tikrit University, Tikrit, Iraq. Email: saja.sulimanfaiad771@st.tu.edu.iq*

[b] *Department of Mathematics, College of Education for women , Tikrit University, Tikrit, Iraq..Email: zian2016@tu.edu.iq*

ARTICLE INFO

Article history:
Received: 30/05/2025
Rrevised form: 21/06/2025
Accepted : 25/06/2025
Available online: 30/09/2025

Keywords:

ideal supra space, ideal supra $\mathcal{Z}$_open set, ideal supra $\mathcal{Z}\_T_i$-space

ABSTRACT

This study introduces the ideal supra $\mathcal{Z}$_open set in ideal supra topological space, where its basic properties are defined and examined through illustrative examples and related theorems and the relations between some separation axioms (T_i, i=0,1,2) are generalized and studied over the spaces under study.

MSC..

https://doi.org/10.29304/jqcsm.2025.17.32386

1. Introduction and Preliminaries

Ideal topology was first introduced by Kuratowski [4] and Vaidyanathswamy [7]. Further Hamlett and Jankovic also studied the properties of ideal topological spaces in [3] and [13]. In 1983, Mashhourm A.S. [2] established supra topological space(U, μ) and studied supra- continuous maps. The properties of supra topological space were generalized by many other researchers as in [12,5]. In 2008, Yasseen, R.B. el at [11] studied separation axioms in supra topology. The ideal supra topological space was studied by researchers Shyamapada elat. [9,1].Tamer, S.A. is first to suggest $\zeta -$ open sets and some types of $\zeta -$ continuous functions on topological spaces [8] . As for researcher, Nadia, she has proven that the $\zeta -$ topological spaces are ζ –compact spaces [5]. In 2024, Saja S. Faiad elat proposed the concepts of supra $\mathcal{Z}$_open sets and studied new properties of them, as well as and supra $\mathcal{Z}$_continuous functions and some of their properties [10].

Definition 1.1 [10]

A subset Z of (U, μ) is said to be supra $\mathcal{Z}$- open sets ($\mathcal{Z}_\mu\_os$), if $Z \cap cl_\mu(H) \neq \emptyset, \forall x \in Z$,there exists non-empty supra open set H content x, such that if $Z=\emptyset \rightarrow Z \cap cl_\mu(H)= \emptyset$ and $Z = H = U \rightarrow Z \cap cl_\mu(H) = U$.

∗Corresponding author

Email addresses: *saja.sulimanfaiad771@st.tu.edu.iq*

Communicated by 'sub etitor'

So, the complement of $\mathcal{Z}_{\mu}\_os$ is supra $\mathcal{Z}$- closed set ($\mathcal{Z}_{\mu}\_cs$) and the collections of all ($\mathcal{Z}_{\mu}\_os$) ($\mathcal{Z}_{\mu}\_cs$) subset of (U, μ) will be denoted by $\mathcal{Z}O(U, \mu)$, $\mathcal{Z}C(U, \mu)$.

Theorem 1.2 [10]: The sets of all $\mathcal{Z}_{\mu}\_os$ is satisfies the conditions of supra topology on U .

1. $\emptyset$,U $\in$ $\mathcal{Z}$O(U,μ).

2. If $\mathcal{Z}_{\alpha} \in \mathcal{Z}$O(U,μ) and x∈∪ $\mathcal{Z}_{\alpha} \rightarrow x \in H_{\alpha}^{*}$ and x $\in Z_{\alpha}^{*}$,$\forall \alpha^{*} \in \wedge$.

Then $(U, \mu_{\mathcal{Z}})$ is called supra $\mathcal{Z}$-topological space

Definition 1.3 [10]

A function f:$(U, \mu_{\mathcal{Z}}) \rightarrow (V, \acute{\mu}_{\mathcal{Z}})$ is called supra $\mathcal{Z}$_continuous ($\mathcal{Z}_{\mu}$_continuous), if the invers image is $\mathcal{Z}_{\mu}$_os in U, for each μ_os in V.

Example 1.4[10]

Let $\mu_u = \{U, \emptyset, \{b\}, \{a, b\}\}$ on U={a, b, c, d}, $\mu_u{}^{c} = \{U, \emptyset, \{a, c\}, \{c\}\}$

and $\mu_v = \{V, \emptyset, \{1,2\}\}$ on $V = \{1,2,3\}$, $\mu_v{}^{c} = \{U, \emptyset, \{3\}\}$. Then

$\mathcal{Z}O(U, \mu) = \{U, \emptyset, \{a\}, \{b\}, \{a, b\}\}$

$\mathcal{Z}O(V, \mu) = \{V, \emptyset, \{1\}, \{2\}, \{1,2\}\}$

Defined $f:(U, \mu_u) \rightarrow (V, \mu_v)$; f (a)=1 , f (b)=2 , f (c)= f (d) =3. So,

$f^{-1}(V)$=U $\in \mathcal{Z}_{\mu}O(U, \mu)$;

$f^{-1}(\emptyset)=\emptyset \in \mathcal{Z}_{\mu}O(U, \mu)$;

$f^{-1}(\{1,2\})$={a, b} $\in \mathcal{Z}_{\mu}O(U, \mu)$. Then f is $\mathcal{Z}_{\mu}$_cont. .

The aim of this paper is to study the set $\mathcal{Z}$ on the ideal supra topological space and to prove some of its properties, as well as to generalize the separation axioms on the space under consideration.

2- $\mathcal{Z}_{\mu}$_Separation axioms in supra topological space

Definition 2.1

A subset non-empty $\mathcal{A}$ of (U, μ), so $\mu_{\mathcal{A}}{}^{*}$ is the class of every intersection of $\mathcal{A}$ every element in U , then($\mathcal{A}, \mu_{\mathcal{A}}{}^{*}$) is supra topological subspace.

Definition 2.2

A (U, $\mu_{\mathcal{Z}}$) is called $\mathcal{Z}_{\mu}\_T_0$-space if and only if for every pair of distinct points x , y ∈ U, ∃ $\mathcal{Z}_{\mu}$_open set that contains one of the points but not the other.

Examples 2.3

Let μ= {∅, U ,{1}} on U={1,2,3}, then $\mathcal{Z}_{\mu}$_O(U, μ)= { ∅, U,{1}}.

So, 1≠2 → {1}∈ $\mathcal{Z}_{\mu}$_O(U, μ)→{1}∈{1}∧{2}∉{1}; we get (U, $\mu_{\mathcal{Z}}$) is $\mathcal{Z}_{\mu}\_T_0$-space. But, if $\mathcal{Z}_{\mu}$_O(U, μ)= { ∅, U,{1},{2},{1,2}} on μ= {∅, U ,{1,2}} then (U, $\mu_{\mathcal{Z}}$) is not $\mathcal{Z}_{\mu}\_T_0$-space.

Proportion 2.4

A space $(U, \mu_{\mathcal{Z}})$ is $\mathcal{Z}_{\mu}$_T_0-space if and only if $cl_{\mathcal{Z}_\mu}\{x\} \neq cl_{\mathcal{Z}_\mu}\{y\}, \forall x, y \in U; x \neq y$.

Proof:

Let U be $\mathcal{Z}_{\mu}$_T_0-space, to prove $cl_{\mathcal{Z}_\mu}\{x\} \neq cl_{\mathcal{Z}_\mu}\{y\}, \forall x, y \in U; x \neq y$.

U is $\mathcal{Z}_{\mu}$_T_0 -space and $x \neq y \longrightarrow \exists \mathbb{G} \in \mu_{\mathcal{Z}}; (x \in \mathbb{G} \wedge y \notin \mathbb{G}) \wedge (x \notin \mathbb{G} \wedge y \in \mathbb{G})$, let $(x \in \mathbb{G} \wedge y \notin \mathbb{G}) \longrightarrow (x \in \mathbb{G} \wedge y \in U / \mathbb{G})$, so

$U/\mathbb{G}$ is $\mathcal{Z}_{\mu}$_os, since $\mathbb{G}$ $\mathcal{Z}_{\mu}$_os $\longrightarrow \{y\} \subseteq U / \mathbb{G} \longrightarrow cl_{\mathcal{Z}_\mu}\{y\} \subseteq cl_{\mathcal{Z}_\mu}(U/\mathbb{G}) = U / \mathbb{G}$ (since $U/\mathbb{G}$ $\mathcal{Z}_{\mu}$_c an $cl_{\mathcal{Z}_\mu}(U/\mathbb{G})) = U / \mathbb{G}$. Thus $cl_{\mathcal{Z}_\mu}(y) \subseteq U/\mathbb{G} \wedge x \in \mathbb{G} \longrightarrow x \nsubseteq U/\mathbb{G}$ $\therefore cl_{\mathcal{Z}_\mu}\{x\} \neq cl_{\mathcal{Z}_\mu}\{y\}$

By similar way if we take $(x \notin \mathbb{G} \wedge y \in \mathbb{G})$

Conversely;

Let $cl_{\mathcal{Z}_\mu}\{x\} \neq cl_{\mathcal{Z}_\mu}\{y\}, \forall x \neq y \in U$ and U be not $\mathcal{Z}_{\mu}$_T_0-space $\longrightarrow \exists x, y \in U; \forall \mathbb{G} \in \mu_{\mathcal{Z}}; x \in \mathbb{G} \longrightarrow y \in \mathbb{G}$, by using defined (2.1)

So, $z \in U; z \in cl_{\mathcal{Z}_\mu}(x)$............(*) $\longrightarrow \forall \mathbb{G} \in \mu_{\mathcal{Z}}; \mathcal{Z}_{\mu} \in \mathbb{G} \wedge \mathbb{G} \cap \{x\} \neq \emptyset$

[since by true : $z \in cl_{\mathcal{Z}_\mu}(\mathcal{A}) \leftrightarrow \forall \mathbb{G} \in \mu_{\mathcal{Z}}; z \in \mathbb{G} \wedge \mathbb{G} \cap \mathcal{A} \neq \emptyset$]

But, $\mathbb{G} \cap \mathcal{A} \neq \emptyset \longrightarrow x \in \mathbb{G}$ (since the only element in $\{x\}$ is x)

$\therefore$ all set contains z must contain x. So, we have the following two statements : each $\mathcal{Z}_{\mu}$_os contains z must contains x and each $\mathcal{Z}_{\mu}$_os contains x must contains y $\longrightarrow$ every $\mathcal{Z}_{\mu}$_os contains z must contains y.

$\longrightarrow \forall \mathbb{G} \in \mu_{\mathcal{Z}}; \mathcal{Z}_{\mu} \in \mathbb{G} \wedge \mathbb{G} \cap \{y\} \neq \emptyset \longrightarrow z \in cl_{\mathcal{Z}_\mu}(y)$..........(*)

$\longrightarrow \forall z \in cl_{\mathcal{Z}_\mu}(x) \longrightarrow z \in cl_{\mathcal{Z}_\mu}(y) \longrightarrow cl_{\mathcal{Z}_\mu}\{x\} \subseteq cl_{\mathcal{Z}_\mu}\{y\}$

By similar way we prove $cl_{\mathcal{Z}_\mu}\{y\} \subseteq cl_{\mathcal{Z}_\mu}\{x\} \longrightarrow cl_{\mathcal{Z}_\mu}\{x\} = cl_{\mathcal{Z}_\mu}\{y\}$C!! . Then U is $\mathcal{Z}_{\mu}$_T_0-space(since $cl_{\mathcal{Z}_\mu}\{x\} \neq cl_{\mathcal{Z}_\mu}\{y\}$).

Definition 2.5

A $(U, \mu_{\mathcal{Z}})$ is $\mathcal{Z}_{\mu}$_T_1-space if and only if, for any two distinct points $x, y \in U$, ,$\exists$ $\mathcal{Z}_{\mu}$_os in U, $\ni$ one contains x but not y, and $\mathcal{Z}_{\mu}$_os in U containing y but not x .

Example 2.6

Let μ= {∅, U,{2},{2,3},{2,4}} on U={2,3,4}; so, $\mathcal{Z}_{\mu}$_O(U, μ)= { ∅, U,{2},{3},{4},{2,3},{2,4},{3,4}},Then, U is $\mathcal{Z}_{\mu}$_T_1-space.

Let 2≠3 $\rightarrow \begin{cases} \exists \mathbb{G} = \{2\} s.t \{2\} \in \{2\} \wedge \{3\} \notin \{2\} \\ \exists \mathbb{H} = \{3\} s.t \{2\} \notin \{3\} \wedge \{3\} \in \{3\} \end{cases}$

Let 2≠4 $\rightarrow \begin{cases} \exists \mathbb{G} = \{2\} s.t \{2\} \in \{2\} \wedge \{4\} \notin \{2\} \\ \exists \mathbb{H} = \{4\} s.t \{2\} \notin \{4\} \wedge \{4\} \in \{4\} \end{cases}$

Let 3≠4 $\rightarrow \begin{cases} \exists \mathbb{G} = \{3\} s.t \{3\} \in \{3\} \wedge \{3\} \notin \{4\} \\ \exists \mathbb{H} = \{4\} s.t \{3\} \notin \{4\} \wedge \{4\} \in \{4\} \end{cases}$

Remark 2.7

$\forall\ \mathcal{Z}_{\mu}\_T_1$-space is $\mathcal{Z}_{\mu}\_T_0$-space. On the contrary, it is not true. As the example

Let $\mu_{\mathcal{Z}}$ ={ U, ∅, {3}, {1,3}} on U={1,2,3},then $\mathcal{Z}_{\mu}\_O(U, \mu_{\mathcal{Z}})$={ V, ∅, {1}, {3}, {1,3}}

Then, U is $\mathcal{Z}_{\mu}\_T_0$-space but not $\mathcal{Z}_{\mu}\_T_1$-space

Theorem 2.8

$(U, \mu_{\mathcal{Z}})$ is $\mathcal{Z}_{\mu}\_T_1$-space if all singleton set in U is $\mathcal{Z}_{\mu}\_cs$.

Proof

Let U be $\mathcal{Z}_{\mu}\_T_1$-space, to prove $\{ x \}\ \mathcal{Z}_{\mu}\_c, \forall x \in U$

i.e., $U / \{ x \}\ \mathcal{Z}_{\mu}\_os$, we must prove $U /\{ x \}$ contains $\text{ꞥbđ}_{\mathcal{Z}_{\mu}}\ \forall y \in U / \{ x \}$

Let $y \in U / \{ x \} \longrightarrow x \neq y \longrightarrow$ U is $\mathcal{Z}_{\mu}\_T_1$-space,

$\longrightarrow \exists\ \mathbb{G}, \mathbb{H}_y \in \mu_{\mathcal{Z}}\ ; (x \in \mathbb{G} \wedge y \notin \mathbb{G}) \wedge (x \notin \mathbb{H}_y \wedge y \in \mathbb{H}_y)$

$y \in \mathbb{H}_y \wedge x \notin \mathbb{H}_y \longrightarrow \{ x \} \cap \mathbb{H}_y = \emptyset \Rightarrow \mathbb{H}_y \subseteq U /\{ x \} \wedge y \in \mathbb{H}_y$

$\longrightarrow \mathbb{H}_y \subseteq U / \{ x \} \forall y \in U / \{ x \} \longrightarrow U /\{ x \}$contains $\text{ꞥbđ}_{\mathcal{Z}_{\mu}}\ \forall y \in U / \{ x \}$.

$\therefore\ U /\{ x \}$contains $\text{ꞥbđ}_{\mathcal{Z}_{\mu}}\ \forall y \in U / \{ x \}$.

$\therefore\ U /\{ x \} \mathcal{Z}_{\mu}\_os \longrightarrow \{ x \}\ \mathcal{Z}_{\mu}\_c\ \forall x \in U$.

Conversely;

Let { x}is $\mathcal{Z}_{\mu}\_c$; $\forall x \in U$. To prove U is $\mathcal{Z}_{\mu}\_T_1$-space, let $x , y \in U$; $x \neq y \longrightarrow \{ x \}, \{y\}$ are $\mathcal{Z}_{\mu}\_c$ set $\longrightarrow U / \{ x \}, U /\{y\}$ are $\mathcal{Z}_{\mu}\_c$ sets, say $\mathbb{G} = U /\{y\}$, $\mathbb{H} = U /\{x\} \longrightarrow (x \in \mathbb{G} \wedge y \notin \mathbb{G}) \wedge (x \notin \mathbb{H} \wedge y \in \mathbb{H})$. Then $(U, \mu_{\mathcal{Z}})$ is $\mathcal{Z}_{\mu}\_T_1$-space

Corollary 2.9 : A $(U, \mu_{\mathcal{Z}})$ is a $\mathcal{Z}_{\mu}\_T_1$-space, then each finite set is $\mathcal{Z}_{\mu}\_c$.

Proof

Let $\mathcal{A}$ be a finite set in $U \longrightarrow \mathcal{A} = \{ x_1, \dots . x_n\} = \cup_{i=1}^{n}\{ x_i\} \longrightarrow \{ x_i\} \in \mathcal{Z}_{\mu}\_C(U, \mu)\ \forall i \longrightarrow \cup_{i=1}^{n}\{ x_i\}\ \mathcal{Z}_{\mu}\_c.$, Then $\mathcal{A}$ is $\mathcal{Z}_{\mu}\_c$.

Definition 2.10

A $(U, \mu_{\mathcal{Z}})$ is said to be a $\mathcal{Z}_{\mu}\_T_2$-space, if and only if for any two distinct points $x , y \in U$, $\exists \mathcal{Z}_{\mu}\_o$ sets $\mathbb{G}$ and $\mathbb{H}$ such that $x \in \mathbb{G}, y \in \mathbb{H}$, and the intersection $\mathbb{G} \cap \mathbb{H} = \emptyset$.

Examples 2.11

1-Let μ= {∅, U,{1,2},{2,3}} on U={1,2,3};

$\mathcal{Z}_{\mu}\_O(U, \mu)$= { ∅, U,{1},{2},{3},{1,2},{1,3},{2,3}}. Then, U is $\mathcal{Z}_{\mu}\_T_2$-space.

2- Let μ= {∅, U,{1,2}} on U={1,2,3,4}, then $\mathcal{Z}_{\mu}\_O(U, \mu)$= { ∅, U,{1},{2},{1,2}} , Then, U is not $\mathcal{Z}_{\mu}\_T_1$-space

Remark 2.12

$\forall\ \mathcal{Z}_{\mu}\_T_2$-space is $\mathcal{Z}_{\mu}\_T_1$-space. On the contrary, it is not true, as the example

$(\mathcal{N}, \mu_{\mathcal{Z}(\mathrm{cof})})$ is $\mathcal{Z}_{\mu}\_T_1$-space. But $(\mathcal{N}, \mu_{\mathcal{Z}(\mathrm{cof})})$ is not $\mathcal{Z}_{\mu}\_T_2$-space [because if $x \neq y, \exists\ \mathbb{G} = \mathcal{N} / \{x\} \in \mu_{\mathcal{Z}(\mathrm{cof})}$, $\mathbb{H} = \mathcal{N} / \{y\} \in \mu_{\mathcal{Z}(\mathrm{cof})}$, but $\mathbb{G} \cap \mathbb{H} \neq \emptyset$].

Proposition 2.13

$\mathcal{Z}_{\mu}\_T_i$-space is a hereditary property. Where i=0,1,2

Proof : prove the $\mathcal{Z}_{\mu}\_T_1$-space.

Let $(U, \mu_{\mathcal{Z}})$ be $\mathcal{Z}_{\mu}\_T_1$-space and $(V, \mu_{\mathcal{Z}'})$ subspace of U, to prove $(V, \mu_{\mathcal{Z}'})$ is $\mathcal{Z}_{\mu}\_T_1$-space, let $x, y \in V; x \neq y \rightarrow x, y \in U$ (since $V \subseteq U$)

$\because$ U is $\mathcal{Z}_{\mu}\_T_1$-space $\rightarrow \exists\ \mathbb{G}, \mathbb{H} \in \mu_{\mathcal{Z}}$; $(x \in \mathbb{G} \wedge y \notin \mathbb{G}) \vee (x \notin \mathbb{H} \wedge y \in \mathbb{H})$

$\longrightarrow \mathbb{G} \cap V \wedge \mathbb{H} \cap V \in \mu_{\mathcal{Z}'}$ (by def. $\mu_{\mathcal{Z}'}$)

$\longrightarrow (x \in \mathbb{G} \cap V \wedge y \notin \mathbb{G} \cap V) \wedge (x \notin \mathbb{H} \cap V \wedge y \in \mathbb{H} \cap V)$.; Then $(V, \mu_{\mathcal{Z}'})$ is a $\mathcal{Z}_{\mu}\_T_1$-space.

Theorem 2.14

$\mathcal{Z}_{\mu}\_T_i$-space is a Topological property. Where i=0,1,2

Proof: prove the $\mathcal{Z}_{\mu}\_T_2$-space

Since $f : (U, \mu_{\mathcal{Z}}) \rightarrow (V, \mu_{\mathcal{Z}'})$ is bijective $\mathcal{Z}_{\mu}$_cont.

Let $x_1, x_2 \in V; x_1 \neq x_2 \longrightarrow f^{-1}(x_1), f^{-1}(x_2) \in V$ [f^{-1} $\mathcal{Z}_{\mu}$_cont.]

$\because$ f onto $\mathcal{Z}_{\mu}$_function $\rightarrow f^{-1}(x_1) \neq \emptyset, f^{-1}(y_2) \neq \emptyset$

$\because$ f 1-1 $\mathcal{Z}_{\mu}$_ function$\rightarrow \exists\ y_1 \in V$; $f^{-1}(x_1) = y_1$ and $\exists\ y_2 \in V$; $f^{-1}(x_2) = y_2$ and $y_1 \neq y_2$ and $y_1, y_2 \in V$. $\because$ V is $\mathcal{Z}_{\mu}\_T_2$-space $\longrightarrow \exists\ \mathbb{H}_1, \mathbb{H}_2 \in \mu_{\mathcal{Z}'}$; $\mathbb{H}_1 \cap \mathbb{H}_2 \neq \emptyset$ $(y_1 \in \mathbb{H}_1 \wedge y_2 \notin \mathbb{H}_1)$;

f is $\mathcal{Z}_{\mu}$_ continuous $\longrightarrow f^{-1}(\mathbb{H}_1) = \mathbb{G}_1, f^{-1}(\mathbb{H}_2) = \mathbb{G}_2 \in \mu_{\mathcal{Z}}$; $\mathbb{G}_1 \cap \mathbb{G}_2 = f^{-1}(\mathbb{H}_1) \cap f^{-1}(\mathbb{H}_2) = f^{-1}(\mathbb{H}_1 \cap \mathbb{H}_2) = f^{-1}(\emptyset) = \emptyset$, $x_1 \in \mathbb{G}_1 \wedge x_2 \in \mathbb{G}_2$. Then U is $\mathcal{Z}_{\mu}\_T_2$-space. By similar we prove, if V is $\mathcal{Z}_{\mu}\_T_2$-space, then U is $\mathcal{Z}_{\mu}\_T_2$-space

3- On ideal supra $\mathcal{Z}$_open set in ideal supra topological space

I generalized and studied the $\mathcal{Z}_{\mu}$_ set on the $(U, \mu, \mathbb{I})$, with proving some theorems and providing examples related. to it . We illustrate this with the following example, which will calculate the $\mathcal{Z}_{\mu}\_o$ set on $(U, \mu, \mathbb{I})$by denoted $\mathcal{Z}_{\mu}\mathbb{I}\_os$.

Remarks 3.1

1.The complement of $\mathcal{Z}_{\mu}\mathbb{I}\_os$ is ideal supra $\mathcal{Z}$_closed set ($\mathcal{Z}_{\mu}\mathbb{I}_cs$).

2.The collections of all ($\mathcal{Z}_{\mu}\mathbb{I}\_os$) ($\mathcal{Z}_{\mu}\mathbb{I}\_cs$) subset of $(U, \mu, \mathbb{I})$ will be denoted by $(\mathcal{Z}O(U, \mu, \mathbb{I}))$, $(\mathcal{Z}C(U, \mu, \mathbb{I}))$.

Example 3.2

Let $\mu = \{\emptyset, U, \{a, b\}, \{a, d\}, \{a, b, d\}\}$ on $U = \{a, b, c, d\}$; $\mu^c = \{\emptyset, U, \{c, d\}, \{b, c\}, \{c\}\}$, with $\mathbb{I} = \{\emptyset, \{b\}\}$. So , $(U, \mu, \mathbb{I}) = \{\emptyset, U, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}$;

P(U)		$\mathcal{Z}_{\mu}\mathbb{I}-os$
$\emptyset$	From def.	**Yes**
U	From def.	**Yes**
$\{a\}$	$\exists$open set contains a and a$\in\{a,b\}$,$\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$	**Yes**
$\{b\}$	$\exists$open set contains b and b $\in\{a,b\}$,$\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$	**Yes**
$\{c\}$	$\nexists$open set contains c and c$\in$ U,$\ni\ \emptyset\neq U=U$ {since $c\in U=U$	**No**
$\{d\}$	$\exists$open set contains d and d $\in\{a,d\}$,$\ni\ \emptyset\neq\{a,d\}\neq U$ and $\mathbb{S}\cap\overline{\{a,d\}}\to\mathbb{S}\cap U\neq\emptyset$	**Yes**
$\{a,b\}$	$\exists$open set contains a and a $\in\{a,b\},\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$ $\exists$open set contains b and b $\in\{a,b\},\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$	**Yes**
$\{a,c\}$	$\exists$open set contains a and a $\in\{a,b\},\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$ $\nexists$open set contains c and c $\in U,\ni\ \emptyset\neq U=U$ {since $c\in U=U$}	**No**
$\{a,d\}$	$\exists$open set contains a and a $\in\{a,b\},\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$ $\exists$open set contains d and d $\in\{a,d\},\ni\ \emptyset\neq\{a,d\}\neq U$ and $\mathbb{S}\cap\overline{\{a,d\}}\to\mathbb{S}\cap U\neq\emptyset$	**Yes**
$\{b,c\}$	$\exists$open set contains b and b $\in\{a,b\},\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$ $\nexists$open set contains c and c $\in U,\ni\ \emptyset\neq U=U$ {since c $\in U=U$}	**No**
$\{b,d\}$	$\exists$open set contains b and b $\in\{a,b\},\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$ $\exists$open set contains d and d $\in\{a,d\},\ni\ \emptyset\neq\{a,d\}\neq U$ and $\mathbb{S}\cap\overline{\{a,d\}}\to\mathbb{S}\cap U\neq\emptyset$	**Yes**
$\{c,d\}$	$\nexists$open set contains c and c $\in U,\ni\ \emptyset\neq U=U$ {since $c\in U=U$} $\exists$open set contains d and d $\in\{a,d\},\ni\ \emptyset\neq\{a,d\}\neq U$ and $\mathbb{S}\cap\overline{\{a,d\}}\to\mathbb{S}\cap U\neq\emptyset$	**No**
$\{a,b,c\}$	$\exists$open set contains a and a $\in\{a,b\},\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$ $\exists$open set contains b and b $\in\{a,b\},\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$ $\nexists$open set contains c and c $\in U,\ni\ \emptyset\neq U=U$ {since c $\in U=U$}	**No**
$\{a,b,d\}$	$\exists$open set contains a and a $\in\{a,b\},\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$ $\exists$open set contains b and b $\in\{a,b\},\ni\ \emptyset\neq\{a,b\}\neq U$ and $\mathbb{S}\cap\overline{\{a,b\}}\to\mathbb{S}\cap U\neq\emptyset$ $\exists$open set contains d and d $\in\{a,d\},\ni\ \emptyset\neq\{a,d\}\neq U$ and $\mathbb{S}\cap\overline{\{a,d\}}\to\mathbb{S}\cap U\neq\emptyset$	**Yes**

$\{\mathbf{a},\mathbf{c},\mathbf{d}\}$	$\begin{cases} \exists\text{open set contains a and } a \in \{a,b\}, \ni\ \emptyset \neq \{a,b\} \neq U \\ \text{and } \mathbb{S} \cap \overline{\{a,b\}} \to \mathbb{S} \cap U \neq \emptyset \\ \nexists\text{open set contains c and } c \in U, \ni\ \emptyset \neq U = U \\ \{\text{since } c \in U = U\} \\ \exists\text{open set contains d and } d \in \{a,d\}, \ni\ \emptyset \neq \{a,d\} \neq U \\ \text{and } \mathbb{S} \cap \overline{\{a,d\}} \to \mathbb{S} \cap U \neq \emptyset \end{cases}$	**No**
$\{\mathbf{b},\mathbf{c},\mathbf{d}\}$	$\begin{cases} \exists\text{open set contains b and } b \in \{a,b\}, \ni\ \emptyset \neq \{a,b\} \neq U \\ \text{and } \mathbb{S} \cap \overline{\{a,b\}} \to \mathbb{S} \cap U \neq \emptyset \\ \nexists\text{open set contains c and } c \in U, \ni\ \emptyset \neq U = U \\ \{\text{since } c \in U = U\} \\ \exists\text{open set contains d and } d \in \{a,d\}, \ni\ \emptyset \neq \{a,d\} \neq U \\ \text{and } \mathbb{S} \cap \overline{\{a,d\}} \to \mathbb{S} \cap U \neq \emptyset \end{cases}$	**No**

$\therefore \mathcal{Z}O(U,\mu,\mathbb{I}) = \Big\{\emptyset, U, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{a,b,d\}\Big\}$,

Remarks 3.3

1.$\forall$ μ_os is $\mu\mathbb{I}\_os$, but the convers is not true as the example (3.2) ,where $\{b\} \in \mu\mathbb{I}\_os$, but $\{b\} \notin \mu\_os$.

2.$\forall$ $\mathbb{I}\_os$ is $\mu\mathbb{I}\_os$, but the convers is not true as the example (3.2) ,where $\{a,b\} \in \mu\mathbb{I}\_os$, but $\{a,d\} \notin \mathbb{I}\_os$.

Proposition 3.4

Every $\mu\mathbb{I}\_o$ ($\mu\mathbb{I}\_c$) sets is $\mathcal{Z}_\mu\mathbb{I}\_o$ ($\mathcal{Z}_\mu\mathbb{I}\_c$) sets

Proof : Let Z be $\mu\mathbb{I}\_o$ sets. Since, $\forall$ μ_os is $\mu\mathbb{I}\_os$ and by using def.(1.1) . Then Z is $\mathcal{Z}_\mu\mathbb{I}\_os$.

But ,the convers is not true as the example (3.2) ,where $\{b,d\} \in \mathcal{Z}_\mu\mathbb{I}$ but $\{b,d\} \notin \mu\mathbb{I}\_os$.

Remark 3.5

A subset Z of space $(U,\mu,\mathbb{I})$ is $\mathcal{Z}_\mu\mathbb{I}_$clopen if Z are both ($\mathcal{Z}_\mu\mathbb{I}\_os$) and($\mathcal{Z}_\mu\mathbb{I}\_cs$),

from the example (3.2), note $\emptyset$,U are $\mathcal{Z}_\mu\mathbb{I}_$clopen sets

Propositions 3.6 : $\forall$ $\mathcal{Z}_{\mu}\_os$ is $\mathcal{Z}_\mu\mathbb{I}\_os$.

Proof : Let Z $\subseteq$ (U,μ) is $\mathcal{Z}_{\mu}\_os$, by using remarks(3.3(1)) and proposition (3.4), then $\mathcal{Z}_{\mu}\_os$ is $\mathcal{Z}_\mu\mathbb{I}\_os$.

Definitions 3.7

1. A subset $\mathcal{A}$ on $(U,\mu,\mathbb{I})$ is ideal supra interior, denoted by $int_{\mathcal{Z}_\mu\mathbb{I}}(\mathcal{A})$, defined as the largest $\mu\mathbb{I}\_os$ contained in $\mathcal{A}$; that is, the union of all $\mu\mathbb{I}\_o$ sets included in $\mathcal{A}$.

From the example 3.2 with $\mathcal{A}$ ={a,b}, then $int_{\mathcal{Z}_\mu\mathbb{I}}(\mathcal{A})$={a ,b}

2. A subset $\mathcal{A}$ on $(U,\mu,\mathbb{I})$ is ideal supra closure, denoted by $cl_{\mathcal{Z}_\mu\mathbb{I}}(\mathcal{A})$, defined as the smallest $\mu\mathbb{I}\_cs$ containing $\mathcal{A}$; that is, the intersection of all $\mu\mathbb{I}\_c$ sets that include $\mathcal{A}$.

From the example 3.2 with $\mathcal{A}$ ={a,b}, then $cl_{\mathcal{Z}_\mu\mathbb{I}}(\mathcal{A})$={a,b,c}

Properties 3.8 : A subset $\mathcal{A}$ of $(U, \mu, \mathbb{I})$,then

1. $int_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A}) \subseteq \mathcal{A}$

2. $\mathcal{A} \in \mu\mathbb{I} \Leftrightarrow int_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A}) = \mathcal{A}$

3. $cl_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A}) \supseteq \mathcal{A}$

4. $\mathcal{A}$ is $\mu\mathbb{I}\_os \Leftrightarrow \mathcal{Z}_{\mu}\mathbb{I}(\mathcal{A}) = \mathcal{A}$

5. $x \in \mathcal{Z}_{\mu}\mathbb{I}(\mathcal{A}) \Leftrightarrow \forall\ \mu\mathbb{I}\_os\ S_x$ containing x , $S_x \cap \mathcal{A} \neq \emptyset$

Proposition 3.9

A subset $\mathcal{A}$ of $(U, \mu, \mathbb{I})$. Then $int_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A}) = U - cl_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A}^c)$.

Proof : Let $x \in int_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A})$ .Then there is $S \in \mu\mathbb{I}$, $\ni$ $x \in S \subset \mathcal{A}$ and $x \notin U - S$, i.e., $x \notin cl_{\mathcal{Z}_{\mu\mathbb{I}}}(U - S)$, since $U - S$ is an $\mu\mathbb{I}\_os$. So $x \notin cl_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A}^c)$, from definitions (3.7), $cl_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A}^c) \subset cl_{\mathcal{Z}_{\mu\mathbb{I}}}(U - S)$ and hence $x \in U - cl_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A}^c)$.

Conversely ; Suppose that $x \in U - cl_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A}^c)$. So $x \notin cl_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A}^c)$, then there is a $\mu\mathbb{I}\_o$ set S_x containing x, such that $S_x \cap (\mathcal{A}^c) = \emptyset$. So $S_x \subset \mathcal{A}$. Therefore, $x \in int_{\mathcal{Z}_{\mu\mathbb{I}}}(\mathcal{A})$. Hence the result.

Now, we generalized the separation axioms through $\mathcal{Z}_{\mu}\mathbb{I}$_open sets in $(U, \mu, \mathbb{I})$.

Definition 3.10

A space $(U, \mu, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_0$-space iff for all pair of distinct points $x, y \in U$, $\exists$ $\mathcal{Z}_{\mu}\mathbb{I}\_os$ that contains one of the points but not the other.

Examples 3.11

1- From the example (2.3), with $\mathbb{I} = \{\emptyset, \{2\}\}$. So , $(U, \mu_{\mathcal{Z}}, \mathbb{I}) = \{\emptyset, U, \{1\}, \{2\}\}$,then $\mathcal{Z}O(U, \mu, \mathbb{I}) = \{\emptyset, U, \{1\}, \{2\}, \{1,2\}\}$, we get $(U, \mu, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_0$-space.

2- From the example (2.3), with $\mathbb{I} = \{\emptyset, \{2\}\}$. So,$(U, \mu, \mathbb{I}) = \{\emptyset, U, \{2\}, \{1,2\}\}$,then $\mathcal{Z}O(U, \mu, \mathbb{I}) = \{\emptyset, U, \{1\}, \{2\}, \{1,2\}\}$. So, $(U, \mu, \mathbb{I})$is not $\mathcal{Z}_{\mu}\mathbb{I}\_T_0$-space.

Theorem 3.12

A space $(U, \mu, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_0$-space iff for every pair of distinct points x , y of U , $cl_{\mathcal{Z}_{\mu\mathbb{I}}}\{x\} \neq cl_{\mathcal{Z}_{\mu\mathbb{I}}}\{y\}$.

Proof : By using proposition (2.4) and remarks (3.3), $\forall\ \mu\_os$ is $\mu\mathbb{I}\_os$, so the proof .

From the example (3.11), we get $cl_{\mathcal{Z}_{\mu\mathbb{I}}}(\{1\}) \neq cl_{\mathcal{Z}_{\mu\mathbb{I}}}(\{2\})$. Hence $(U, \mu, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_0$-space.

Definitions 3.13 : A space $(U, \mu, \mathbb{I})$ is said to be

1- $\mathcal{Z}_{\mu}\mathbb{I}\_T_1$-space iff for any two distinct points x , $y \in U$, $\exists$ $\mathcal{Z}_{\mu}\mathbb{I}\_os$ in U $\ni$ one contains x but not y and $\mathcal{Z}_{\mu}\mathbb{I}\_os$ in U containing y but not x.

2- $\mathcal{Z}_{\mu}\mathbb{I}\_T_2$-space iff for any two distinct points $x, y \in U$, $\exists\ \mathcal{Z}_{\mu}\mathbb{I}\_os$ $\mathbb{G}$ and $\mathbb{H}$, $\ni x \in \mathbb{G}, y \in \mathbb{H}$, and the intersection $\mathbb{G} \cap \mathbb{H} = \emptyset$.

Examples 3.14

1- From the example (2.6), with $\mathbb{I} = \{\emptyset, \{3\}\}$. So , $(U, \mu, \mathbb{I}) = \{\emptyset, U, \{2\}, \{3\}, \{2,3\}, \{2,4\}\}$,then

$\mathcal{Z}O(U, \mu, \mathbb{I})= \{ \emptyset, U, \{2\}, \{3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$; Then $(U, \mu, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_1$-space.

2- From the example (2.11), with $\mathbb{I} = \{\emptyset, \{3\}\}$. So $(U, \mu, \mathbb{I})= \{U, \emptyset, \{3\}, \{1,2\}, \{2,3\}\}$,then $\mathcal{Z}_{\mu}\_O(U, \mu)= \{ \emptyset, U, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$. Then $(U, \mu, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_2$-space.

Theorem 3.15

Every $\mathcal{Z}_{\mu}\mathbb{I}\_T_2$-space is $\mathcal{Z}_{\mu}\mathbb{I}\_T_0$-space.

Proof : Let $(U, \mu, \mathbb{I})$ be $\mathcal{Z}_{\mu}\mathbb{I}\_T_2$-space, and $x, y \in U, x \neq y$ then, $\exists$ two $\mathcal{Z}_{\mu}\mathbb{I}\_os$ $\mathbb{G}, \mathbb{H} \subseteq U, \ni x \in \mathbb{G}, y \in \mathbb{H}, \mathbb{G} \cap \mathbb{H} = \emptyset$. Since $\mathbb{G} \cap \mathbb{H} = \emptyset$. That is mean $x \in \mathbb{G}$ and $x \notin \mathbb{H}, y \notin \mathbb{G}, y \in \mathbb{H}$.

Hence $(U, \mu, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_0$-space.

Theorem 3.16

Every $\mathcal{Z}_{\mu}\mathbb{I}\_T_i$-space is $\mathcal{Z}_{\mu}\mathbb{I}\_T_{i-1}$-space. On the contrary, it is not true ,where i=0,1,2

Proof : proving $\mathcal{Z}_{\mu}\mathbb{I}\_T_2$-space is $\mathcal{Z}_{\mu}\mathbb{I}\_T_1$-space.

Let $(U, \mu, \mathbb{I})$ be $\mathcal{Z}_{\mu}\mathbb{I}\_T_2$-space, and $x, y \in U, x \neq y$ then, $\exists$ two$\mathcal{Z}_{\mu}\mathbb{I}_os$

$\mathbb{G}, \mathbb{H} \subseteq U, \ni x \in \mathbb{G}$ and $x \notin \mathbb{H}, y \notin \mathbb{G}, y \in \mathbb{H}, \mathbb{G} \cap \mathbb{H} = \emptyset$. That is mean, $\exists\ \mathcal{Z}_{\mu}\mathbb{I}_$os $\mathbb{G} \subseteq U, \ni x \in \mathbb{G}$ and $y \notin \mathbb{G}$. Hence $(U, \mu, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_1$-space..

Example 3.17

From the remark(2.7) with $\mathbb{I} = \{\emptyset, \{1\}\}$, So $(U, \mu, \mathbb{I})= \{U, \emptyset, \{1\}, \{3\}, \{1,3\}\}$;

$\mathcal{Z}O(U, \mu, \mathbb{I})= \{V, \emptyset, \{1\}, \{3\}, \{1,3\}\}$. Then, $(U, \mu, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_0$-space but not $\mathcal{Z}_{\mu}\mathbb{I}\_T_1$-space.

Proposition 3.18

Let $(U, \mu, \mathbb{I})$ be $\mathcal{Z}_{\mu}\mathbb{I}\_T_1$-space iff for each $x \in U$, $\{x\}$ is $\mu\mathbb{I}\_cs$.

Proof : Let $(U, \mu, \mathbb{I})$ be $\mu\mathbb{I}$_topological space, we prove that $\{x\}^c$ is $\mu\mathbb{I}\_os$ in U. Let $a \in \{x\}^c, a \neq x$ then by (def. (3.13)), $\exists\ \mathbb{G}_a$ is $\mu\mathbb{I}\_os$ in U where $\mathbb{G}_a$does not contain x. Hence; $a \in \mathbb{G}_a \in \{x\}^c$ and $\{x\}^c = \{\mathbb{G}_a : a \in \{x\}^c\}$. This means $\{x\}^c$ is a union of each $\mu\mathbb{I}\_o$ sets thus $\{x\}^c$ is $\mu\mathbb{I}\_os$. Then $\{x\}$ is $\mu\mathbb{I}\_cs$.

Conversely; Let $\{x\}$ is $\mu\mathbb{I}\_cs$ in U and let $\{a, b\}\in U$ where $a \neq b$ then $a \in \{b\}^c, b \in \{a\}^c$ and $\{b\}^c, \{a\}^c$are $\mu\mathbb{I}\_o$ sets in U. Hence $(U, \mu, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_1$-space.

Theorems 3.19

A space $(U, \mu, \mathbb{I})$ be any $\mathcal{Z}_{\mu}\mathbb{I}\_T_i$-space , then each $(\mathcal{A}, \mu_{\mathcal{A}}{}^*, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_i$-space, where i= 0,1,2.

Proof; 1- To prove $(\mathcal{A}, \mu_{\mathcal{A}}{}^*, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_1$-space

Let $\mathcal{A} \subseteq U$ and $a_1, a_2 \in \mathcal{A}, \ni a_1 \neq a_2$, $\exists$ two $\mathcal{Z}_{\mu}\mathbb{I}\_\_os$ $\mathbb{G}^*, \mathbb{H}^* \subseteq U \ni a_1 \in \mathbb{G}^*$ and $a_1 \in \mathcal{A}$, then $a_1 \in \mathcal{A} \cap \mathbb{G}^* = \mathbb{G}^{**}$ and $a_2 \in \mathbb{H}^*$ then $a_2 \in \mathcal{A} \cap \mathbb{H}^* = \mathbb{H}^{**}$. Therefore $(\mathcal{A}, \mu_{\mathcal{A}}{}^*, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_1$-space

2- To prove $(\mathcal{A}, \mu_{\mathcal{A}}{}^{*}, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_2$-space

Let $a_1, a_2 \in \mathcal{A}$, $\ni$ $a_1 \neq a_2$, since $\mathcal{A} \in U$, so $a_1, a_2 \in U$, then $\exists$ two $\mu\mathbb{I}\_os$ $\mathbb{G}^*, \mathbb{H}^* \subseteq U, \ni$ $a_1 \in \mathbb{G}^*$, $a_2 \in \mathbb{H}^*$ and $\mathbb{G}^* \cap \mathbb{H}^* = \emptyset$. then $a_1 \in \mathcal{A} \cap \mathbb{G}^* \in (\mu_{\mathcal{A}}{}^{*}, \mathbb{I})$, $a_2 \in \mathcal{A} \cap \mathbb{H}^* \in (\mu_{\mathcal{A}}{}^{*}, \mathbb{I})$, and $(\mathcal{A} \cap \mathbb{G}^*) \cap (\mathcal{A} \cap \mathbb{H}^*) = \emptyset$. Therefore $(\mathcal{A}, \mu_{\mathcal{A}}{}^{*}, \mathbb{I})$is $\mathcal{Z}_{\mu}\mathbb{I}\_T_2$-space.

Remark 3.20

The figure represents the logical relationships some the separation axioms in $(\mathrm{U}, \mu, \mathbb{I})$.

$$\begin{array}{ccc}
\mu\mathbb{I}_T_2\text{- space} & \longrightarrow & \mathcal{Z}_{\mu}\mathbb{I}_T_2\text{- space} \\
\downarrow & & \downarrow \\
\mu\mathbb{I}_T_1\text{- space} & \longrightarrow & \mathcal{Z}_{\mu}\mathbb{I}_T_1\text{- space} \\
\downarrow & & \downarrow \\
\mu\mathbb{I}_T_0\text{- space} & \longrightarrow & \mathcal{Z}_{\mu}\mathbb{I}_T_0\text{- space}
\end{array}$$

Figure (3-1): The relationships some the separation axioms in $(\mathrm{U}, \mu, \mathbb{I})$

The converse of the above remark is need not be true in general, is shown in the

Example 3.21

From the example (2.11,(2)), with $\mathbb{I} = \{\emptyset, \{2\}\}$. So, $(U, \mu, \mathbb{I}) = \{\emptyset, \mathrm{U}, \{2\}, \{1,2\}\}$,then

$\mathcal{Z}\mathrm{O}(U, \mu, \mathbb{I})$= { $\emptyset$, U,{1},{2},{1,2}};

Then $(U, \mu, \mathbb{I})$ is not $\mathcal{Z}_{\mu}\mathbb{I}\_T_1$-space.

Remarks 3.22

1-From the example (3.11,(2)), we get $(U, \mu, \mathbb{I})$ is $\mu\mathbb{I}\_T_0$-space but not $\mu\mathbb{I}\_T_1$-space.

And also, we get $(U, \mu, \mathbb{I})$ is $\mu\mathbb{I}\_T_0$-space but not $\mathcal{Z}_{\mu}\mathbb{I}\_T_0$-space.

2-From the example (3.11,(1)), we get $(U, \mu, \mathbb{I})$ is $\mathcal{Z}_{\mu}\mathbb{I}\_T_0$-space but not $\mathcal{Z}_{\mu}\mathbb{I}\_T_1$-space.

3- From the remark (2.12), with $\mathbb{I}$ we get is $\mathcal{Z}_{\mu}\mathbb{I}\_T_1(\mu\mathbb{I}\_T_2)$-space but not $\mathcal{Z}_{\mu}\mathbb{I}\_T_2(\mathcal{Z}_{\mu}\mathbb{I}\_T_2)$-space.

4-From the example (3.21), we get $(U, \mu, \mathbb{I})$ is $\mu\mathbb{I}\_T_1(\mu\mathbb{I}\_T_2)$-space but not $\mathcal{Z}_{\mu}\mathbb{I}\_T_1(\mathcal{Z}_{\mu}\mathbb{I}\_T_2)$-space.

5- The figure represents the relationships between some the separation axioms in $(\mathrm{U}, \mu, \mathbb{I})$ But we note that the converse is not necessarily true.

$$\begin{array}{ccccc}
\mathcal{Z}_{\mu}\mathbb{I}_T_2\text{-space} & \leftarrow & \mathcal{Z}_{\mu}_T_2\text{-space} & \leftarrow & \mu_T_2\text{-space} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{Z}_{\mu}\mathbb{I}_T_1\text{-space} & \leftarrow & \mathcal{Z}_{\mu}\mathbb{I}_T_1\text{-space} & \leftarrow & \mu_T_1\text{-space} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{Z}_{\mu}\mathbb{I}_T_0\text{-space} & \leftarrow & \mathcal{Z}_{\mu}_T_0\text{-space} & \leftarrow & \mu_T_0\text{-space}
\end{array}$$

Figure (3-2): The relationships some the separation axioms in $(\mathrm{U}, \mu, \mathbb{I})$

Conclusion: In our work we have provided new concepts supra $\mathcal{Z}$ on the ideal supra topological space and to prove some of its properties, as well as to generalize the separation axioms on the space under consideration.

References

[1] A. A. Abdulfatah and H. J. Ali, "Some results on supra semi preopen sets," to be published in AIP Conf. Proc., (2024).

[2] A. S. Mashhour, "On supra topological spaces," Indian J. Pure Appl. Math., vol. 14, (1983), pp. 502–510.

[3] D. Jankovic and T. R. Hamlett, "New topologies from old via ideals," Amer. Math. Monthly, vol. 97, no. 4, (1990), pp. 295–310.

[4] K. Kuratowski, Topology, vol. 1. New York, NY, USA: Academic Press, 1966.

[5] N. A. Hameed and T. H. Jassim, "ζ-Compactness in ζ- Topological spaces," M.Sc. thesis, Coll. Educ. for Women, Tikrit Univ., (2023).

[6] O. R. Sayed and T. Noiri, "On supra-b–open sets and supra b-continuity on topological spaces," Eur. J. Pure Appl. Math., vol. 3, no. 2, (2010), pp. 295–302.

[7] R. Vaidyanathaswamy, Set topology. Boston, MA, USA: Courier Corporation, (1960).

[8] S. A. Tamer and T. H. Jasim, "ζ-Open Sets and ζ-Continuity," Tikrit J. Pure Sci., vol. 28, no. 3,(2023), pp. 96–100.

[9] S. Modak and S. Mistry, "Ideal on supra topological space," Int. J. Math. Anal., vol. 6, no. 1, (2012), pp. 1–10.

[10] S. S. Faiad and R. B. Yasseen, "Supra topological spaces via supra Z-open," in AIP Conf. Proc. (2nd ICSR-2024), Tikrit Univ., accepted for publication, 2024.

[11] T. H. Jassim, R. B. Yasseen, and N. E. Arif, "On some separation axioms of supra topological spaces," Tikrit J. Pure Sci., vol. 13, no. 2, (2008), pp. 59–62..

[12] T. M. Al-Shami, "Some results related to supra topological spaces," J. Adv. Stud. Topol., vol. 7, no. 4, (2016), pp. 283–294.

[13] T. R. Hamlett, "Ideals in topological spaces and the set operator ψ," Boll. Un. Mat. Ital., vol. 7, (1990), pp. 863–874.