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On Jordan*- Centralizers On Gamma Rings With Involution

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Abstract

Let M be a 2-torsion free Γ-ring with involution satisfies the condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$.an additive mapping *: M→Mis called Involution if and only if $(a\alpha b)^*=b^*\alpha a^*$and $(a^*)^*=a$. In section one of this paper ,we prove if M be a completely prime Γ-ring and T:M→M an additive mapping such that $T(a\alpha a)=T(a)\alpha a^*$ (resp., $T(a\alpha a)=a^*\alpha T(a)$)holds for all $a\in M,\alpha\in\Gamma$.Then T is an anti- left *centralizer or M is commutative (res.,an anti- right* centralizer or M is commutative) and so every Jordan* centralizer on completely prime Γ-ring M is an anti- *centralizer or M is commutative. In section two we prove that every Jordan* left centralizer (resp., every Jordan* right centralizer) on Γ-ring has a commutator right non-zero divisor(resp., on Γ-ring has a commutator left non-zero divisor)is an anti- left *centralizer(resp., is an anti- right *centralizer) and so we prove that every Jordan* centralizer on Γ-ring has a commutator non –zero divisor is an anti-* centralizer .

Key wards :Γ-ring,involution, prime Γ-ring,semi-prime Γ-ring , left centralizer, Left* centralizer ,Right centralizer, Right* centralizer, centralizer, Jordan *centralizer.

1-Introduction

Throughout this paper,M will represent Γ-ring with center Z .In [8] B.Zalar proved that anyJordan left (resp.,right)centralizer on a 2-torsion free semi-prime ring is a left (resp.,right)Centralizer . In [3] authors proved that anyJordan left (resp.,right)σ- centralizer on a 2-torsion free R has a commutator right (resp., left) non- zero divisor is a left (resp.,right) σ-Centralizer . In [7] Vukman proved that if R is2-torsion free semi-prime ring and T:R→R be an additive mapping such that $2T(x^2)=T(x)x+xT(x)$ holds for all $x,y\in R$ .Then T is left and right centralizer.In [6]Rajaa C.Shaheen defined Jordan centralizer on Γ-ring and showed that the existence of a non-zero Jordan centralizer Ton a 2-torsion free completely prime Γ-ring M which satisfies the condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$ implies either T is centralizer or M is commutative Γ-ring.We should mentioned the ***reader that the concept of Γ-ring was*** introduced by Nobusawa[5] and generalized by Barnes[1],as follows

Let *M and* Γ be additive abelian groups**, *M*** is called **a Γ-ring if for any *x,y,z* $\in$*M and* α, β $\in$ Γ,**the following conditions are satisfied

(1) $x\alpha y \in M$

(2) $(x+y)\alpha z = x\alpha z + y\alpha z$

$x(\alpha+\beta)z = x\alpha z + x\beta z$

$x\alpha(y+z) = x\alpha y + x\alpha z$

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$

many properties of Γ-ring were obtained by many research such as [2]

Let A,B be subsets of a Γ-ringM and Λ a subset of Γ we denote $A\Lambda B$ the subset of M consisting of all finite sum of the form $\sum a_i\lambda_i b_i$ where $a_i \in A, b_i \in B$ and $\lambda_i \in \Lambda$ .Aright ideal(resp.,left ideal) of aΓ-ring M is an additive subgroup I of M such that $I\Gamma M \subset I$(resp.,$M\Gamma I \subset I$).If I is a right and left ideal inM,then we say that I is an ideal .M is called a 2-torsion free if 2x=0 implies x=0 for all $x \in M$.AΓ-ringM is called prime if $a\Gamma M\Gamma b=0$ implies a=0 or b=0 and M is called completely prime if $a\Gamma b=0$ implies a=0 or b=0(a,b $\in$M),Since $a\Gamma b\Gamma a\Gamma b \subset a\Gamma M\Gamma b$,then every completely prime Γ-ring is prime.AΓ-ring M is called semi-prime if $a\Gamma M\Gamma a=0$ implies a=0and M is called completely semi-prime if $a\Gamma a=0$ implies a=0($a \in M$)

Let R be a ring,.A left(right) centralizer of R is an additive mapping $T:R \to R$ which satisfies T(xy)=T(x)y(T(xy)=xT(y)) for all $x,y \in R$.A Jordan centralizer be an additive mapping T which satisfies $T(x \circ y)=T(x) \circ y = x \circ T(y)$.

A Centralizer of R is an additive which is both left and right centralizer.An easy computation shows that every centralizer is also a Jordan centralizer.Many Papers work about the problem every Jordan centralizer be centralizer such as in[8] .In [6] Rajaa work this problem on some kind of Γ-ring.In this paper we define Jordan *centralizer on Γ-ring with involution* and study this concept on some kind of Γ-ring with involution.

Now ,we shall give the following definition which are basic in this paper.

Definition1.2

Let M be a Γ-ring with involution* and let $T:M \to M$ be an additive map ,T is called

Left* centralizer of M,if for any a,b $\in$M and $\alpha \in \Gamma$, the following condition satisfy $T(a\alpha b)=T(a)\alpha b^*$,

Right* centralizer of M,if for any a,b $\in$M and $\alpha \in \Gamma$, the following condition satisfy

$T(a\alpha b)=a^*\alpha T(b)$,

Jordan left* centralizer if for all $a \in M$ and $\alpha \in \Gamma$, the following condition satisfy

$T(a\alpha a)=T(a)\alpha a^*$

Jordan Right* centralizer if for all $a\in M$ and $\alpha\in\Gamma$, the following condition satisfy

$T(a\alpha a)=a^*\alpha T(a)$

Jordan* centralizer of M,if for any a,b $\in M$ and $\alpha\in\Gamma$, the following condition satisfy $T(a\alpha b+b\alpha a)=T(a)\alpha b^*+b^*\alpha T(a)=a^*\alpha T(b)+T(b)\alpha a^*$.

Now we shall prove the following Lemmas which are necessarily to prove our main result in this paper.

Lemma 1.3

Let M be a 2-torsion free Γ-ring with involution* and let $T:M\rightarrow M$ be an additive mapping which satisfies $T(a\alpha a)=T(a)\alpha a^*$,(resp., $T(a\alpha a)=a^*\alpha T(a)$) for all $a\in M$ and $\alpha\in\Gamma$,then the following statement holds for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$,

(i) $T(a\alpha b+b\alpha a)=T(a)\alpha b^*+T(b)\alpha a^*$
(resp., $T(a\alpha b+b\alpha a)=a^*\alpha T(b)+b^*\alpha T(a)$)

(ii) Especially if M is 2-torsion free and $a\alpha b\beta c=a\beta b\alpha c$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$ then
$T(a\alpha b\beta a)=T(a)\alpha b^*\beta a^*$ (resp., $T(a\alpha b\beta a)=a^*\alpha b^*\beta T(a)$)

(iii) $T(a\alpha b\beta c+c\alpha b\beta a)=T(a)\alpha b^*\beta c^*+T(c)\alpha b^*\beta a^*$.
(resp., $T(a\alpha b\beta c+c\alpha b\beta a)=a^*\alpha b^*\beta T(c)+c^*\alpha b^*\beta T(a)$

Proof

(i) Since $T(a\alpha a)=T(a)\alpha a^*$ for all $a\in M$ and $\alpha\in\Gamma$,…….(1)

Replace a by a+b in (1),we get

$T(a\alpha b+b\alpha a)=T(a)\alpha b^*+T(b)\alpha a^*$……………………………….(2)

(ii) by replacing b by $a\beta b+b\beta a$, $\beta\in\Gamma$

$W=T(a\alpha(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha a)$

$=T(a)\alpha(a\beta b+b\beta a)^*+T(a\beta b+b\beta a)\alpha a^*$

$=T(a)\alpha(a\beta b)^*+T(a)\alpha(b\beta a)^*+(T(a)\beta b^*+T(b)\beta a^*)\alpha a^*$

Since * is involution,then

$W=T(a)\alpha(b^*\beta a^*)+T(a)\alpha(a^*\beta b^*)+T(a)\beta b^*\alpha a^*+T(b)\beta a^*\alpha a^*$

Since $a\alpha b\beta c=a\beta b\alpha c$,then

$W=T(a)\alpha(a^*\beta b^*)+2T(a)\alpha(b^*\beta a^*)+T(b)\beta a^*\alpha a^*$

On the other hand

$W=T(a\alpha(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha a)$

$=T(a\alpha(a\beta b)+a\alpha(b\beta a)+(a\beta b)\alpha a+(b\beta a)\alpha a)$

$=T(a\alpha a\beta b+b\beta a\alpha a)+2T(a\alpha b\beta a)$

$=T(a)\alpha a^*\beta b^*+T(b)\beta a^*\alpha a^*+2T(a\alpha b\beta a)$

By comparing these two expression of W, we get

$2T(a\alpha b\beta a) = 2T(a)\ \alpha b^{*}\beta a^{*}$

Since M is 2-torsion free ,then

$T(a\alpha b\beta a) = T(a)\ \alpha b^{*}\beta a^{*}$……………………………(3)

(iii)In (3) replace a by a+c ,to get

$T(a\alpha b\beta c+c\alpha b\beta a)=T(a)\ \alpha b^{*}\beta c^{*}+T(c)\ \alpha b^{*}\beta a^{*}$………………(4)

Theorem 1.4

Let M be a 2-torsion free completely prime Γ-ring which satisfy the condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$, α, $\beta\in\Gamma$, and let $T:M\rightarrow M$ be an additive mapping which satisfies $T(a\alpha a)=T(a)\ \alpha a^{*}$,for all $a\in M$ and $\alpha\in\Gamma$,then $T(a\alpha b)=T(b)\ \alpha a^{*}$, for all $a ,b\in M$ and $\alpha\in\Gamma$ or M is commutative Γ-ring.

Proof

By [Lemma 1.3,(iii)] ,we have

$T(a\alpha b\beta c+c\alpha b\beta a)=T(a)\ \alpha b^{*}\beta c^{*}+T(c)\ \alpha b^{*}\beta a^{*}$

Replace c by $b\alpha a$

$W=T(a\alpha b\beta(b\alpha a)+(b\alpha a)\alpha b\beta a)$

$=T(a)\ \alpha b^{*}\beta\ a^{*}\alpha b^{*} +T(b\alpha a)\ \alpha b^{*}\beta a^{*}$

On the other hand

$W= T((a\alpha b)\beta(b\alpha a)+(b\alpha a)\alpha(b\beta a))$

$= T(a)\alpha b^{*}\beta b^{*}\alpha a^{*}+T(b\alpha a)\ \beta a^{*}\alpha b^{*}$

By comparing these two expression of W, we get

$T(b\alpha a)\beta(a\alpha b-b\ \alpha\ a)^{*} + T(a)\alpha b^{*}\ \beta\ (b\alpha a-a\ \alpha\ b)^{*}=0$

$T(b\alpha a)\beta(a\alpha b-b\ \alpha\ a)^{*} - T(a)\alpha b^{*}\ \beta\ (a\alpha b-b\ \alpha\ a)^{*}=0$

$(T(b\alpha a)- T(a)\alpha b^{*})\beta(a\alpha b-b\ \alpha\ a)^{*}=0$…………………(5)

Since M is completely prime Γ-ring ,then

either $T(b\alpha a)- T(a)\alpha b^{*}=0$ or $(a\alpha b-b\ \alpha\ a)=0$

if $T(b\alpha a)- T(a)\alpha b^{*}=0$then $T(b\alpha a)= T(a)\alpha b^{*}$ so T is an anti-left *centralizers.

and if $a\alpha b-b\ \alpha\ a=0$ for all a, $b\in M$ and $\alpha\in\Gamma$,then M is commutative Γ-ring

Theorem 1.5

Let M be a 2-torsion free completely prime Γ-ring which satisfy the condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$, α, $\beta\in\Gamma$, and and let $T:M\rightarrow M$ be an additive mapping which satisfies $T(a\alpha a)= a^{*}\ \alpha\ T(a)$for all $a\in M$ and $\alpha\in\Gamma$,then $T(a\alpha b)= b^{*}\ \alpha\ T(a)$ for all $a,b\in M$ and $\alpha\in\Gamma$ or M is commutative Γ-ring.

Proof

From[Lemma 1.3,(iii)],we have for all $a,b,c\in M$ and α, $\beta\in\Gamma$,

$T(a\alpha b\beta c+c\alpha b\beta a)=a^*\alpha b^*\beta T(c)+c^*\alpha b^*\beta T(a)$……………(6)

In (6) replace c by $a\alpha b$, then

$W= T(a\alpha b\beta(a\alpha b)+(a\alpha b)\alpha b\beta a)$

$=a^*\alpha b^*\beta T(a\alpha b)+ b^*\alpha a^*\beta b^*\alpha T(a)$

on the other hand

$W = T(a\alpha b\beta(a\alpha b)+(a\alpha b\beta(b\alpha a))$

$=b^*\alpha a^*\beta T(a\alpha b)+a^*\alpha b^*\beta b^*\alpha T(a)$

by comparing these two expression of W ,we get

$(a\alpha b-b\alpha a)^*\beta(T(a\alpha b)-b^*\alpha T(a))=0$……………………………(7)

since M is completely prime Γ-ring,then

either $(T(b\alpha a)-b^*\alpha T(a))=0 \Rightarrow T(a\alpha b)=b^*\alpha T(a)$ and so T is an anti- right *centralizers or $a\alpha b-b\alpha a=0 \Rightarrow a\alpha b=b\alpha a \Rightarrow$ M is commutative Γ-ring◙

<u>Corollary 1.6</u>:- Every Jordan* centralizer of 2-torsion free completely prime Γ-ring M which satisfy the condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$, $\alpha, \beta\in\Gamma$, is an anti-*centralizer on M or M is commutative.

2-Jordan* Centralizers On Some Gamma Ring

Theorem 2.1

Let M be a 2-torsion free Γ-ring which satisfy the condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$, $\alpha, \beta\in\Gamma$ and has a commutator right non-zero divisor and let $T:M\rightarrow M$ be an additive mapping which satisfies $T(a\alpha a)= T(a)\alpha a^*$ for all $a\in M$ and $\alpha\in\Gamma$,then $T(a\alpha b)= T(b)\alpha a^*$ for all $a,b\in M$ and $\alpha\in\Gamma$.

Proof

from (5),we have

$(T(b\alpha a)- T(a)\alpha b^*)\beta(a\alpha b-b\alpha a)^*=0$

if we suppose that

$\delta(b,a)= T(b\alpha a)- T(a)\alpha b^*$ and $[a,b]^*=(a\alpha b-b\alpha a)^*$

then $\delta(b,a)\beta[a,b]^*=0$ for all $a,b\in M$ and $\alpha,\beta\in\Gamma$……………(9)

Since M has a commutator right non-zero divisor ,then $\exists x,y\in M$, $\alpha\in\Gamma$ such that if for every $c\in M$, $\beta\in\Gamma$

$c\beta[x,y]=0\Rightarrow c=0$

since * is involution ,we have $\delta(y,x)\beta[x,y]=0$ and so $\delta(x,y)=0$………………(10)

replace a by a+x

$\delta(b,a+x)\beta[a+x,b]^*=0$ and so by () and ()

$\delta(b,x)\beta[a,b]^*+\delta(b,a)\beta[x,b]^*=0$……………..(11)

Now replace b by b+y

$\delta(b+y,x)\ \beta\ [a,b+y]^*+\delta(b+y,a)\ \beta\ [x,b+y]^*=0$

and so by (10) and (11),we get

$\delta(b,x)\ \beta\ [a,b]^*+\delta(y,x)\ \beta\ [a,b]^*+\delta(b,x)\ \beta\ [a,y]^*+\delta(y,x)\ \beta\ [a,y]^*+$

$\delta(b,a)\ \beta\ [x,b]^*+\delta(y,a)\ \beta\ [x,b]^*+\delta(b,a)\ \beta\ [x,y]^*+\delta(y,a)\ \beta\ [x,y]^*=0$

by (11),we get

$\delta(a,b)\ \beta\ [x,y]^*-\delta(x,y)\ \beta\ [a,y]^*=0$

then

$\delta(a,b)\ \beta\ [x,y]^*=0$,and so $\delta(a,b)=0$ for all $a,b\in M$ and $\alpha\in\Gamma$

$T(a\alpha b)=T(b)\alpha a^*\Rightarrow T$ Is anti- left *centralizer of M.

Theorem 2.2

Let M be a 2-torsion free Γ-ring with involution which satisfy the condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$, α, $\beta\in\Gamma$ and has a commutator left non-zero divisor and let T:M→M be an additive mapping which satisfies $T(a\alpha a)=a^*\ \alpha\ T(a)$ for all $a\in M$ and $\alpha\in\Gamma$,then $T(a\alpha b)=a^*\ \alpha\ T(b)$ for all $a,b\in M$ and $\alpha\in\Gamma$.

Proof

From[Lemma 1.3,(iii)],we have

$T(a\ \alpha\ b\ \beta\ c+c\ \alpha\ b\ \beta\ a)=a^*\ \alpha\ b^*\ \beta\ T(c)+c^*\ \alpha\ b^*\ \beta\ T(a)$……………(12)

In (12) replace c by $b\ \alpha\ a$,then

$W=T(a\ \alpha\ b\ \beta\ (b\ \alpha\ a)+(b\ \alpha\ a)\ \alpha\ b\ \beta\ a)$

$=a^*\alpha\ b^*\beta\ T(b\ \alpha\ a)+b^*\alpha\ a^*\beta\ b^*\alpha\ T(a)$

on the other hand

$W=T(a\ \alpha\ (b\ \beta\ b)\ \alpha\ a+(b\ \alpha\ a)\ \alpha\ (b\ \beta\ a))$

$=a^*\alpha\ b^*\beta\ b^*\alpha\ T(a)+b^*\alpha\ a^*\beta\ T(b\alpha a)$

by comparing these two expression of W ,we get

$a^*\alpha\ b^*\beta\ (T(b\alpha a)-b^*\alpha T(a))-b^*\alpha\ a^*\beta\ (T(b\alpha a)-b^*\alpha T(a))=0$

then if we suppose $B(b,a)=(T(b\alpha a)-b^*\alpha T(a))$

$[a,b]^*\beta\ B(b,a)=[a,b]^*\beta\ B(a,b)=0$ for all $a,b\in M$,α, $\beta\in\Gamma$ ……………………(13)

Since M has a commutator left non-zero divisor then $\exists\, x,y\in M$, $\alpha\in\Gamma$ such that if for every $c\in M$, $\beta\in\Gamma$, $[x,y]\ \beta\ c=0\Rightarrow c=0$

then by (13),we have

$[x,y]\ \beta\ B(x,y)=0\Rightarrow B(x,y)=0$…………………(14)

in (13) replace a by a+x

$[a+x,b]^*\ \beta\ B(a+x,b)=0$

then by (13)

$[x,y]^*\beta\ B(a,b)+[a,b]^*\beta\ B(x,b)=0$…………………(15)

Now replace b by b+y

[x,b+y] *β B(a,b+y)+[a,b+y] *β B(x,b+y)=0
then by using (14) and (15),we get
[x,y] *βB(a,b)=0
and since [x,y] is a commutator left non-zero divisor then
B(a,b)=0$\Rightarrow$T(aαb)=a*αT(b) which is mean that T is an anti- right *centralizer

Corrolary2.3

Let M be a 2-torsion freeΓ-ring with involution which satisfy the condition $x\alpha y\beta z=x\beta y\alpha z$ for all x,y,z$\in$M ,α, $\beta\in\Gamma$, has a commutator non-zero divisor and let T:M→M be a Jordan *centralizer then T is *centralizer

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الخلاصة

قدمنا في هذا البحث دراسة حول تطبيق جوردان المركزي على بعض الحلقات من نوع كاما