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Certain geometric properties define on meromorphic via Bernardi integral operator on complex Hardy space

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ABSTRACT

The aim goal of this paper to study the bernardi integral operator on a new sub class of analytic meromorphic function of Hilbert space and more specifically on the hardy space H^2 , and we obtain the coefficient estimates for the new subclass with certain geometric properties as growth and distortion also discussed radii of starlikeness and convexity functions in the end the convexity of linear combination with the class is discussed.

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1. Introduction

The investigation of meromorphic univalent functions plays a main role in complex analysis, especially in the study of geometric function classes. The family of meromorphic starlike functions was first introduced and developed by Pommerenke [1], with further substantial contributions later provided by Mogra et al. [2] and Abdelrahman M. Yehia [3] . This area of research relies heavily on the concept of differential subordination, originally formulated by Miller and Mocanu [4], providing researchers an effective means to use concepts to study analytic and meromorphic functions.

Let the function of the form

$$Ƒ(\omega) = \omega^{-1}[1 + \sum_{m=1}^{\infty} c_m \omega^{m+1}] \qquad , c_m \in ¢ \tag{1.1}$$

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is analytic in the open unit disc without center point $\Omega =\{ \omega: \omega \in ¢ ,0 < \|\omega\| < 1 \}$ and belong to the class $\mathcal{M}$. let $\mathcal{M}_c^*(p)$ be a subclass of meromorphic convex and $\mathcal{M}_s^*(p)$ be a subclass of meromorphic starlike respectively of order p (see also the recent works [5], [6][7]). we can say that the function Ƒ of the form (1.1) in a subclass $\mathcal{M}_s^*(p)$ if it holding

$$\mathrm{Re}\left(-\left(1+\ \omega\frac{\acute{\acute{F}}(\omega)}{\acute{F}(\omega)}\right)\right) > p\,,\ \ p \in [0\,,1) \quad ,\omega \in \Omega\,.$$

And $F \in \mathcal{M}_c^*(p)$ if it holding

$$\mathrm{Re}\left(-\omega\frac{\acute{F}(\omega)}{F(\omega)}\right) > p\,, \qquad p \in [0\,,1) \quad ,\omega \in \Omega\,.$$

There are a many perversely studies on a class meromorphic functions specifically the subclass $\mathcal{M}^*(p)$, $p \in [0\,,1)$ by [8], [9], [10], [11], [12].

let $T(\omega)$ and $F(\omega)$ are two analytic functions in Ω , the function $F(\omega)$ is said to be subordinate to $T(\omega)$ written by $F(\omega) \prec T(\omega)$ if there exist a schwars function $M(\omega) \in \Omega$ s.t $F(\omega) = T\big(M(\omega)\big)$ when $M(0) = 0$ $|M(\omega)| < 1 , \omega \in \Omega$. Our operator is called bernardi operator [13], [14], [15], [16] which is $\mathcal{D}\rho : \mathcal{M} \rightarrow \mathcal{M}$ define as

$$\mathcal{D}\rho\big(F(\omega)\big) = \int_0^\omega \frac{\beta+\Upsilon}{\omega\beta} z^{\Upsilon-1} F(z)\, dz \quad , \ \beta = 1,2,3, \ldots \tag{1.2}$$

And

$$\mathcal{D}\rho\big(F(\omega)\big) = \ \omega^{-\beta}\left[1+\sum_{m=1}^{\infty}\left(\frac{\beta+\Upsilon}{m+\Upsilon}\right)c_m\omega^{m+\beta}\right] \tag{1.3}$$

If $\beta = 1$ and $\Upsilon = 1$ for $F(\omega)$

We get

$$\mathcal{D}\rho\big(F(\omega)\big) = \ \omega^{-1}\left[1+\sum_{m=1}^{\infty}\left(\frac{2}{m+1}\right)c_m\omega^{m+1}\right] \tag{1.4}$$

The operator in (1.4) is called libera integral operator, it`s worth noting this operator was studied [17], in this paper we`ll using the operator in (1.3) when $\beta = 1$ as

$$\mathcal{D}\rho\big(F(\omega)\big) = \ \omega^{-1}\left[1+\sum_{m=1}^{\infty}\left(\frac{1+\Upsilon}{m+\Upsilon}\right)c_m\omega^{m+\beta}\right] \tag{1.5}$$

The form in (1.5) is called bernardi integral operator, we say that $F \in \mathcal{M}$ belong to $\mathcal{M}_p^*[c\,,\partial,\Upsilon,\mathcal{R},p,\varrho\,] \cap \mathcal{M} = \mathcal{M}_p^*[c\,,\partial,\Upsilon,\mathcal{R},p,\varrho\,]$.

Definition (1.1): in the case of $-1 \le \partial < \mathcal{R} \le 1$, $\Upsilon \ge 1$, $\mathrm{Re}(p) > 0$, $\varrho = \frac{\Gamma(m)}{(m-1)!}$ the function $F \in \mathcal{M}$ in the class $\mathcal{M}_p^*[c\,,\partial,\Upsilon,\mathcal{R},p,\varrho\,]$ if:

$$1-\frac{1}{p}\left(\frac{\omega[\mathcal{D}\rho(\varrho,\Upsilon,\omega\,)]^{`}}{\mathcal{D}\rho(\varrho,\Upsilon,\omega\,)}\right)+1 \prec \frac{1+\mathcal{R}\omega}{1+\partial\omega}$$

In the other word equivalently

$$\left\|\frac{\frac{\omega[\mathcal{D}\rho(\varrho,\Upsilon,\omega)]'}{\mathcal{D}\rho(\varrho,\Upsilon,\omega)}+1}{\partial\frac{\omega[\mathcal{D}\rho(\varrho,\Upsilon,\omega)]'}{\mathcal{D}\rho(\varrho,\Upsilon,\omega)}+\mathrm{p}\mathcal{R}+\partial(\mathrm{p}-1)}\right\|<1 \qquad (1.6)$$

For each Ƒ in the subclass $\mathcal{M}_p^*[c, \partial, \Upsilon, \mathcal{R}, p, \varrho]$ be of the form

$$Ƒ(\omega) = \omega^{-1}[1 + \sum_{m=1}^{\infty}|c_m|\omega^{m+1}] \qquad (1.7)$$

Notation and background (1.1) (Hilpert spaces of meromorphic functions**)**

The rady space H^2 is the space of meromorphic function[18], [19] Ƒ ; $\Omega \to ¢$ s.t

$$\|Ƒ\|_{H2}^2 = \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|Ƒ(r\omega)|^2 d\omega < \infty$$

If Ƒ of the form (1.1) have simple pole we have

$$\|Ƒ\|_{H2}^2 = \sum_{m=1}^{\infty}|c_m|^2$$

2. Main Result

At the began we supplies the enough conditions to show the formula (1.7) which is the form of meromorphic function in the class $\mathcal{M}_p^*[c, \partial, \Upsilon, \mathcal{R}, p, \varrho]$. and we study the closure, distortion and growth with it`s bounds, linear combinations at last the extremal points.

Theorem 2.1) the coefficient estimate

let Ƒ(ω) of the formula (1.7) class then $Ƒ \in \mathcal{M}_p^*[c, \partial, \Upsilon, \mathcal{R}, p, \varrho]$ iff;

$$\left\|\sum_{m=1}^{\infty}[(m+1)(1-\partial) - |p|(\mathcal{R}-\partial)]\left(\frac{1+\Upsilon}{m+\Upsilon}\right)^{\varrho} c_m\right\| \le |p|(\mathcal{R}-\partial) \qquad (2.1)$$

Proof: let $Ƒ \in \mathcal{M}_p^*[c, \partial, \Upsilon, \mathcal{R}, p, \varrho]$ s.t Ƒ(ω) of the formula (1.7) then from (1.6) we have

$$\left\|\frac{\omega\,[\mathcal{D}\rho(\varrho,\Upsilon,\omega)\,Ƒ(\omega)]' + \mathcal{D}\rho(\varrho,\Upsilon,\omega)\,Ƒ(\omega)}{\partial\omega[\mathcal{D}\rho(\varrho,\Upsilon,\omega)\,Ƒ(\omega)]' + p(\mathcal{R}-\partial)[\mathcal{D}\rho(\varrho,\Upsilon,\omega)\,Ƒ(\omega)]}\right\|$$

$$= \left\|\frac{\sum_{m=1}^{\infty}(m+1)\left(\frac{1+\Upsilon}{m+\Upsilon}\right)^{\varrho}\|c_m\|\omega^{m+1}}{p(\mathcal{R}-\partial)+\sum_{m=1}^{\infty}\partial(m+1)\left(\frac{1+\Upsilon}{m+\Upsilon}\right)^{\varrho}\|c_m\|\omega^{m+1}}\right\| < 1$$

But we have that $\|Re(\omega)\| \le \|\omega\|$ consequently

$$Re\left(\frac{\sum_{m=1}^{\infty}(m+1)\left(\frac{1+\Upsilon}{m+\Upsilon}\right)^{\varrho}\|c_m\|\omega^{m+1}}{p(\mathcal{R}-\partial)+\sum_{m=1}^{\infty}\partial(m+1)\left(\frac{1+\Upsilon}{m+\Upsilon}\right)^{\varrho}\|c_m\|\omega^{m+1}}\right) < 1 \qquad (2.2)$$

Regard $Re(\omega) = v \in [0.1)$ the devisor is greater than 0. take $\omega \to 1^-$ Eq (2.2) we get:

$$\left\|\sum_{m=1}^{\infty}[(m+1)(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{1+\Upsilon}{m+\Upsilon}\right)^{\varrho}\right\| \leq |p|(\mathcal{R}-\partial)$$

The first hand said is complete, conversely, we noting that:

$$\left\|\frac{\frac{\omega\,[\mathcal{D}\rho(\varrho,\Upsilon,\omega)\,\digamma(\omega)]^{`}}{\mathcal{D}\rho(\varrho,\Upsilon,\omega)\,\digamma(\omega)}+1}{\partial\frac{\omega\,[\mathcal{D}\rho(\varrho,\Upsilon,\omega)\,\digamma(\omega)]^{`}}{\mathcal{D}\rho(\varrho,\Upsilon,\omega)\,\digamma(\omega)}+\partial+p(\mathcal{R}-\partial)}\right\|$$

$$=\left\|\frac{\sum_{m=1}^{\infty}(m+1)\left(\frac{1+\Upsilon}{m+\Upsilon}\right)^{\varrho}}{p(\mathcal{R}-\partial)+\sum_{m=1}^{\infty}[\partial(m+1)+p(\mathcal{R}-\partial)]\left(\frac{1+\Upsilon}{m+\Upsilon}\right)^{\varrho}\|c_m\|}\right\| \leq 1$$

By using Eq (2.1) if Regard $\|\omega\| = 1$ we get $\digamma \in \mathcal{M}_p^*[c\,,\partial,\Upsilon,\mathcal{R},p,\varrho\,]$ the convers hand said is hold

Corollary (1.2) if the function of the formal (1.7) belong to the class $\mathcal{M}_p^*[c\,,\partial,\Upsilon,\mathcal{R},p,\varrho\,]$ then:

$$\|c_m\| \leq \frac{\|p\|(\mathcal{R}-\partial)}{[(m+1)(1-\partial)-\|p\|(\mathcal{R}-\partial)]\left(\frac{1+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\ \forall m \geq 1$$

the result is sharp for

$$\digamma(\omega) = \frac{1}{\omega}+\frac{\|p\|(\mathcal{R}-\partial)}{[(m+1)(1-\partial)-\|p\|(\mathcal{R}-\partial)]\left(\frac{1+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\omega^m \quad \forall m \geq 1$$

Theorem 2.2) growth and distortion

Let the function $\digamma$ of the form (1.7) belong to $\mathcal{M}_p^*[c\,,\partial,\Upsilon,\mathcal{R},p,\varrho\,]$ then for $\|\omega\| = \lambda < 1$

we receive:

i) $$\lambda^{-1}-\frac{\|p\|(\mathcal{R}-\partial)}{[2(1-\partial)-\|p\|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\lambda \leq \|\digamma(\omega)\| \leq \lambda^{-1}+\frac{|p|(\mathcal{R}-\partial)}{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\lambda$$

ii)) $$\lambda^{-2}-\frac{\|p\|(\mathcal{R}-\partial)}{[2(1-\partial)-\|p\|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\lambda \leq \left\|\digamma^{`}(\omega)\right\| \leq \lambda^{-2}+\frac{|p|(\mathcal{R}-\partial)}{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\lambda$$

There exist $(3-2\,\partial)-|p|(\mathcal{R}-\partial) > 1$

Proof : let $\digamma(\omega) \in \mathcal{M}_p^*[c\,,\partial,\Upsilon,\mathcal{R},p,\varrho\,]$ then from th.2 we get

$$[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}]\sum_{m=1}^{\infty}\frac{m!}{\Gamma m}|c_m| \leq [(m+1)(1-\partial)-\|p\|(\mathcal{R}-\partial)]\left(\frac{1+\Upsilon}{m+\Upsilon}\right)^{\varrho} \leq |p|(\mathcal{R}-\partial)$$
$\forall\, m \geq 1$

Implies

$$\sum_{m=1}^{\infty}|c_m| \leq \frac{|p|(\mathcal{R}-\partial)}{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\,\partial \tag{2.3}$$

therefore

$$\|F(\omega)\| \leq \|\omega^{-1}\| + \|\omega\| \sum_{m=1}^{\infty}|c_m| \leq \|\omega^{-1}\| + \frac{|p|(\mathcal{R}-\partial)}{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\|\omega\| \quad \text{and}$$

$$\|F(\omega)\| \geq \|\omega^{-1}\| - \|\omega\| \sum_{m=1}^{\infty}|c_m| \geq \|\omega^{-1}\| - \frac{|p|(\mathcal{R}-\partial)}{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\|\omega\|$$

Differentiate with respect ω each side of Eq (1.7) to get

$$\|F'(\omega)\| \leq \|\omega^{-2}\| + \|\omega\| \sum_{m=1}^{\infty}|c_m| \leq \|\omega^{-2}\| + \frac{|p|(\mathcal{R}-\partial)}{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\|\omega\|$$

and

$$\|F'(\omega)\| \geq \|\omega^{-2}\| - \|\omega\| \sum_{m=1}^{\infty}|c_m| \geq \|\omega^{-2}\| - \frac{|p|(\mathcal{R}-\partial)}{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\|\omega\| \quad \forall\, m \geq 1$$

complete

Theorem 2.3) the radii of starlike-ness and convexity

let F in (1.7) in the class $\mathcal{M}_p^*[c, \partial, \Upsilon, \mathcal{R}, p, \varrho]$ therefore we have:

i) a function F is meromorphically starelike of order p in the unit disk $\|\omega\| < \lambda_1$ in other word

$$\mathrm{Re}\left(-\frac{\omega F'(\omega)}{F(\omega)}\right) > p\,, \quad (\|\omega\| < \lambda_1\,, p \in [0,1))$$

where

$$\lambda_1 = \inf_{m\geq 1}\left\{\frac{(1-p)[(m+1)(1-\partial)-|p|(\mathcal{R}-\partial)]}{(m+p)\,|p|(\mathcal{R}-\partial)}\right\} \tag{2.4}$$

ii) a function F is meromorphically convex of order p in the disk $|\omega| < \lambda_2$ in other word

$$\mathrm{Re}\left(-\left(1+\frac{\omega F'(\omega)}{F(\omega)}\right)\right) > p\,, (\|\omega\| < \lambda_2\,, p \in [0,1))$$

where

$$\lambda_2 = \inf_{m\geq 1}\left\{\frac{(1-p)[(m+1)(1-\partial)-|p|(\mathcal{R}-\partial)]}{(m+p)|p|m(\mathcal{R}-\partial)}\right\}^{\frac{1}{m+1}} \tag{2.5}$$

Proof: i) by (1.7) we obtain

$$\left\|\frac{1+\frac{\omega F'(\omega)}{F(\omega)}}{2p-1+\frac{\omega F'(\omega)}{F(\omega)}}\right\| \leq \frac{\sum_{m=1}^{\infty}(m+1)|c_m|\,\|\omega\|^{m+1}}{(2-2p)-\sum_{m=1}^{\infty}(2p-1+m)|c_m|\,\|\omega\|^{m+1}}$$

Hence

$$\left\|\frac{1+\frac{\omega \acute{F}(\omega)}{F(\omega)}}{2p-1+\frac{\omega \acute{F}(\omega)}{F(\omega)}}\right\| \leq 1 \ , \ p \in [0,1)$$

If

$$\sum_{m=1}^{\infty} \left(\frac{m+p}{1-p}\right)|c_m| \ \|\omega\|^{m+1} \leq 1 \qquad (2.6)$$

its satisfies by th.1 Eq (2.6) satisfy if

$$\left(\frac{m+p}{1-p}\right)\|\omega\|^{m+1} \leq \frac{[(m+1)(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}{|p|(\mathcal{R}-\partial)}$$

Hence

$$\|\omega\| \leq \left\{\frac{(1-p)[(m+1)(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}{|p|(\mathcal{R}-\partial)(m+p)}\right\}^{\frac{1}{m+1}}$$

By (2.4) we have $\|\omega\| < \lambda_1$

ii) by (1.7) we get

$$\left\|\frac{2+\frac{\omega \acute{\acute{F}}(\omega)}{\acute{F}(\omega)}}{2p+\frac{\omega \acute{\acute{F}}(\omega)}{\acute{F}(\omega)}}\right\| \leq \frac{\sum_{m=1}^{\infty}(m+1)|c_m| \ \|\omega\|^{m+1}}{\frac{1}{m}[(2-2p)-\sum_{m=1}^{\infty}(2p-1+m)|c_m| \ \|\omega\|^{m+1}]}$$

Hence we get

$$\left\|\frac{2+\frac{\omega \acute{\acute{F}}(\omega)}{\acute{F}(\omega)}}{2p+\frac{\omega \acute{\acute{F}}(\omega)}{\acute{F}(\omega)}}\right\| \leq 1 \ , \ p \in [0,1)$$

Now If

$$\sum_{m=1}^{\infty} m\left(\frac{m+p}{1-p}\right)|c_m| \ \|\omega\|^{m+1} \leq 1 \qquad (2.7)$$

its satisfies by th.1 if

$$m\left(\frac{m+p}{1-p}\right)\|\omega\|^{m+1} \leq \frac{[(m+1)(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}{|p|(\mathcal{R}-\partial)}$$

Then

$$\|\omega\| \leq \left\{\frac{(1-p)[(m+1)(1-\partial)-\|p\|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}{|p|m(\mathcal{R}-\partial)(m+p)}\right\}^{\frac{1}{m+1}}$$

Thus

By (2.4) we have $|\omega| < \lambda_2$

Theorem 2.4) (convex linear combination)

the closed of the class $\mathcal{M}_p^*[c, \partial, \Upsilon, \mathcal{R}, p, \varrho]$ under convex linear combinations is hold.

Proof : let the functions of the form

$Ƒ(\omega) = \frac{1}{\omega} + \sum_{m=1}^{\infty}|c_m|\, \omega^m \quad (m = 1, 2)$

Are belong to $\mathcal{M}_{\varrho}^{*}[c , \vartheta\,]$ it is sufficient to explain that the function Ʈ which defined by

$Ʈ(\omega) = (1 - \delta)Ƒ_1(\omega) + \delta Ƒ_2(\omega) \quad \forall \delta \in [0,1]$

is belong to $\mathcal{M}_{p}^{*}[c , \partial, \Upsilon, \mathcal{R}, p, \varrho\,]$ since

$$Ʈ(\omega) = \frac{1}{\omega} + \sum_{m=1}^{\infty}[(1 - \delta)\left|c_{m,1}\right| + \delta\left|c_{m,2}\right|]\omega^m \quad \delta \in [0,1] \tag{2.8}$$

Given in th.(2.1)

$[(m + 1)(1 - \partial) - |p|(\mathcal{R} - \partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}\left[(1 - \delta)\left|c_{m,1}\right| + \delta\left|c_{m,2}\right|\,\right]$

$=[(m + 1)(1 - \partial) - |p|(\mathcal{R} - \partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}(1 - \delta)\left|c_{m,1}\right|$

$+[(m + 1)(1 - \partial) - |p|(\mathcal{R} - \partial)]\left(\frac{2 + \Upsilon}{m + \Upsilon}\right)^{\varrho}\delta\left|c_{m,2}\right|$

$\leq (1 - \delta)|p|(\mathcal{R} - \partial) + \delta|p|(\mathcal{R} - \partial) = \|p\|(\mathcal{R} - \partial)$

Thus

$Ʈ(\omega) \in \mathcal{M}_{p}^{*}[c , \partial, \Upsilon, \mathcal{R}, p, \varrho\,]$

Theorem 2.5) [Extreme points of the class]

suppose $Ƒ_0(\omega) = \frac{1}{\omega}$ &

$Ƒ_m(\omega) = \frac{1}{\omega} + \frac{|p|(\mathcal{R}-\partial)}{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\omega^m \qquad m \geq 1$

Then $Ƒ \in \mathcal{M}_{p}^{*}[c , \partial, \Upsilon, \mathcal{R}, p, \varrho\,]$ Iff it can be written by the form

$$Ƒ(\omega) = \frac{1}{\omega} + \sum_{m=1}^{\infty} r_m Ƒ_m(\omega) \tag{2.9}$$

s.t $r_m \geq 0 \quad , \quad \sum_{m=0}^{\infty} r_m = 1$

proof: suppose that the function $Ƒ(\omega)$ as the form (2.9) then

$Ƒ(\omega) = \frac{1}{\omega} + \sum_{m+1}^{\infty} r_m \frac{|p|(\mathcal{R}-\partial)}{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}\omega^m$

Implies

$[2(1 - \partial) - |p|(\mathcal{R} - \partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho} \times r_m \frac{|p|(\mathcal{R}-\partial)}{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}$

$= \sum_{m+1}^{\infty} r_m\, |p|(\mathcal{R} - \partial) = (1 - r_0)|p|(\mathcal{R} - \partial) \leq |p|(\mathcal{R} - \partial)$

The constraint of eq.(2.1) is true thus $\mathfrak{F} \in \mathcal{M}_p^*[c, \partial, \Upsilon, \mathcal{R}, p, \varrho]$ and

$$|c_m| \le \frac{|p|(\mathcal{R}-\partial)}{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}} \qquad m \ge 1$$

We place

$$r_m = \frac{[2(1-\partial)-|p|(\mathcal{R}-\partial)]\left(\frac{2+\Upsilon}{m+\Upsilon}\right)^{\varrho}}{|p|(\mathcal{R}-\partial)} |c_m| \; m \ge 1$$

But

$$r_0 = 1 - \sum_{m=1}^{\infty} r_m$$

Consequently

$$\mathfrak{F}(\omega) = \sum_{m=1}^{\infty} r_m \mathfrak{F}_m(\omega)$$

The prove is complete

3. Conclusion

In present paper we insert a generalization subclass of meromorphic which is related to bernardi operator in Hardy space, and studied some geometric properties as growth and distortion and others were looked.

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