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On Some Relationships Between Spectral Sequences And The New Exact Sequences

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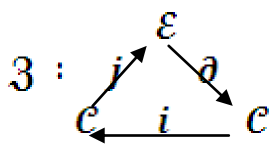
Abstract

In this paper I introduce and study relationships between spectral sequences and the new exact sequences . I will show that the new exact sequence is the second derived exact couple .

Let



be an exact couple , we derive $\mathfrak{Z}$ to get the second exact couple $\mathfrak{Z}^1 = NES$ which is the new exact sequence . the process of derivation can be iterated indefinitely, yielding an infinite sequence of exact couples .

I establish some results ,for examples ;

The class of all exact couples and morphisms between these couples formed a category $\mathfrak{T} = (\mathfrak{Z}, \mathcal{F})$. There is a functor from the category of $cw-complexes$ into $\mathfrak{T}$.

If $K \equiv L$ $(are\ cw-complexes)$, $then$ $\mathfrak{Z}^r(K) \cong \mathfrak{Z}^r(L)$ $\forall r, r = 0, 1, 2, \cdots$.

If K be any $(n-1)-connected\ complex$, then $\mathcal{E}^r_{p,q} = 0$ $\forall r$ $\& \forall p, p = 1,2,\cdots,n-1$, and $\mathcal{E}^r_{n,q} \cong \mathcal{C}^r_{n,q}$ $\forall r$.

Introduction

In this work I introduce and study relationships between spectral sequences and the new exact sequences (in short NES) introduced in [D] . I mean by the term " complex " in the sequel a " connected cw-complex ", see [H] .

This work contains two sections ; in first section , I construct a spectral sequence from exact couples and I construct a categories whose objects are exact couples and morphisms between these couples .

In second section ,I establish some results about our work , some of these results are purely algebraic and others depend on the topology of space , for examples ;

If K be any $(n-1)-connected\ complex$, then $\mathcal{E}^r_{p,q}=0 \quad \forall r\ \&\ \forall p, p=1,2,\cdots,n-1$, and $\mathcal{E}^r_{n,q}\cong\mathcal{C}^r_{n,q} \quad \forall r$.

Section 1 (Construction)

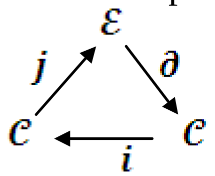
Let K be a cw-complex . For each integer p and q , let $\mathcal{E}_{p,q}\ be\ \pi_{p+q}(K^p,K^{p-1})\ if\ q\geq -p$ and zero otherwise , and $\mathcal{C}_{p,q}\ be\ \pi_{p+q}(K^p)\ if\ q\geq -p$ and zero otherwise .

Consider the following sequence which is known to be exact , see [D] ,

$$\cdots\to\pi_{p+q}(K^{p-1})\xrightarrow{i_{p,q}}\pi_{p+q}(K^p)\xrightarrow{j_{p,q}}\pi_{p+q}(K^p,K^{p-1})\xrightarrow{\partial_{p,q}}\pi_{p+q-1}(K^{p-1})\to\cdots$$

which forms a first exact couple ,

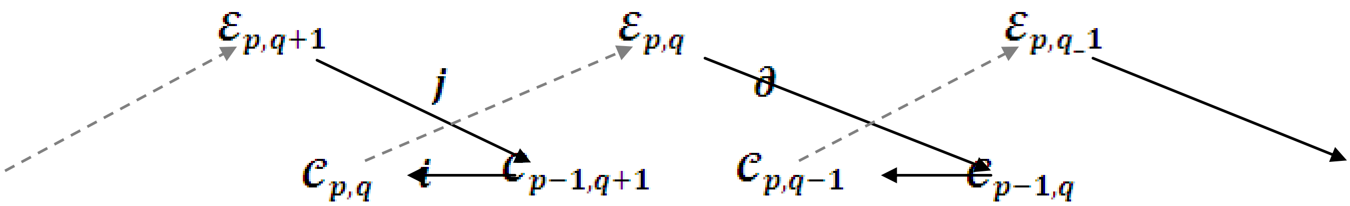
$\mathfrak{Z}$:



where

$\partial\ whose\ degree\ (-1,0)$
$i\ whose\ degree\ (1,-1)$
$j\ whose\ degree\ (0,0)$

and



From this exact couple, we construct a second exact couple, which is itself the new exact sequence (NES), (see [D]), by taking

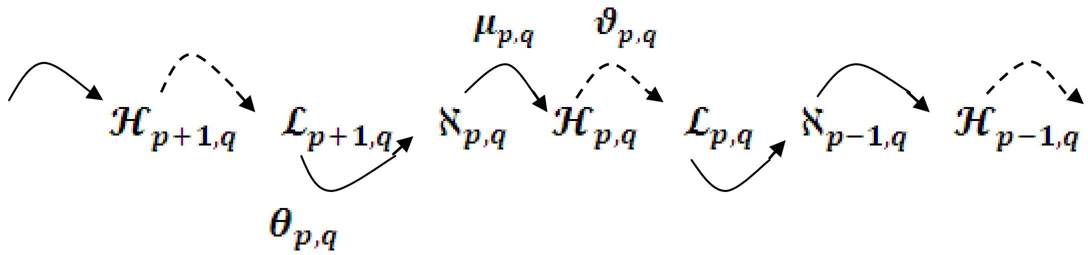
$$\mathcal{E}^1_{p,q}=\ker.d_{p,q}/im.d_{p-1,q} \quad (=\mathcal{H}_{p,q})$$

$$\mathcal{C}^1_{p,q}=\mathcal{C}_{p,q}/im.\partial_{p+1,q} \quad (=\aleph_{p,q})$$

$$\mathcal{C}^1_{p,q}=\ker.j_{p,q} \quad (=\mathcal{L}_{p,q})$$

where $d_{p,q} = j_{p-1,q} \circ \partial_{p,q}$.

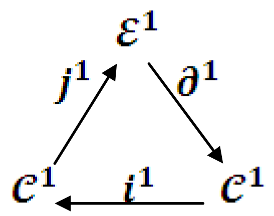
and



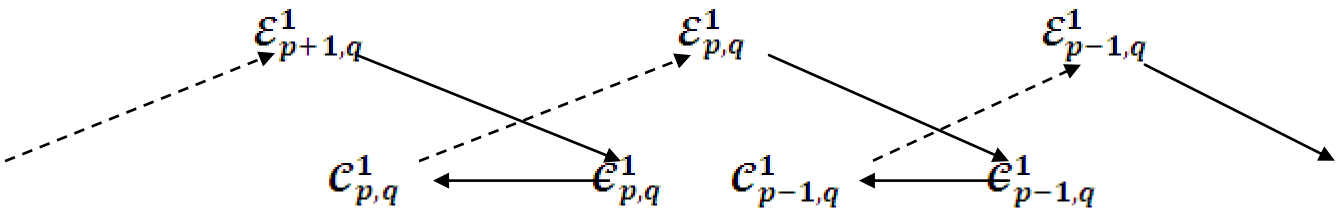
It is easy to see that the NES is an exact sequence, (for details see [D])

Hence we have a second exact couple ;

$\mathfrak{Z}^1$:



where



Now, we will construct a third exact couple .

Let $d^1_{p,q} = j^1 \circ \partial^1 : \mathcal{E}^1_{p,q} \to \mathcal{E}^1_{p-1,q}$.

It is clear that $d^1 \circ d^1 = 0$, implies that $\mathcal{E}^1$ is a chain complex .

Denote

$$\mathcal{E}^2 = {ker.d^1}/{im.d^1}$$

$$\mathcal{C}^2 = ker.j^1 = im.i^1$$

$$\mathcal{C}^2 = {\mathcal{C}^1}/{ker.i^1}$$

Define

$$i^2 : \mathcal{C}^2 \longrightarrow \mathcal{C}^2 \quad by \quad i^2(x) = [x], \qquad \forall x \in \mathcal{C}^2 ,$$

$$j^2 : \mathcal{C}^2 \longrightarrow \mathcal{E}^2 \quad by \quad j^2([y]) = [j^1(y)], \quad \forall [y] \in \mathcal{C}^2 ,$$

and

$$\partial^2 : \mathcal{E}^2 \longrightarrow \mathcal{C}^2 \quad by \quad \partial^2([z]) = \partial^1(z), \quad \forall [z] \in \mathcal{E}^2 .$$

Remark 1.1

The homomorphisms i^2 , j^2 *and* ∂^2 are well defined .

To prove j^2 is well defined ;

Let $[y] \in \mathcal{C}^2 = \mathcal{C}^1/_{ker.i^1}$, $y \in \mathcal{C}^1 \Rightarrow j^1(y) \in \mathcal{E}^1$

$$\Rightarrow \quad d^1\left(j^1(y)\right) = j^1 \circ \partial^1\left(j^1(y)\right) = 0$$

$$\Rightarrow \quad j^1(y) \in ker.d^1 \quad \Rightarrow \quad [j^1(y)] \in \mathcal{E}^2$$

Now , suppose $[y_1] = [y_2]$

$$\Rightarrow \quad y_1 - y_2 \in ker.i^1 = im.\partial^1$$

$$\Rightarrow \exists\, x \in \mathcal{E}^1 \quad such\,that \quad \partial^1(x) = y_1 - y_2$$

$$\Rightarrow \quad j^1(y_1 - y_2) = j^1\partial^1(x) \in im.d^1 \quad \Rightarrow \quad [j^1(y_1)] = [j^1(y_2)] .$$

To prove ∂^2 is well defined ;

Let $[z] \in \mathcal{E}^2 = ker.d^1/_{im.d^1}$, $z \in ker.d^1$

$$\Rightarrow \quad 0 = d^1(z) = j^1 \circ \partial^1(z) \quad \Rightarrow \quad \partial^1(z) \in ker.j^1 = im.i^1 = \mathcal{C}^2 .$$

Now , suppose $[z_1] = [z_2]$

$$\Rightarrow \quad z_1 - z_2 \in im.d^1 , \Rightarrow \exists\, y \in \mathcal{E}^1 \quad such\,that \quad d^1(y) = z_1 - z_2$$

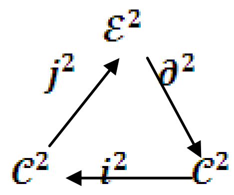
$$\Rightarrow \quad \partial^1(z_1 - z_2) = \partial^1\left(d^1(y)\right) = \partial^1\left(j^1 \circ \partial^1(y)\right) = 0$$

$$\Rightarrow \quad \partial^1(z_1) = \partial^1(z_2) .$$

And it is clear that i^2 is well defined ■

Theorem 1.2

The diagram



is an exact couple .

proof

First , to prove ; $im.j^2 = ker.\partial^2$.

Let $a \in \mathcal{C}^2$, $a = [x]$, *where* $x \in \mathcal{C}^1$

$$\partial^2\left(j^2(a)\right) = \partial^2([j^1(x)]) = \partial^1\left(j^1(x)\right) = 0 , \Rightarrow im.j^2 \subseteq ker.\partial^2 .$$

Let $b \in \mathcal{E}^2$, $b = [z]$, *where* $z \in ker.d^1$

assume that $\partial^2(b) = 0 \Rightarrow \partial^1(z) = 0 \Rightarrow z \in ker.\partial^1 = im.j^1$

$$\Rightarrow \quad z = j^1(y) \quad for\,some \quad y \in \mathcal{C}^1 \Rightarrow [y] \in \mathcal{C}^2$$

so we let $j^2([y]) = [j^1(y)] = [z] = b \quad \Rightarrow \quad ker.\partial^2 \subseteq im.j^2$.

Second , to prove ; $im.i^2 = ker.j^2$.

Let $a \in \mathcal{C}^2$, $a = [x]$, $where \;\; x \in \mathcal{C}^1$

assume $j^2(a) = 0 \quad \Rightarrow \quad [j^1(x)] = 0 \quad in \;\; \mathcal{E}^2 = {ker.d^1}/{im.d^1}$

$\Rightarrow j^1(x) \in im.d^1 \quad so\;that \quad d^1(y) = j^1(x) \; for\;some \; y \in \mathcal{E}^1$

thus $j^1\partial^1(y) = j^1(x) \quad \Rightarrow \quad \left(x - \partial^1(y)\right) \in ker.j^1 = im.i^1 = \mathcal{C}^2$

so we let $i^2\left(x - \partial^1(y)\right) = [x] = a \quad \Rightarrow \quad ker.j^2 \subseteq im.i^2$.

Let $x \in \mathcal{C}^2 = im.i^1 = ker.j^1$

$\Rightarrow \quad j^2\left(i^2(x)\right) = j^2([x]) = [j^1(x)] = 0 \quad \Rightarrow \quad im.i^2 \subseteq ker.j^2$.

Final , to prove ; $im.\partial^2 = ker.i^2$.

Let $b \in \mathcal{E}^2$, $b = [z]$, $where \;\; z \in ker.d^1$

$\Rightarrow \quad i^2\left(\partial^2(b)\right) = i^2\left(\partial^2([z])\right) = i^2\left(\partial^1(z)\right) = 0 \quad \Rightarrow im.\partial^2 \subseteq ker.i^2$.

Let $x \in \mathcal{C}^2 = im.i^1 = ker.j^1$

assume $i^2(x) = [x] = 0 \quad in \quad \mathcal{C}^2 = {\mathcal{C}^1}/{ker.i^1}$

$\Rightarrow \quad x \in ker.i^1 = im.\partial^1 \quad \Rightarrow \quad \partial^1(y) = x \; , \; for\;some \;\; y \in \mathcal{E}^1$

so $d^1(y) = j^1\partial^1(y) = j^1(x) = 0 \quad \Rightarrow \quad y \in ker.d^1$

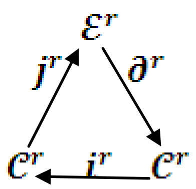
Let $[y] \in \mathcal{E}^2 \quad \Rightarrow \quad \partial^2([y]) = \partial^1(y) = x \quad \Rightarrow \quad x \in im.\partial^2$

$\Rightarrow \quad ker.i^2 \subseteq im.\partial^2$ ■

Remark 1.3

The exact couple $\mathfrak{Z}^1$ is called the derived couple of the original exact couple $\mathfrak{Z}$. the process of derivation can be iterated indefinitely, yielding an infinite sequence of exact couples ;

$\mathfrak{Z}^r$:

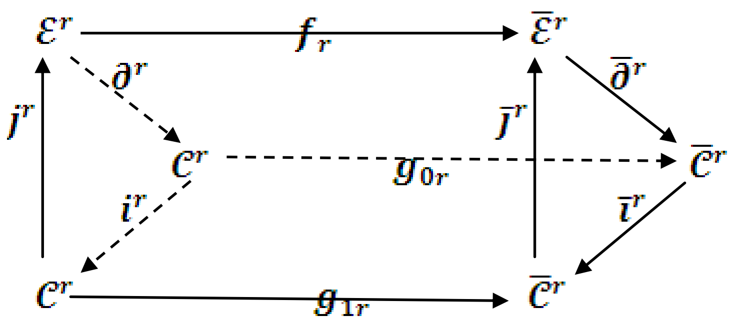


$r = 0, 1, 2, \cdots$

such that $\mathfrak{Z}^0 = \mathfrak{Z}$, $\mathfrak{Z}^1 = NES \;\; and \;\; \mathfrak{Z}^{r+1}$ is the derived couple of $\mathfrak{Z}^r$.
The endomorphism $d^r = j^r \circ \partial^r$ has the property that $d^r \circ d^r = 0$,
so that $\mathcal{E}^r$ is a chain complex under d^r, whose homology group is $\mathcal{E}^{r+1}$.
In this way we obtain a spectral sequence .

Definition 1.4

Define a morphism between two exact couple $\mathfrak{Z}^r , \bar{\mathfrak{Z}}^r$ $\mathcal{F}_r : \mathfrak{Z}^r \longrightarrow \bar{\mathfrak{Z}}^r$, we mean a family of homomorphisms (f_r, g_r) showing in the following diagram



such that

$$\bar{\partial}^r \circ f_r = g_{0r} \circ \partial^r$$
$$\bar{i}^r \circ g_{0r} = g_{1r} \circ i^r$$
$$\bar{j}^r \circ g_{1r} = f_r \circ j^r$$

So that $\mathcal{F}_r$ is a chain morphism and induces a morphism $\mathcal{F}_r^1 = \mathcal{F}_{r+1}$, and f_r^1 , g_r^1 defined a maps between the derived couples , such that ;

$$f_{r+1} : \mathcal{E}^{r+1} \to \bar{\mathcal{E}}^{r+1} \quad defined\ by \quad f_{r+1}([z]) = [f_r(z)] \ ,\forall [z] \in \mathcal{E}^{r+1} .$$
$$g_{0r+1} : \mathcal{C}^{r+1} \to \bar{\mathcal{C}}^{r+1} \quad defined\ by \quad g_{0r+1}(x) = g_{0r}(x) \ ,\forall x \in \mathcal{C}^{r+1} .$$
$$g_{1r+1} : \mathcal{C}^{r+1} \to \bar{\mathcal{C}}^{r+1} \quad defined\ by \quad g_{1r+1}([x]) = [g_{1r}(x)] \ ,\forall x \in \mathcal{C}^{r+1}.$$

Now, let $\mathcal{F}_r : \mathfrak{Z}^r \longrightarrow \bar{\mathfrak{Z}}^r$ and $\mathcal{F}_r^* : \bar{\mathfrak{Z}}^r \longrightarrow \bar{\bar{\mathfrak{Z}}}^r$ are two morphisms , we define the composition of morphisms as following ;

$$\mathcal{F}_r^* \circ \mathcal{F}_r = (f_r^* \circ f_r , g_r^* \circ g_r) : \mathfrak{Z}^r \longrightarrow \bar{\bar{\mathfrak{Z}}}^r$$

it is easy to show that $\mathcal{F}_r^* \circ \mathcal{F}_r$ is a homomorphism .

Remark 1.5

The homomorphisms f_{r+1} , g_{0r+1} and g_{1r+1} are well defined .

To prove f_{r+1} is well defined ;

Let $[z] \in \mathcal{E}^{r+1} = {ker.d^r}/{im.d^r}$ $, z \in ker.d^r \subseteq \mathcal{E}^{r+1}$.

$$\begin{aligned} \bar{d}^r(f_r(z)) = \bar{j}^r \circ \bar{\partial}^r(f_r(z)) &= \bar{j}^r(g_{0r} \circ \partial^r)(z) \\ &= \bar{j}^r\left(g_{0r}(\partial^r(z))\right) = \bar{j}^r(g_{1r}(\partial^r(z))) \\ &= (\bar{j}^r \circ g_{1r})(\partial^r(z)) = (f_r \circ j^r)(\partial^r(z)) \\ &= f_r(j^r \circ \partial^r(z)) = f_r(d^r(z)) = 0 \end{aligned}$$

$\Rightarrow \; f_r(z) \in ker.\bar{d}^r \;\Rightarrow\; [f_r(z)] \in \bar{\mathcal{E}}^{r+1}$ .x

Assume $[z_1] = [z_2]$

$\Rightarrow \; z_1 - z_2 \in im.d^r$ that is $d^r(y) = z_1 - z_2$ for some $y \in \mathcal{E}^r$

$$\Rightarrow f_r(z_1 - z_2) = f_r(d^r(y)) = f_r(j^r \circ \partial^r(y)) = (f_r \circ j^r)\partial^r(y)$$
$$= (\bar{j}^r \circ g_{1r})\partial^r(y) = \bar{j}^r\left(g_{1r}(\partial^r(y))\right) = \bar{j}^r(g_{0r}(\partial^r(y)))$$
$$= \bar{j}^r\left(\bar{\partial}^r \circ f_r(y)\right) = \bar{d}^r(f_r(y))$$

but $f_r(y) \in \bar{\mathcal{E}}^r \;\Rightarrow\; f_r(z_1 - z_2) \in im.\bar{d}^r$

and $\bar{\mathcal{E}}^{r+1} = {ker.\bar{d}^r}/{im.\bar{d}^r}$, that is $[f_r(z_1)] = [f_r(z_2)]$.

To prove g_{0r+1} is well defined ;

Let $x \in \mathcal{C}^{r+1} = ker.j^r = im.i^{r+1} \subseteq \mathcal{C}^r$

$$\bar{j}^r(g_{0r}(x)) = \bar{j}^r(g_{1r}(x)) = f_r \circ j^r(x) = f_r(0) = 0$$

$\Rightarrow \; g_{0r}(x) \in ker.\bar{j}^r \;\Rightarrow\; g_{0r+1}(x) \in \bar{\mathcal{C}}^{r+1}$

Assume $x_1 = x_2 \;\Rightarrow\; g_{0r}(x_1) = g_{0r}(x_2) \;\Rightarrow\; g_{0r+1}(x_1) = g_{1r+1}(x_2)$.

To prove g_{1r+1} is well defined ;

Let $[x] \in \mathcal{C}^{r+1} = {\mathcal{C}^r}/{ker.i^r}$, $x \in \mathcal{C}^r \;\Rightarrow\; g_{1r}(x) \in \bar{\mathcal{C}}^r$

$\Rightarrow \; [g_{1r}(x)] \in \bar{\mathcal{C}}^{r+1} = {\bar{\mathcal{C}}^r}/{ker.\bar{i}^r}$

Assume $[x_1] = [x_2] \;\Rightarrow\; x_1 - x_2 \in ker.i^r \subseteq \bar{\mathcal{C}}^r$

$$\bar{i}^r(g_{1r}(x_1 - x_2)) = \bar{i}^r(g_{0r}(x_1 - x_2)) = g_{1r} \circ i^r(x_1 - x_2) = 0$$

$\Rightarrow \; g_{1r}(x_1 - x_2) \in ker.\bar{i}^r \;\Rightarrow\; [g_{1r}(x_1)] = [g_{1r}(x_2)]$ ■

We shall describe $\mathcal{F}_r : \mathfrak{Z}^r \longrightarrow \bar{\mathfrak{Z}}^r$ is an isomorphism , if and only if , f_r , g_{0r} & g_{1r} are an isomorphisms . We shall describe $\mathfrak{Z}^r$ as isomorphic to $\bar{\mathfrak{Z}}^r$ and write $\mathfrak{Z}^r \cong \bar{\mathfrak{Z}}^r$, if and only if , there is an isomorphism $\mathcal{F}_r : \mathfrak{Z}^r \longrightarrow \bar{\mathfrak{Z}}^r$.

Remark 1.6

The relation , $\cong$, is an equivalence relation .

Section 2 (Results and Conclusion)

From remark 1.3 , definition 1.4 and (theorem 3 in [D]), we have the following theorem

Theorem 2.1

The class of all exact couples and the homomorphisms between these couples forms a category $\mathfrak{T} = (\mathfrak{Z}^r, \mathcal{F}_r)$.

From remark 1.3 , definition 1.4 , (theorem6 in [D]) and (see [S]) we have the following theorem

Theorem 2.2

There is a functor from the category of $cw-complexes$ into $\mathfrak{T}$. Denote it by $\mathfrak{F}$, where $\mathfrak{F}(K) = \mathfrak{Z}^r(K)$ and $\mathfrak{F}(f) = \mathcal{F}_r$.

Theorem 2.3

In the second exact couple $\mathfrak{Z}^1(= NES)$.
If K is $(n-1)-connected\ complex$. Then $\aleph_{n+1,q}(K)$ is an extension of $\mathcal{H}_{n+1,q}(K)$ by $\mathcal{L}_{n+1,q}(K) / \vartheta_{n+2,q}(\mathcal{H}_{n+2,q}(K))$.

Proof

From our property of NES [D] , we have
$\aleph_{n,q}(K) \cong \mathcal{H}_{n,q}(K)$ and $\mu_{n+1,q}$ $is\ onto$, we extract from $NES(= \mathfrak{Z}^1)$ the subsequence

$$\cdots \longrightarrow \mathcal{H}_{n+2,q}(K) \xrightarrow{\vartheta_{n+1,q}} \mathcal{L}_{n+1,q}(K) \xrightarrow{\theta_{n+1,q}} \aleph_{n+1,q}(K) \xrightarrow{\mu_{n+1,q}} \mathcal{H}_{n+1,q}(K) \longrightarrow 0$$

from exactness we have $ker.\mu_{n+1,q} = \theta_{n+1,q}(\mathcal{L}_{n+1,q}(K))$
but $\theta_{n+1,q}$ maps $\mathcal{L}_{n+1,q}(K)$ $onto$ $\theta_{n+1,q}(\mathcal{L}_{n+1,q}(K))$
with $ker.\theta_{n+1,q} = \vartheta_{n+1,q}(\mathcal{H}_{n+2,q}(K))$
then from *fundamental homomorphism theorem* , we have

$$\theta_{n+1,q}\left(\mathcal{L}_{n+1,q}(K)\right) \cong \mathcal{L}_{n+1,q}(K) / ker.\theta_{n+1,q}$$

$$= \mathcal{L}_{n+1,q}(K) / \vartheta_{n+2,q}(\mathcal{H}_{n+2,q}(K))$$

but $\mu_{n+1,q}\left(\aleph_{n+1,q}(K)\right) = \mathcal{H}_{n+1,q}(K)$

Then

$$\mathcal{H}_{n+1,q}(K) \cong \aleph_{n+1,q}(K) / ker.\mu_{n+1,q} = \aleph_{n+1,q}(K) / \theta_{n+1,q}(\mathcal{L}_{n+1,q}(K))$$

Therefore

$$\mathcal{H}_{n+1,q}(K) \cong \frac{\aleph_{n+1,q}(K)}{\mathcal{L}_{n+1,q}(K)/\vartheta_{n+2,q}\left(\mathcal{H}_{n+2,q}(K)\right)} \quad ■$$

Corollary 2.4

If $q = 0$, then $\mathcal{H}_{n+1}(K) \cong \frac{\pi_{n+1}(K)}{\mathcal{L}_{n+1}(K)/\vartheta_{n+2}(\mathcal{H}_{n+2}(K))}$.

Proof

From $NES(=\mathfrak{Z}^1)$, we have $\aleph_{p,0} \cong \pi_p(K)$, and from above theorem (Th. 2.3) the corollary is hold ■

From remark 1.3 , definition 1.4 and (theorem 7 in [D]), we have the following lemma

Lemma 2.5

Let $f, f^* : K \longrightarrow L$ be homotopic $(f \approx f^*)$, then $\mathfrak{F}(f) = \mathfrak{F}(f^*) : \mathfrak{Z}^r(K) \longrightarrow \mathfrak{Z}^r(L)$ $(i.e.\ \ \mathcal{F}_r = \mathcal{F}_r^*)$.

Theorem 2.6

If $K \equiv L$, *then* $\mathfrak{Z}^r(K) \cong \mathfrak{Z}^r(L)$ $\forall r, r = 0, 1, 2, \cdots$.

proof

From [D] , we have $\mathfrak{Z}^1(K) \cong \mathfrak{Z}^1(L)$, but $\mathfrak{Z}^{r+1}$ derivative from $\mathfrak{Z}^r$, $\forall r, r = 0, 1, 2 \cdots$. Therefore $\mathfrak{Z}^r(K) \cong \mathfrak{Z}^r(L)$ $\forall r, r = 0, 1, 2, \cdots$ ■

Theorem 2.7

If $\mathcal{C}^1_{p,q} \cong \overline{\mathcal{C}}^1_{p,q}$, $\forall p, q$ *in* $\mathfrak{Z}^1$. *Then* $\mathfrak{Z}^r \cong \overline{\mathfrak{Z}}^r$ $\forall r, r = 1, 2, \cdots$.

Proof

From *five lemma theorem*

(In a commutative diagram of abelian groups ,

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \varepsilon \\ A^* & \xrightarrow{i^*} & B^* & \xrightarrow{j^*} & C^* & \xrightarrow{k^*} & D^* & \xrightarrow{l^*} & E^* \end{array}$$

if the two rows are exact and α, β, δ *and* ε are isomorphisms , then γ is an isomorphism also) see [H] . ■

Theorem 2.8

If K is a $cw-complex$ such that K^{n-1} consists of a single $0-cell\ e^0$. Then

(1) $\mathcal{C}^r_{p,q} = 0 \ , \forall r, r = 0,1,2,\cdots \ ,\forall p, p \le n-1$,

(2) $\mathcal{C}^r_{n,q} \cong \mathcal{E}^r_{n,q} \ , \forall r, r = 0,1,2,\cdots$.

Proof

Consider the second exact couple (NES) of K ;

$$\cdots \to \aleph_{n+1,q} \to \mathcal{H}_{n+1,q} \to \mathcal{L}_{n,q} \to \aleph_{n,q} \to \mathcal{H}_{n,q} \to \mathcal{L}_{n-1,q} \to \aleph_{n-1,q} \to \cdots$$

We have , (see [D])

$$\mathcal{L}_{p,q} = \aleph_{p,q} = 0 \quad if \quad p \le n-1 ,$$

and $\mathcal{L}_{n,q} = 0 \quad , \quad \aleph_{n,q} \cong \mathcal{H}_{n,q}$.

Then we have the following exact sequence ;

$$\cdots \xrightarrow{\partial^1} \mathcal{L}_{n+1,q} \xrightarrow{i^1} \aleph_{n+1,q} \xrightarrow{j^1} \mathcal{H}_{n+1,q} \xrightarrow{\partial^1} 0 \xrightarrow{i^1} \aleph_{n,q} \xrightarrow{\cong} \mathcal{H}_{n,q} \to 0$$

return to second exact couple , we have

$$\mathcal{C}^1_{p,q} = \mathcal{L}_{p,q} = 0 \quad if \ p \le n ,$$

$$\mathcal{C}^1_{p,q} = \aleph_{p,q} = 0 \quad if \ p \le n-1 ,$$

and $\mathcal{C}^1_{n,q} = \aleph_{n,q} \cong \mathcal{H}_{n,q} = \mathcal{E}^1_{n,q}$.

In third exact couple , we have

$$\mathcal{E}^2_{p,q} = {ker.d^1_{p,q}}\Big/{im.d^1_{p+1,q}} \quad , \ where \ \ d^1_{p,q} = j^1_{p,q} \circ \partial^1_{p,q}$$

$$\mathcal{C}^2_{p,q} = ker.j^1_{p,q} = im.i^1_{p,q}$$

$$\mathcal{C}^2_{p,q} = {\aleph_{p,q}}\Big/{im.\vartheta_{p+1,q}} = {\mathcal{C}^1_{p,q}}\Big/{ker.i^1_{p,q}}$$

Now

$$\mathcal{C}^2_{p,q} = ker.j^1_{p,q} = 0 \quad if \ p \le n-1 ,$$

$$\mathcal{C}^2_{n,q} = ker.j^1_{n,q} = 0 \ ,$$

$$\mathcal{C}^2_{p,q} = {\mathcal{C}^1_{p,q}}\Big/{ker.i^1_{p,q}} = 0 \quad if \ p \le n-1 ,$$

$$d^1_{n,q} = j^1_{n-1,q} \circ \partial^1_{n,q} = 0 \quad \Rightarrow \quad ker.d^1_{n,q} = \mathcal{H}_{n,q} = \mathcal{E}^1_{n,q} ,$$

$$d^1_{n+1,q} = j^1_{n,q} \circ \partial^1_{n+1,q} = 0 \quad \Rightarrow \quad im.d^1_{n+1,q} = 0 .$$

Then

$$\mathcal{E}^2_{n,q} = {ker.d^1_{n,q}}\Big/{im.d^1_{n+1,q}} = {\mathcal{H}_{n,q}}\Big/{0} = \mathcal{E}^1_{n,q} \cong \mathcal{C}^1_{n,q}$$

But

$$\mathcal{C}^2_{n,q} = \mathcal{C}^1_{n,q} \big/ ker.\, i^1_{n,q} = \mathcal{C}^1_{n,q} \big/ 0 = \mathcal{C}^1_{n,q}$$

Then

$$\mathcal{E}^2_{n,q} \cong \mathcal{C}^2_{n,q} .$$

Therefore the third exact couple is

$$\cdots \to \mathcal{C}^2_{n+1,q} \to \mathcal{C}^2_{n+1,q} \to \mathcal{E}^2_{n+1,q} \to 0 \to \mathcal{C}^2_{n,q} \xrightarrow{\cong} \mathcal{E}^2_{n,q} \to 0 .$$

Continue in this way , we have for each $r = 0, 1, 2, \cdots$,

$$\cdots \to \mathcal{C}^r_{n+1,q} \to \mathcal{C}^r_{n+1,q} \to \mathcal{E}^r_{n+1,q} \to 0 \to \mathcal{C}^r_{n,q} \to \mathcal{E}^r_{n,q} \to 0 .$$

That is

$$\mathcal{C}^r_{p,q} = 0 \qquad \forall r, r = 0, 1, 2, \cdots, \forall p, p \le n-1 ,$$

and $\quad \mathcal{C}^r_{n,q} \cong \mathcal{E}^r_{n,q} \qquad \forall r, r = 0, 1, 2, \cdots$ ■

From above theorems (Th. 2.6 & Th. 2.8) , and two lemmas in [D] (Le. 2.14 & Le.2.15) , we have the following theorem

Theorem 2.9

If K be any $(n-1) - connected\ complex$. Then

(1) $\mathcal{C}^r_{p,q} = 0$, $\forall r, r = 0, 1, 2, \cdots$, $\forall p, p \le n-1$,

(2) $\mathcal{C}^r_{n,q} \cong \mathcal{E}^r_{n,q}$, $\forall r, r = 0, 1, 2, \cdots$.

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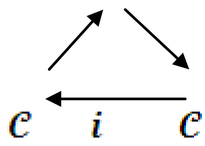
الخلاصة

هذا البحث يصف علاقة ربط بين مفهوم المتتالية الطيفية والمتتالية الجديدة المضبوطة .
وجدت أن المتتالية الجديدة المضبوطة هي المشتقة الثانية للثنائي المضبوط .
ليكن

$\mathcal{E}$

$\mathfrak{Z} : \quad j \quad \partial$



ثنائي مضبوط . نشتق من $\mathfrak{Z}$ الثنائي المضبوط (الثاني) $\mathfrak{Z}^1 = \mathbf{NES}$ الّذي هو المتتالية الجديدة المضبوطة ونشتق من $\mathfrak{Z}^1$ الثنائي المضبوط (الثالث) $\mathfrak{Z}^2$ ونستمر بهذا العمل . عملية الاشتقاق هذهِ تولد متتالية غير منتهية من الثنائيات المضبوطة .

حصلنا على بعض النتائج ، منها :

يشكل صف كل الثنائيات المضبوطة والتشاكلات بين هذهِ الثنائيات فصيلة $\mathfrak{T} = (\mathfrak{Z}, \mathcal{F})$ ، يوجد مقرن من فصيلة المجمعات cw الى $\mathfrak{T}$.

أذا كان كلاً من $\mathbf{K}, \mathbf{L}$ مجمعاً متصلاً وكانا متكافئان هوموتوبياً فـأن

$\mathfrak{Z}^{\mathbf{r}}(\mathbf{K}) \cong \mathfrak{Z}^{\mathbf{r}}(\mathbf{L}) \qquad \forall \mathbf{r}, \mathbf{r} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \cdots$.

أذا كان K مجمعا متصلا من نوع $(n-1)$ فـأن

$\mathcal{E}^r_{p,q} = \mathbf{0} \quad \forall r \ \& \ \forall p, p = 1,2,\cdots,n-1$. $\mathcal{E}^r_{n,q} \cong \mathcal{C}^r_{n,q} \quad \forall r$, and