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Applications on Geometric Function Theory using Symmetric Geometric Properties for a New Subclass of k-valent Functions

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ABSTRACT

This study introduces a new class $\mathcal{MS}_{\mathrm{k}}(\alpha,\lambda,\beta,\gamma)$ of k-valent analytic functions constructed through the Hadamard product within the open unit disk $U=\{z:z\in\mathbb{C},|z|<1\}$. The paper explores several geometric characteristics of this class, including coefficient bounds, distortion and growth behavior, closure properties, radii related to convexity and starlikeness, as well as the determination of extreme points and weighted mean with some applications.

MSC..

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1. Introduction

Geometric function theory is one of the most striking areas in mathematical analysis that has raised interest of many researchers since the beginning of 20th century. Teodor et al. [24], this branch of complex analysis is highly fascinating because of its applications in other fields, like model mathematical physics, fluid dynamics, fractional calculus, linear and nonlinear integreable system theory and theory of partial differential equations. Geometric Function Theory deals with the theory of univalent and k-valent functions and it is associated with geometry properties of analytic functions. The theory of k-valent functions occupies a central position in complex analysis, supported by its wide applications in mathematical modeling, engineering, and physical sciences (see [1,10,12,13,23]). Recent developments emphasize the importance of studying special classes of k-valent analytic

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functions that exhibit richer geometric and structural properties when constructed through the Hadamard product (convolution). These constructions provide insight into important geometric features such as coefficient.

Consider the family $\mathcal{A}_k$ of functions that are k-valent and analytic in the open unit disk $U = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}$, normalized in a standard form:

$$f(z) = z^k + \sum_{n=k+1}^{\infty} a_n z^n, \quad \{z \in U, k \in \mathbb{N} = \{1,2,3, \dots\}\}. \tag{1.1}$$

In this paper, we focus on a specific subclass $\Im_k$ consisting of functions $f \in \mathcal{A}_k$, which are analytic and k-valent in the open unit disk U of the form:

$$f(z) = z^k - \sum_{n=k+1}^{\infty} a_n z^n, \quad a_n \geq 0\,, k \in \mathbb{N}. \tag{1.2}$$

Recent advancements in the theory of k-valent functions further highlight their critical role in complex analysis an its applications [4,13,19,20,23,27,28,29].

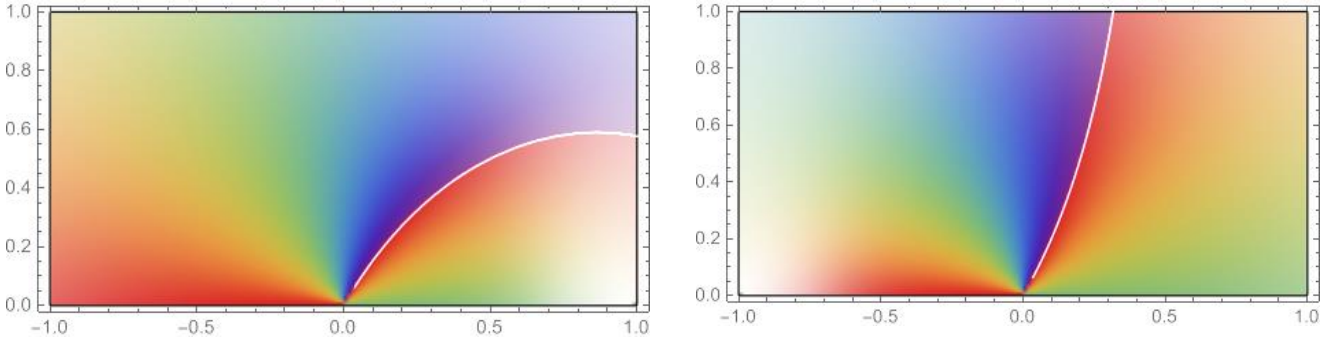


Figure 1: 2D Plot Representing complex function behaviors for k-valent classes: $\frac{z^k}{z-1}$, $\frac{z^k}{z+1}$, $k = 3$

And can be expressed using a particular representation involving Hadamard product (or convolution) of two functions f is defined by (1.2) and g

$$g(z) = z^k - \sum_{n=k+1}^{\infty} b_n z^n, \quad b_n \geq 0\,, k \in \mathbb{N}\,, \tag{1.3}$$

is provided by

$$(f * g)(z) = z^k - \sum_{n=k+1}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.4}$$

For $f(z) \in \Im_k$, denote by $S_k^*(\alpha)$ also $K_k(\alpha)$ the classes of k-valently starlike also convex functions of order α also $0 \leq \alpha < k$, in that order (see [5,6,14,15,16]), characterized by:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \tag{1.5}$$

and

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha. \tag{1.6}$$

It is derived from equations (1.5) and (1.6) that

$$f(z) \in K_k(\alpha) \leftrightarrow \frac{zf'(z)}{k} \in S_k^*(\alpha). \tag{1.7}$$

Refer to Goodman [9] for further insight.

Also, several authors defined $C_k(\beta,\alpha)$ and $C_k^*(\beta,\alpha)$ as the categories of k-valently close-to-convex functions also encompass quasi-convex functions of order β also type α satisfying, in that order (see [2,3,11]) with k = 1,

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} > \beta, \qquad (g(z) \in S_k^*(\alpha), \beta \geq 0, \alpha < k), \tag{1.8}$$

$$Re\left\{\frac{\left(zf'(z)\right)'}{g'(z)}\right\} > \beta, \qquad (g(z) \in K_k(\alpha), \beta \geq 0, \alpha < k). \tag{1.9}$$

It follows from (1.8) and (1.9) that

$$f(z) \in C_k^*(\beta,\alpha) \leftrightarrow \frac{zf'(z)}{k} \in C_k(\beta,\alpha). \tag{1.10}$$

This research is structured to Initially, delineate the new subclass of k-valent functions precisely, followed by a detailed examination of their coefficient inequality, symmetric distortion and growth properties, closure theorems, and radii of starlikeness and convexity. Further sections are dedicated to discussing applications in fluid dynamics, signal processing, image warping, and cryptographic systems, illustrating the broad utility of the newly defined function class.

2. Coefficient Inequality

Definition (2.1): A function $f \in \Im_k$ is said to belong to the subclass $\mathcal{M}S_k(\alpha,\lambda,\beta,\gamma)$ if and only if

$$\left|\frac{\alpha z^{2-k}(f*g)''(z) - (\lambda-\beta)z^{1-k}(f*g)'(z)}{z^{1-k}(f*g)'(z) + \gamma z^{-k}(f*g)(z) - k(\lambda-\beta)}\right| < 1, \tag{2.1}$$

where $z \in U, 0 \leq \alpha \leq 1, 0 \leq \lambda \leq \beta \leq 1, \gamma \geq 0$ and $k \in \mathbb{N} = \{1,2,3,\dots\}$.

The subsequent theorem provides the requisite and adequate condition for the function $f(z) \in \mathcal{M}S_k(\alpha,\lambda,\beta,\gamma)$.

Theorem (2.1): Let $f \in \Im_k$. Then $f \in \mathcal{M}S_k(\alpha,\lambda,\beta,\gamma)$ if and only if

$$\sum_{n=k+1}^{\infty} \left[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\right]a_n b_n \leq \gamma + k(\alpha(1-k)+1), \tag{2.2}$$

where $z \in U, 0 \leq \alpha \leq 1, 0 \leq \lambda \leq \beta \leq 1, \gamma \geq 0$ also $k \in \mathbb{N} = \{1,2,3,\dots\}$.

Proof: Assume that the inequality (2.2) is valid. Consider $|z| = 1$ then, we possess

$$|\alpha z^{2-k}(f*g)''(z) - (\lambda-\beta)z^{1-k}(f*g)'(z)| - |z^{1-k}(f*g)'(z) + \gamma z^{-k}(f*g)(z) - k(\lambda-\beta)|$$

$$= \left|\alpha k(k-1) - (\lambda-\beta)k - \sum_{n=k+1}^{\infty} a_n b_n[\lambda-\beta-\alpha(n-1)]n\, z^{n-k}\right| - \left|k - \sum_{n=k+1}^{\infty} a_n b_n (n+\gamma) z^{n-k} + \gamma - k(\lambda-\beta)\right|$$

$$\leq \sum_{n=k+1}^{\infty} \left[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\right]a_n b_n - (k+\gamma) + \alpha k(k-1) \leq 0,$$

by hypothesis using the principle of maximum modulus, we get $f \in \mathcal{M}S_k(\alpha,\lambda,\beta,\gamma)$.

Conversely, assume that $f \in \mathcal{M}S_k(\alpha,\lambda,\beta,\gamma)$. Then the inequality must hold for all n, and we can write

$$\left|\frac{\alpha z^{2-k}(f*g)''(z)-(\lambda-\beta)z^{1-k}(f*g)'(z)}{z^{1-k}(f*g)'(z)+\gamma z^{-k}(f*g)(z)-k(\lambda-\beta)}\right| = \left|\frac{\alpha k(k-1)-(\lambda-\beta)k-\sum_{n=k+1}^{\infty}a_n b_n[\lambda-\beta-\alpha(n-1)]n\, z^{n-k}}{k-\sum_{n=k+1}^{\infty}a_n b_n(n+\gamma)z^{n-k}+\gamma-k(\lambda-\beta)}\right| < 1,$$

because $Re(z) \leq |z|$ For any z , it follows that

$$Re\left\{\frac{\alpha k(k-1)-(\lambda-\beta)k-\sum_{n=k+1}^{\infty}a_n b_n[\lambda-\beta-\alpha(n-1)]n\, z^{n-k}}{k-\sum_{n=k+1}^{\infty}a_n b_n(n+\gamma)z^{n-k}+\gamma-k(\lambda-\beta)}\right\} < 1. \quad (2.3)$$

Permitting $z \to 1^{-}$, in terms of real numbers, therefore allowing us to express (2.3) as

$$\sum_{n=k+1}^{\infty}\left[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\right]a_n b_n \leq \gamma + k(\alpha(1-k)+1).$$

Ultimately, sharpness ensues if we proceed

$$f(z) = z^k - \frac{\gamma + k(\alpha(1-k)+1)}{\left[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\right]b_n}z^n, \qquad n \geq k+1. \qquad (2.4)$$

Corollary (2.1): If f characterized by (1.2) is classified within the category $\mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$. Subsequently

$$a_n \leq \frac{\gamma + k(\alpha(1-k)+1)}{\left[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\right]b_n}, \quad n \geq k+1, k \in \mathbb{N}. \qquad (2.5)$$

The parity in (2.4) is achieved regarding the function f specified in (2.4).

Let $\alpha = \beta = \lambda = 1$ and $\gamma = 0$ in theorem (2.1), we get the following corollary:

Corollary (2.2): If f defined by (1.2) is in the subclass $\mathcal{MS}_k(1,1,1,0)$, then

$$a_n \leq \frac{p(2-k)}{[n(2-n)]b_n}, \quad n \geq k+1, k \in \mathbb{N}.$$

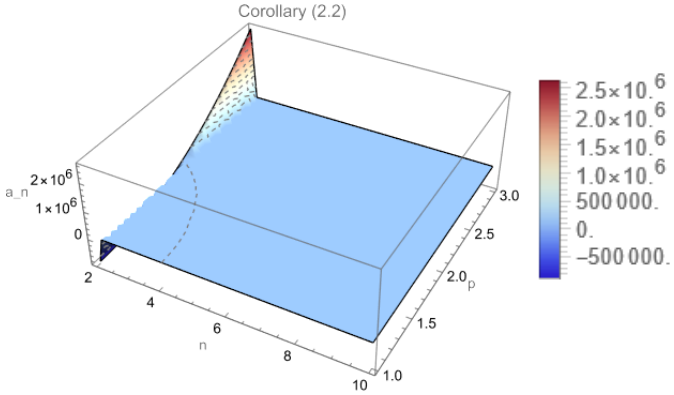


Figure 2 : Plot 3D Corollary (2.2)

Let $\alpha = \beta = \lambda = 0$ and $\gamma = 1$ in theorem (2.1), we get the following corollary:

Corollary (2.3): If f defined by (1.2) is in the subclass $\mathcal{MS}_k(0,0,0,1)$, then

$$a_n \leq \frac{1+k}{2n\, b_n}. \quad n \geq k+1, k \in \mathbb{N}.$$

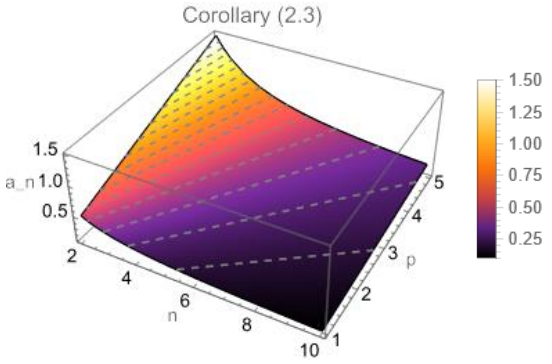


Figure 3 : Plot 3D Corollary (2.3)

3. Symmetric Distortion and Growth Properties

We obtain symmetric distortion and growth theorems for the subclass $\mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$.

Theorem (3.1): If $f \in \mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$, then for $|z| = r < 1$ the following inequalities hold:

$$r^k - \frac{\gamma + k(\alpha(1-k)+1)}{[(1+\lambda-\beta)+\gamma]b_{k+1}} r^{k+1} \le |f(z)| \le r^k + \frac{\gamma + k(\alpha(1-k)+1)}{[(1+\lambda-\beta)+\gamma]b_{k+1}} r^{k+1} \tag{3.1}$$

and

$$kr^{k-1} - \frac{(k+1)(\gamma + k(\alpha(1-k)+1))}{[(1+\lambda-\beta)+\gamma]b_{k+1}} r^k \le |f'(z)| \le kr^{k-1} + \frac{(k+1)(\gamma + k(\alpha(1-k)+1))}{[(1+\lambda-\beta)+\gamma]b_{k+1}} r^k. \tag{3.2}$$

Proof: Since $f \in \mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$, then by theorem (2.1), the coefficients satisfy

$$\sum_{n=1}^{\infty} a_{n+k} \le \frac{\gamma + k(\alpha(1-k)+1)}{[(1+\lambda-\beta)+\gamma]b_{k+1}}.$$

Hence

$$|f(z)| \le |z|^k + \sum_{n=1}^{\infty} a_{n+k} |z|^{n+k} \le r^k + r^{k+1} \frac{\gamma + k(\alpha(1-k)+1)}{[(1+\lambda-\beta)+\gamma]b_{k+1}}$$

and

$$|f(z)| \ge |z|^k - \sum_{n=1}^{\infty} a_{n+k} |z|^{n+k} \ge r^k - r^{k+1} \frac{\gamma + k(\alpha(1-k)+1)}{[(1+\lambda-\beta)+\gamma]b_{k+1}}.$$

Thus proof (3.1). Now we need to proof (3.2)

$$|f(z)| \le |z|^k + \sum_{n=1}^{\infty} a_{n+k} |z|^{n+k}$$

$$|f'(z)| \le k|z|^{k-1} + \sum_{n=1}^{\infty} a_{n+k}(n+k) |z|^{n+k-1}$$

$$\le kr^{k-1} + \frac{(k+1)(\gamma + k(\alpha(1-k)+1))}{[(1+\lambda-\beta)+\gamma]b_{k+1}} r^k$$

and

$$|f(z)| \geq |z|^k - \sum_{n=1}^{\infty} a_{n+k} |z|^{n+k}$$

$$|f'(z)| \geq k|z|^{k-1} - \sum_{n=1}^{\infty} a_{n+k}(n+k) |z|^{n+k-1}$$

$$\geq kr^{k-1} - \frac{(k+1)(\gamma + k(\alpha(1-k)+1))}{[(1+\lambda-\beta)+\gamma]b_{k+1}} r^k.$$

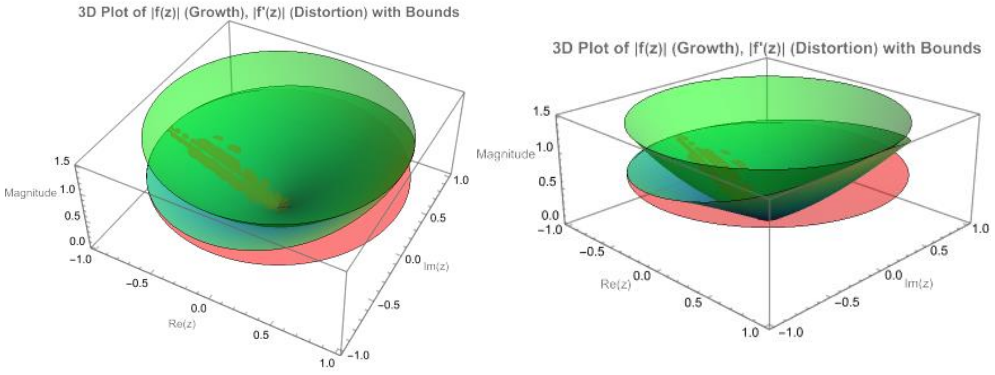


Figure 4 : 3D Plot of Symmetric Distortion and Growth Bounds

4. Closure Property

In the subsequent theorems, we demonstrate the closure property for the class $\mathcal{MS}_k(\alpha, \lambda, \beta, \gamma)$.

Theorem (4.1): Let the sequence of functions $f_j \in \mathcal{MS}_k(\alpha, \lambda, \beta, \gamma)$ for all $j = 1,2,\cdots,\ell$ defined by:

$$f_j(z) = z^k - \sum_{n=k+1}^{\infty} a_{n,j}\, z^n, \qquad \left(a_{n,j} \geq 0, n \in \mathbb{N}, j = 1,2,\cdots \ell\right).$$

Then the function defined by:

$$\hbar(z) = z^k - \sum_{n=k+1}^{\infty} e_n\, z^n, \qquad (\, e_n \geq 0, n \in \mathbb{N})$$

is also a constituent of the class $\mathcal{MS}_k(\alpha, \lambda, \beta, \gamma)$, where

$$e_n = \frac{1}{\ell} \sum_{j=1}^{\ell} a_{n,j}\,, \qquad n = 1,2,3,\cdots.$$

Proof: Because each $f_j \in \mathcal{MS}_k(\alpha, \lambda, \beta, \gamma)$, then by theorem (2.1) that

$$\sum_{n=k+1}^{\infty} \left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right] a_{n,j} b_n \leq \gamma + k(\alpha(1-k)+1),$$

for every $j = 1,2,\cdots \ell$. Hence

$$\sum_{n=k+1}^{\infty} \left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right] e_n b_n = \sum_{n=k+1}^{\infty} \left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right] b_n \left(\frac{1}{\ell} \sum_{j=1}^{\ell} a_{n,j}\right)$$

$$= \frac{1}{\ell}\sum_{j=1}^{\ell}\left(\sum_{n=k+1}^{\infty}\left[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\right]b_n a_{n,j}\right)$$

$$\leq \frac{1}{\ell}\sum_{j=1}^{\ell}\big(\gamma+k(\alpha(1-k)+1)\big) = \gamma+k(\alpha(1-k)+1).$$

According to theorem (2.1), it consequently follows that $\hbar(z) \in \mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$.

Theorem (4.2): The class $\mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$ is a convex set.

Proof: Let $f_1(z)$ and $f_2(z)$ be the arbitrary elements of $\mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$. Then for every ε $(0<\varepsilon<1)$, we shall demonstrate that

$$(1-\varepsilon)f_1(z)+\varepsilon f_2(z) \in \mathcal{MS}_k(\alpha,\lambda,\beta,\gamma).$$

Thus, we have

$$(1-\varepsilon)f_1(z)+\varepsilon f_2(z) = z^k - \sum_{n=k+1}^{\infty}\left[(1-\varepsilon)a_{n,1}+\varepsilon a_{n,2}\right]z^n.$$

Hence

$$= \sum_{n=k+1}^{\infty}\frac{\left[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\right]b_n}{\gamma+k(\alpha(1-k)+1)}\left[(1-\varepsilon)a_{n,1}+\varepsilon a_{n,2}\right]$$

$$= (1-\varepsilon)\sum_{n=k+1}^{\infty}\frac{\left[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\right]b_n}{\gamma+k(\alpha(1-k)+1)}a_{n,1} + \varepsilon\sum_{n=k+1}^{\infty}\frac{\left[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\right]b_n}{\gamma+k(\alpha(1-k)+1)}a_{n,2} \leq 1.$$

This completes the proof.

5. Radii of Starlikeness and Convexity

In the subsequent theorems, we examine the radii of starlikeness and convexity.

Theorem (5.1): Consider $f \in \mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$. Then f is k-valent starlike of order σ $(0 \leq \sigma < k)$ in the disk $|z|<r=r_1$, in which

$$r_1 = \inf_{n\geq k+1}\left[\frac{(k-\sigma)\left[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\right]b_n}{[\gamma+k(\alpha(1-k)+1)](n-\sigma+2k)}\right]^{\frac{1}{n+k}}. \tag{5.1}$$

The result is sharp for the function f defined by (2.4).

Proof: It is adequate to demonstrate that

$$\left|\frac{zf'(z)}{f(z)}+k\right| \leq k-\sigma \tag{5.2}$$

but

$$\left|\frac{zf'(z)+kf(z)}{f(z)}\right| \leq \frac{\sum_{n=k+1}^{\infty}(n+k)a_n|z|^{n+k}}{1-\sum_{n=k+1}^{\infty}a_n|z|^{n+k}}.$$

Consequently, (5.2) will be fulfilled if

$$\frac{\sum_{n=k+1}^{\infty}(n+k)a_n|z|^{n+k}}{1-\sum_{n=k+1}^{\infty}a_n|z|^{n+k}} \le k-\sigma,$$

or if

$$\sum_{n=k+1}^{\infty}\frac{n-\sigma+k}{k-\sigma}a_n|z|^{n+k} \le 1. \tag{5.3}$$

Since $f \in \mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$, we have

$$\sum_{n=k+1}^{\infty}\frac{\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]a_nb_n}{\gamma+k(\alpha(1-k)+1)} \le 1.$$

Hence, (5.3) will true if

$$\frac{n-\sigma+k}{k-\sigma}|z|^{n+k} \le \frac{\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]a_nb_n}{\gamma+k(\alpha(1-k)+1)},$$

or equivalently

$$|z| \le \left[\frac{(k-\sigma)\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]b_n}{[\gamma+k(\alpha(1-k)+1)](n-\sigma+2k)}\right]^{\frac{1}{n+k}}, n \ge k+1.$$

Theorem (5.2): Let $f \in \mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$. Then f is k-valent convex of order σ $(0 \le \sigma < k)$ in the disk $|z| < r = r_2$, where

$$r_2 = \inf_{n\ge k+1}\left[\frac{k(k-\sigma)\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]b_n}{n[\gamma+k(\alpha(1-k)+1)](n-\sigma+2k)}\right]^{\frac{1}{n+k}}. \tag{5.4}$$

The result is sharp for the function f defined by (2.4).

Proof: It is adequate to demonstrate that

$$\left|\frac{zf''(z)}{f'(z)}+k+1\right| \le k-\sigma, \tag{5.5}$$

but

$$\left|\frac{zf''(z)+(k+1)f'(z)}{f'(z)}\right| \le \frac{\sum_{n=k+1}^{\infty}n(n+k)a_n|z|^{n+k}}{k-\sum_{n=k+1}^{\infty}a_n|z|^{n+k}}.$$

Consequently, (5.5) will be fulfilled if

$$\frac{\sum_{n=k+1}^{\infty}n(n+k)a_n|z|^{n+k}}{k-\sum_{n=k+1}^{\infty}a_n|z|^{n+k}} \le k-\sigma,$$

or if

$$\sum_{n=k+1}^{\infty}\frac{n(n-\sigma+k)}{k(k-\sigma)}a_n|z|^{n+k} \le 1. \tag{5.6}$$

Due to $f \in \mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$, will be true if

$$\frac{n(n-\sigma+k)}{k(k-\sigma)}|z|^{n+k} \le \frac{\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]a_nb_n}{\gamma+k(\alpha(1-k)+1)},$$

or equivalently

$$|z| \leq \left[\frac{k(k-\sigma)\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]b_n}{n[\gamma+k(\alpha(1-k)+1)](n-\sigma+2k)}\right]^{\frac{1}{n+k}}, \qquad n \geq k+1.$$

6. Weighted mean and Extreme points

In the next theorem, we discuss the extreme points for the functions in the subclass $\mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$

Theorem (6.1): Let $f_k(z) = z^k$ and $f_n(z) = z^k - \frac{\gamma+k(\alpha(1-k)+1)}{\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]b_n} z^n,\ n \geq k+1$. Then $f \in \mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$ if and only if can be articulated as

$f(z) = \sum_{n=k+1}^{\infty} \mathcal{W}_n f_n(z),\ \mathcal{W}_n \geq 0$ and $\sum_{n=k+1}^{\infty} \mathcal{W}_n = 1 - \mathcal{W}_k$.

|**Proof:** Let $f(z) = \sum_{n=k+1}^{\infty} \mathcal{W}_n f_n(z),\ \mathcal{W}_n \geq 0$ and $\sum_{n=k+1}^{\infty} \mathcal{W}_n = 1 - \mathcal{W}_k$

$$= z^k - \sum_{n=k+1}^{\infty} \mathcal{W}_n \frac{\gamma+k(\alpha(1-k)+1)}{\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]b_n} z^n.$$

Since

$$\sum_{n=k+1}^{\infty} \frac{\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]b_n}{\gamma+k(\alpha(1-k)+1)} \cdot \frac{\gamma+k(\alpha(1-k)+1)}{\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]b_n} \mathcal{W}_n$$

$$= \sum_{n=k+1}^{\infty} \mathcal{W}_n = 1 - \mathcal{W}_k \leq 1.$$

Hence $f \in \mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$.

Conversely, suppose that $f \in \mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$. As

$$a_n \leq \frac{\gamma+p(\alpha(1-k)+1)}{\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]b_n}, \quad n \geq k+1, \qquad k \in \mathbb{N}.$$

Setting

$$\mathcal{W}_n = a_n \frac{\left[n\left(1+\lambda-\beta+\alpha(1-n)\right)+\gamma\right]b_n}{\gamma+k(\alpha(1-k)+1)}$$

and

$$\mathcal{W}_k = 1 - \sum_{n=k+1}^{\infty} \mathcal{W}_n,$$

it therefore follows that

$$f(z) = \sum_{n=k+1}^{\infty} \mathcal{W}_n f_n(z).$$

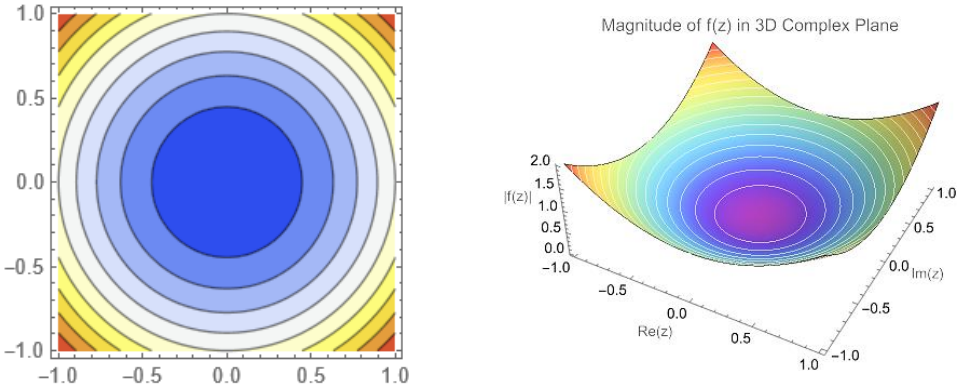


Figure 5 : Plot 2D,3D Extreme points

Definition (6.1): Let f_1 and f_2 attend the subclass $\mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$. The weighted mean w_t of f_1 and f_2 is expressed as

$$w_t(z)=\frac{1}{2}[(1-t)f_1(z)+(1+t)f_2(z)], \qquad 0<t<1. \tag{6.1}$$

Theorem (6.2): The weighted mean of any two functions f_1 and f_2 in $\mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$ is also belongs to the subclass $\mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$.

Proof: By Definition (6.1), we possess

$$w_t(z)=\frac{1}{2}[(1-t)f_1(z)+(1+t)f_2(z)]=z^k-\sum_{n=k+1}^{\infty}\frac{1}{2}\big[(1-t)a_{n,1}+(1+t)a_{n,2}\big]z^n.$$

Since f_1 and f_2 are in the class $\mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$, so by theorem (2.1), we get

$$\sum_{n=k+1}^{\infty}\big[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\big]a_{n,1}b_n\le\gamma+k(\alpha(1-k)+1)$$

and

$$\sum_{n=k+1}^{\infty}\big[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\big]a_{n,2}b_n\le\gamma+k(\alpha(1-k)+1).$$

Hence,

$$\sum_{n=k+1}^{\infty}\big[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\big]b_n\left(\frac{1}{2}\big[(1-t)a_{n,1}+(1+t)a_{n,2}\big]\right)$$

$$=\frac{1}{2}(1-t)\sum_{n=k+1}^{\infty}\big[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\big]b_na_{n,1}+\frac{1}{2}(1+t)\sum_{n=k+1}^{\infty}\big[n\big(1+\lambda-\beta+\alpha(1-n)\big)+\gamma\big]b_na_{n,2}$$

$$\le\gamma+k(\alpha(1-k)+1).$$

Therefore, $w_t(z)\in\mathcal{MS}_k(\alpha,\lambda,\beta,\gamma)$.

7. Applications of the New Subclass of k-valent Functions Defined via Hadamard Product

The newly investigated class of k-valent functions, defined through the Hadamard product, demonstrates broad applicability across various scientific and engineering domains. Owing to their geometric and analytic characteristics, these functions contribute significantly to modeling and optimization tasks in the following areas:

7.1 Application in Fluid Dynamics

The radii of starlikeness and convexity associated with the studied class (Theorems 5.1 and 5.2) offer crucial insights into modeling the flow of incompressible fluids over complex geometries with multiple singularities. These radii act as natural boundary constraints, optimizing fluid transport systems and aiding in the analysis of phenomena such as boundary layer separation in both aerodynamics and hydrodynamics. Such boundary conditions are essential in modern fluid dynamics simulations, as discussed in [2].

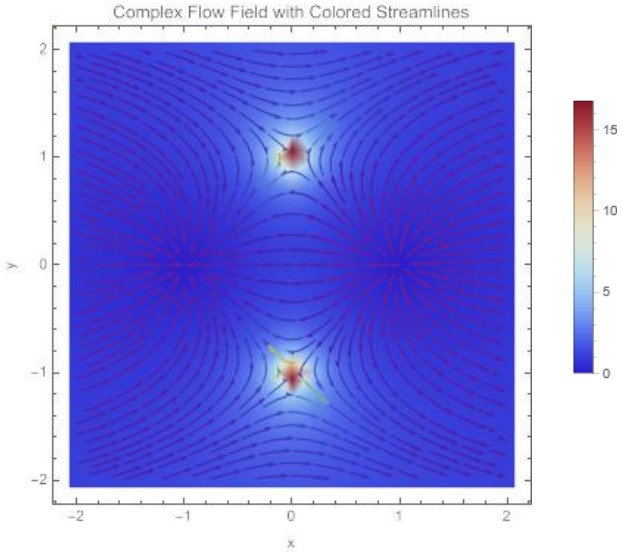


Figure 6 : Plot 2D complex application fluid dynamic

7.2 Application in Signal Processing

Due to the convolutional structure of the Hadamard product, the defined function class is well-suited for frequency-domain filtering operations. It enables the design of analytic filters with controlled distortion, thereby improving signal clarity and fidelity in systems involving amplitude and phase modulation. Hadamard based analytic filters have been proven to be effective in various signal processing techniques in recent studies [10].

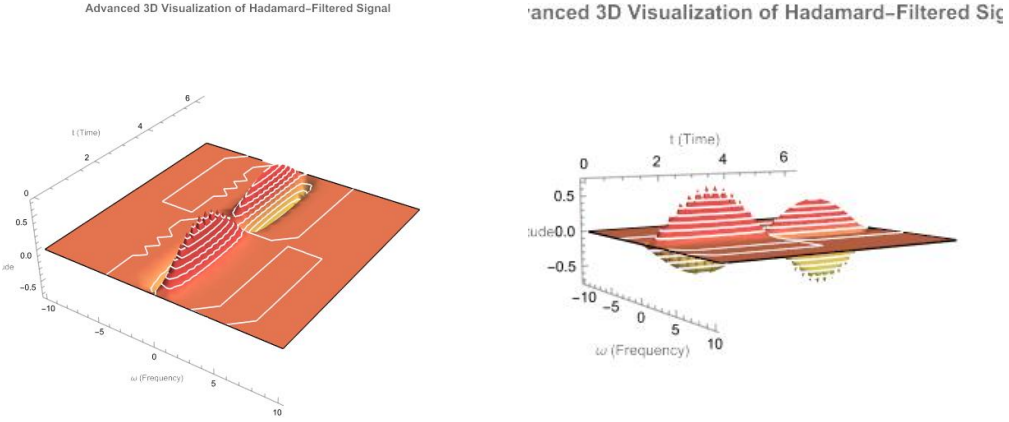


Figure 7 : Plot 3D complex application Signal Processing

7.3 Application in Image Warping and Conformal Mapping

The distortion and the growth properties of this class of functions (Theorem 3.1) have an important application to image deformations within augmented and virtual reality. These properties allow to obtain realistic and structurally correct modifications, that are crucial in interactive environments with high geometric fidelity needs. The use of conformal maps in the context of virtual and augmented reality systems is discussed in [25,26] as well.

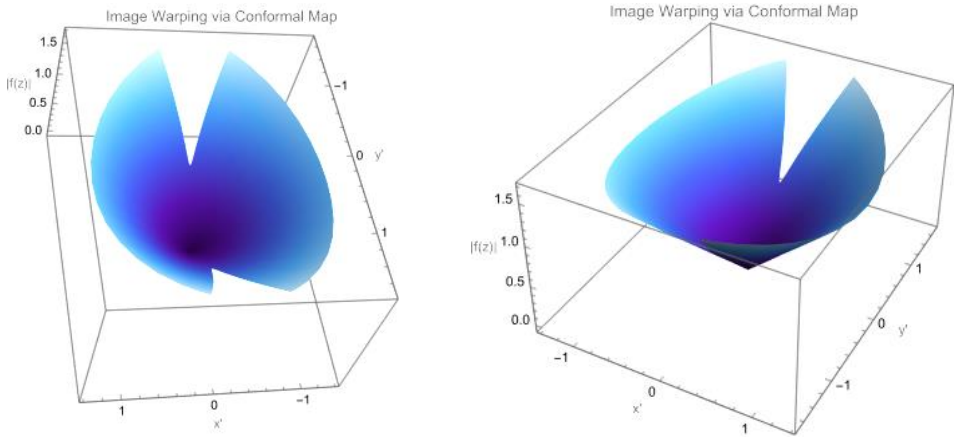


Figure 8 : Plot 3D complex application Image Warping and Conformal Mapping

7.4 Application in Cryptographic Systems

The analytic properties and extremal properties of the new class of p-valent functions established by the authors could be exploited as a mathematical framework that can be applicable in cryptographic context. Especially the complicated geometry aspect, as well as the fact that these functions are multivalent, has lead to the construction of nonlinear transformations which are sensitive to the initial conditions—nice feature to have for the secure communication protocols. In particular, the Hadamard product-based construction makes it possible to construct complex analytic mappings that are suitable as cryptographic tools in stream ciphers and pseudorandom number generators. Furthermore, as was the case with the extremal point characterization (Theorem 6.1) and the weighted mean (Theorem 6.2), one can design key-dependent transformation schemes with free recourse to both unpredictability and strong diffusion properties. These instantiations are also sound with respect to the basic needs of modern cryptographic systems, contrary, in particular for the case of 4 properties of the SWIDH phenomenon, and in a cryptographic context This work is based on elliptic curve cryptography and chaos-based encryption techniques. The inclusion of such classes of functions into cryptographic algorithms could improve the security since it would become more resistant to linear and differential attacks. Another recent study, entitled Hadamard Product Arguments and Their Applications, available in the IACR ePrint archive shows how Hadamard products can be applied to the design of efficient and secure cryptographic protocols. This work supports the relevance of Hadamard-based constructions in Cryptography [8].

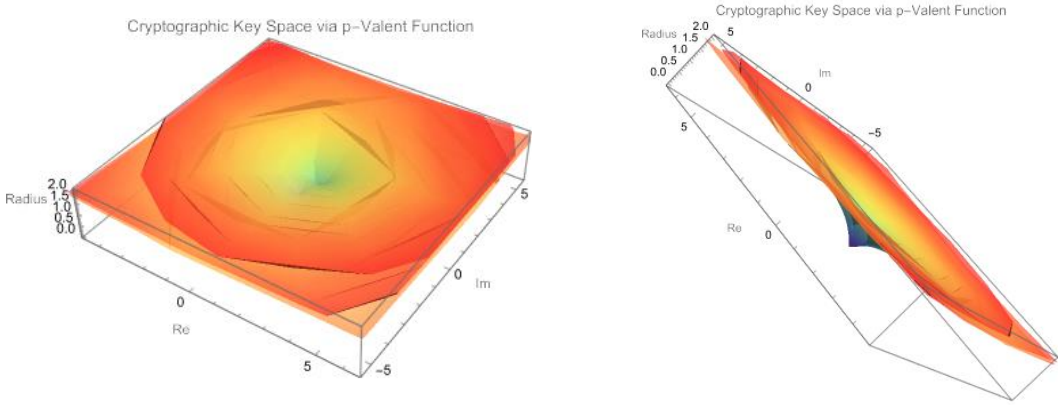


Figure 9 : Plot 3D complex application Cryptographic Systems

8. Conclusions and Future Studies

8.1 Conclusions

In this paper, we define and study a new class of k-valent functions defined by Hadamard product in the open unit disk. Some geometric properties of the class are established with rigorous analysis, including coefficient inequalities, bounds on distortion and growth, closure theorems, radii of starlikeness and convexity, extreme points, and weighted means. The sharpness of results is demonstrated with some extremal functions. These findings help further broaden an understanding of multivalent function theory while laying a foundation for further exploration in complex analysis applications where Hadamard products meet with geometric function theory. Future research, based on the recent findings in multivalent function theory, can apply these results further toward more generalized classes or other branches like dynamic systems or signal processing to develop new and improved models in engineering and applied sciences [7,13,14,17,18].

8.2 Future Studies

- Investigation of Fractional-Order k-valent Functions Extend the current class to fractional-order derivatives and integrals, analyzing how the Hadamard product interacts with non-integer operators in complex analysis.
- Development of Computational Models Build and test computational algorithms for approximating and simulating the geometric properties (like distortion and growth) of the new k-valent function class.
- Applications in Modern Fluid Mechanics Apply the geometric properties of the defined function class to model complex, real-world fluid flow phenomena, including turbulence and multi-phase flow dynamics.
- Integration with Cryptographic Key Generation Explore the use of extremal properties of the k-valent class in designing secure key generation protocols for post-quantum cryptography.

Extension to Multidimensional Complex Spaces Generalize the newly defined function class from the complex plane to higher-dimensional complex manifolds, investigating how the Hadamard product behaves in several complex variables.

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