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k-Fractional Extension of h-Hadamard Integrals and Derivatives

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ABSTRACT

In this paper, we present a generalization types of the h-Hadamard integrals and derivatives parameterized by k. The h-Hadamard integrals and derivatives are themselves generalizations of the Hadamard integrals and derivatives, defined in relation to a continuous function h. We have also developed the concepts of k-gamma and k-beta, which correlate with the definitions of k-fractional h-Hadamard integrals and derivatives. The paper includes various theorems, propositions, properties, and illustrative examples.

MSC..

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1. Introduction

The classical calculus concepts of differentiation and integration are generalized by using fractional numbers. In [1, 2, 3–5,7] many authors introduce several types of fractional integral and derivative. For fractional integrals and derivatives have various applications in various branches of science [8–10, 12, 13, 14]. The type of fractional integral and derivative depend on another function in the kernel of the fractional operator. This type of fractional is considered generalized for the concept of fractional operator. In [11], Kilbas et al. introduce h-fractional Riemann-Liouville, and in [6], K. K.BALACHANDRAN introduces the h-fractional Hadamard operator. Another method to generalize the concept of fractional operator by using k-function ,in the first time, in [15], Mubeen and Habibullah introduce a special k-fractional integral of the Riemann-Liouville, in this method, there is a need for developed Gamma and Beta function. In [16-19], Gamma function and Beta function are developed by parameter k. This development acts as a generalization of fractional integrals and derivatives by using k-functions, to use the k-function for generalizing fractional integrals and derivatives. $C_{[a,b]}$ stands for Banach space of all continuous function defined on [a,b].

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2.Fundamental Concepts

2.1 Definition [6]:

Let ϕbe a function defined on the interval [a,b] and $\alpha > 0$ then :

The formulas :

$$I_H^{\alpha,a^+}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (\ln x - \ln u)^{\alpha-1}\phi(u)\frac{du}{u} \quad , \quad x \in [a,b] \tag{2.1}$$

is called the left Hadamard integral and

$$I_H^{\alpha,b^-}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\ln u - \ln x)^{\alpha-1}\phi(u)\frac{du}{u} \quad , \quad x \in [a,b] \tag{2.2}$$

is called the right Hadamard integral.

2.2 Definition [6]:

Let f be a function defined on the interval [a,b] and $\alpha > 0$ then :

The formulas :$I_{H,h}^{\alpha,a^+}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\frac{h'(u)}{h(u)}\right)(\ln h(x) - \ln h(u))^{\alpha-1}\phi(u)du \quad , \quad x \in [a,b]$ (2.3)

is called the h-left Hadamard integral.

and

$$I_{H,h}^{\alpha,b^-}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\frac{h'(u)}{h(u)}\right)(\ln h(u) - \ln h(x))^{\alpha-1}\phi(u)du \quad , \quad x \in [a,b] \tag{2.4}$$

is called the h-right Hadamard integral.

Now , we introduce a developed version of Definition(1.2) by using parameter $k > 0$. In [ ], Diaz and Pariguan introduce the special k-functions for gamma and beta functions as the following definition.

2.3 Definition [16,17]: Let $\alpha, \beta \in R \text{ and } k > 0$ then the formula

$$\Gamma_k(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y^k}\,dy$$

is called the k-gamma function, and

$$\mathrm{B}_k(\alpha,\beta) = \frac{1}{k}\int_0^1 y^{\frac{\alpha}{k}-1}(1-y)^{\frac{\beta}{k}-1}\,dy$$

is called the k-Beta function

2.4 Some properties of k-gamma and k-Beta function [16,17]:

i. $\Gamma_k(\mu + k) = \mu\Gamma_k(\mu)$;

ii. $\Gamma_k(k) = 1$

iii. $\Gamma_k(n) = \frac{\Gamma\left(\frac{n}{k}\right)}{k}$

iv. $B_k(\mu,\pi) = \frac{\Gamma_k(\mu)\Gamma_k(\pi)}{\Gamma_k(\mu+\pi)}$.

3. k-h-Fractional Hadamard Integral and derivative

In this section, we give a new fractional integral and derivative , namely k-h-fractional Hadamard integral and derivative as generalized for Caputo-Hadamard fractional, Riemann-Liouville fractional and Erdelyi-Kober fractional.

3.1 Definition:

Let ϕ be a continuous function in a Banach space $C_{[a,b]}$ then :

$$I_{H,h}^{k,\alpha,a^+}\phi(x) = \frac{1}{\mathrm{k}\Gamma_k(\alpha)}\int_a^x \left(\frac{h'(u)}{h(u)}\right)(\ln h(x) - \ln h(u))^{\frac{\alpha}{k}-1}\phi(u)du \quad , \quad x \in [a,b] \tag{3.1}$$

is called the k-h-left Hadamard integral. And

$$I_{H,h}^{k,\alpha,b^-}\phi(x) = \frac{1}{\mathrm{k}\Gamma_k(\alpha)}\int_x^b \left(\frac{h'(u)}{h(u)}\right)(\ln h(u) - \ln h(x))^{\frac{\alpha}{k}-1}\phi(u)du \quad , \quad x \in [a,b] \tag{3.2}$$

is called the k-h-right Hadamard integral.

3.2 Definition:

Let ϕ be a continuous function in a Banach space $C_{[a,b]}$ and $n-1<\alpha<n$, $k<n$ then :

$$D_{H,h}^{k,\alpha,a^+}\phi(x) = \left(\frac{h(u)}{h'(u)}\frac{d}{dx}\right)^{\frac{n}{k}} I_{H,h}^{k,n-\alpha,a^+}\phi(x)$$

$$= \frac{1}{\mathrm{k}\Gamma_k(n-\alpha)}\left(\frac{h(u)}{h'(u)}\frac{d}{dx}\right)^{\frac{n}{k}}\int_a^x \left(\frac{h'(u)}{h(u)}\right)(\ln h(x) - \ln h(u))^{\frac{n-\alpha}{k}-1}\phi(u)du \quad , \quad x \in [a,b] \tag{3.3}$$

is called k-h-left Hadamard dervitive. And

$$D_{H,h}^{k,\alpha,b^-}\phi(x) = \left(\frac{h(u)}{h'(u)}\frac{d}{dx}\right)^{\frac{n}{k}} I_{H,h}^{k,\alpha,b^-}\phi(x)$$

$$= \frac{1}{\mathrm{k}\Gamma_k(n-\alpha)}\left(\frac{h(u)}{h'(u)}\frac{d}{dx}\right)^{\frac{n}{k}}\int_x^b \left(\frac{h'(u)}{h(u)}\right)(\ln h(u) - \ln h(x))^{\frac{n-\alpha}{k}-1}\phi(u)du \quad , \quad x \in [a,b] \tag{3.4}$$

is called k-h-right Hadamard derivative.

3.3 A special cases of k-h-left (right) Hadamard integral(dervitive).

1. K=1, h(x)=x we get a classical Hadamard fractional integral(derivative)
2. K=1, $h(x) = e^x$, we get a Riemann-Liouville fractional integral(derivative)
3. K=1, $h(x) = x^c$, c *is constant* , we obtain a generalization of Hadamard fractional integral(derivative).

3.4 Example:

Let $\phi(x) = c$, $h(x) = x^m$, *where c is constant* and $1<\alpha<2$, $k<1$ then :

$$I_{H,x^m}^{\alpha,a^+}(\,c) = \frac{1}{\mathrm{k}\Gamma_k(\alpha)}\int_a^x \left(\frac{mu^{m-1}}{u^m}\right)(m\ln x - m\ln u)^{\frac{\alpha}{k}-1}(\,c\,)\,du \quad , \quad x \in R$$

$$= \frac{m^2 c}{\mathrm{k}\Gamma_k(\alpha)}\int_a^x \left(\frac{1}{u}\right)(\ln x - \ln u)^{\frac{\alpha}{k}-1}\,du = -\frac{m^2 c}{\alpha\Gamma_k(\alpha)}(\ln x - \ln u)^{\frac{\alpha}{k}}\Big|_a^x = \frac{m^2 c}{\alpha\Gamma_k(\alpha)}(\ln x - \ln a)^{\frac{\alpha}{k}}$$

Also we can calculate the k-h-left Hadamard fractional derivative as following

$$D_{H,x^m}^{k,\alpha,a^+}(c)=\left(\frac{1}{m}x\frac{d}{dx}\right)^{\frac{2}{k}} I_{H,x^m}^{k,2-\alpha,a^+}(c)=-\frac{m}{\mathrm{k}\Gamma_k(2-\alpha)}\left(\frac{1}{m}x\frac{d}{dx}\right)^{\frac{2}{k}}\int_a^x\left(\frac{1}{u}\right)(\ln x-\ln u)^{\frac{2-\alpha}{k}-1}(c)du$$

$$=-\frac{mc}{(2-\alpha)\Gamma_k(1-\alpha)}\left(\frac{1}{m}x\frac{d}{dx}\right)^{\frac{2}{k}}(\ln x-\ln u)^{\frac{2-\alpha}{k}}\Big|_a^x$$

$$=\frac{mc}{(2-\alpha)\Gamma_k(2-\alpha)}\left(\frac{1}{m}x\frac{d}{dx}\right)^{\frac{2}{k}}(\ln x-\ln a)^{\frac{2-\alpha}{k}}$$

The following Figure show k-h-left fractional Hadamard derivative where $h(x)=x\,, c=2, a=2$

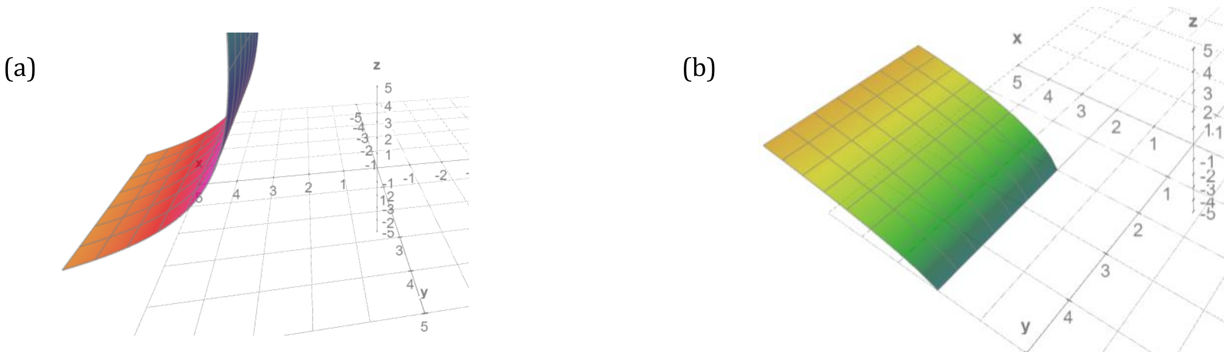

Fig.1 (a) k-h-left fractional Hadamard derivative (b) k-h-left fractional Hadamard Integral ,where $n=2, \alpha=1, k=2$,

3.5 Theorem:

Let ϕ be a continuous function defined on [a,b] , $n-1<\alpha,\beta<n$ and $h\in C^n[a,b]$ then:

$$I_{H,h}^{k,\alpha,a^+} I_{H,h}^{k,\beta,a^+}\phi(x)=I_{H,h}^{k,\beta,a^+} I_{H,h}^{k,\alpha,a^+}\phi(x)=I_{H,h}^{k,\alpha+\beta,a^+}\phi(x) \tag{3.5}$$

Proof:

$$I_{H,h}^{k,\alpha,a^+} I_{H,h}^{k,\beta,a^+}\phi(x)=\frac{1}{\mathrm{k}^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x\left(\frac{h'(u)}{h(u)}\right)(\ln h(x)-\ln h(u))^{\frac{\alpha}{k}-1}\int_v^x\left(\frac{h'(v)}{h(v)}\right)(\ln h(u)-\ln h(v))^{\frac{\beta}{k}-1}\phi(v)dudv$$

$$=\frac{1}{\mathrm{k}^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x\left(\frac{h'(v)}{h(v)}\right)\phi(v)\int_v^x\left(\frac{h'(u)}{h(u)}\right)(\ln h(x)-\ln h(u))^{\frac{\alpha}{k}-1}(\ln h(u)-\ln h(v))^{\frac{\beta}{k}-1}dudv \tag{3.6}$$

Suppose that $z=\frac{\ln h(u)-\ln h(v)}{\ln h(x)-\ln h(v)}$, then $\quad z(\ln h(x)-\ln h(v))=\ln h(u)-\ln h(v)$

$$dz=\left(\frac{h'(u)}{h(u)}\right)\frac{du}{\ln h(x)-\ln h(v)},\quad z^{\frac{\beta}{k}-1}=\left(\frac{\ln h(u)-\ln h(v)}{\ln h(x)-\ln h(v)}\right)^{\frac{\beta}{k}-1},$$

$(1-z)^{\frac{\alpha}{k}-1}=\left(\frac{\ln h(x)-\ln h(u)}{\ln h(x)-\ln h(v)}\right)^{\frac{\alpha}{k}-1}$ thus

$$z^{\frac{\beta}{k}-1}(\ln h(x)-\ln h(v))^{\frac{\beta}{k}-1}=(\ln h(u)-\ln h(v))^{\frac{\beta}{k}-1}$$

$$(1-z)^{\frac{\alpha}{k}-1}(\ln h(x)-\ln h(v))^{\frac{\alpha}{k}-1}=(\ln h(x)-\ln h(u))^{\frac{\alpha}{k}-1}$$

$$dz(\ln h(x)-\ln h(v))=\left(\frac{h'(u)}{h(u)}\right)du$$

Let's simplify the internal integral in Equation (3.6) using the terms z and 1-z , we obtain that

$$\int_{v}^{x}\left(\frac{h'(u)}{h(u)}\right)(\ln h(x)-\ln h(u))^{\frac{\alpha}{k}-1}(\ln h(u)-\ln h(v))^{\frac{\beta}{k}-1}du=(\ln h(x)-\ln h(v))^{\frac{\alpha+\beta}{k}-1}\int_{0}^{1}(1-z)^{\frac{\alpha}{k}-1}\,z^{\frac{\beta}{k}-1}\quad dz$$

$$=k(\ln h(x)-\ln h(v))^{\frac{\alpha+\beta}{k}-1}B_k(\alpha,\beta) \tag{3.7}$$

By substation Eq.(3.7) in Eq.(3.6) we get that:

$$I_{H,h}^{k,\alpha,a^+}I_{H,h}^{k,\beta,a^+}\phi(x)=\frac{B_k(\alpha,\beta)}{\mathrm{k}\Gamma_k(\alpha)\Gamma_k(\beta)}\int_{a}^{x}\left(\frac{h'(v)}{h(v)}\right)\ (\ln h(x)-\ln h(v))^{\frac{\alpha+\beta}{k}-1}\ \ \phi(v)\ dv$$

$$=\frac{1}{\mathrm{k}\Gamma_k(\alpha+\beta)}\int_{a}^{x}\left(\frac{h'(v)}{h(v)}\right)\ (\ln h(x)-\ln h(v))^{\frac{\alpha+\beta}{k}-1}\ \ \phi(v)\ dv$$

$$I_{H,h}^{k,\alpha,a^+}I_{H,h}^{k,\beta,a^+}\phi(x)=I_{H,h}^{k,\alpha+\beta,a^+}\phi(x)$$

By similarly method we can prove that $I_{H,h}^{k,\beta,a^+}I_{H,h}^{k,\alpha,a^+}\phi(x)=I_{H,h}^{k,\alpha+\beta,a^+}\phi(x)$

Therefor $I_{H,h}^{k,\alpha,a^+}I_{H,h}^{k,\beta,a^+}\phi(x)=I_{H,h}^{k,\beta,a^+}I_{H,h}^{k,\alpha,a^+}\phi(x)=I_{H,h}^{k,\alpha+\beta,a^+}\phi(x)$

Theorem 3.6:

Let f be a continuous function defined on [a,b] ,and $n-1<\alpha,\beta<n$ and $h\in C^n[a,b]$ then:

$$D_{H,h}^{k,\alpha,a^+}I_{H,h}^{k,\alpha,a^+}\phi(x)=\phi(x) \tag{3.8}$$

Where $C^n[a,b]$ is a Banach space of all continuous functions and differentiable n-times.

Proof:

$$D_{H,h}^{k,\alpha,a^+}I_{H,h}^{k,\alpha,a^+}\phi(x)=\left(\frac{h(u)}{h'(u)}\frac{d}{dx}\right)^{\frac{n}{k}}I_{H,h}^{k,n-\alpha,a^+}I_{H,h}^{k,\alpha,a^+}\phi(x)=\left(\frac{h(u)}{h'(u)}\frac{d}{dx}\right)^{\frac{n}{k}}I_{H,h}^{k,n,a^+}\phi(x)$$

$$=\frac{1}{\mathrm{k}\Gamma_k(n)}\left(\frac{h(u)}{h'(u)}\frac{d}{dx}\right)^{\frac{n}{k}}\int_{a}^{x}\left(\frac{h'(u)}{h(u)}\right)(\ln h(x)-\ln h(u))^{\frac{n}{k}-1}\phi(u)du$$

By using partition method for above integral we obtain

$$=\frac{1}{\mathrm{n}\Gamma_k(n)}\left(\frac{h(u)}{h'^{(u)}}\frac{d}{dx}\right)^{\frac{n}{k}}\left((\ln h(x)-\ln h(a))^{\frac{n}{k}}\phi(a)+\int_{a}^{x}(\ln h(x)-\ln h(u))^{\frac{n}{k}}\phi'(u)du\right)$$

Now , We analyze the quantity $\left(\frac{h(u)}{h'^{(u)}}\frac{d}{dx}\right)^{\frac{n}{k}}=\left(\frac{h(u)}{h'^{(u)}}\frac{d}{dx}\right)^{\frac{n}{k}-1}\left(\frac{h(u)}{h'^{(u)}}\frac{d}{dx}\right)$ and the factor $\left(\frac{h(u)}{h'^{(u)}}\frac{d}{dx}\right)$ begins to acts it first.

$$=\frac{1}{\mathrm{k}\Gamma_k(n)}\left(\frac{h(u)}{h'^{(u)}}\frac{d}{dx}\right)^{\frac{n}{k}-1}\left((\ln h(x)-\ln h(a))^{\frac{n}{k}-1}\phi(a)+\int_{a}^{x}(\ln h(x)-\ln h(u))^{\frac{n}{k}-1}\phi'(u)du\right)$$

$$=\frac{\left(\frac{n}{k}-1\right)}{\Gamma\left(\frac{n}{k}\right)}\left(\frac{h(u)}{h'^{(u)}}\frac{d}{dx}\right)^{\frac{n}{k}-2}\left((\ln h(x)-\ln h(a))^{\frac{n}{k}-2}\phi(a)+\int_{a}^{x}(\ln h(x)-\ln h(u))^{\frac{n}{k}-2}\phi'(u)du\right)$$

if we continue in this process we obtain

$=\frac{\Gamma\left(\frac{n}{k}\right)}{\Gamma\left(\frac{n}{k}\right)}\left(\phi(a)+\int_a^x\phi'(u)du\right)=\phi(x)$, then the proof is perform.

3.7 Proposition:

Let ϕ be a continuous function defined on [a,b] , $n, m \in N$, $n-1<\alpha<n$ and $h \in C^{n+m}[a,b]$ then:

$$\left(\frac{h(u)}{h'(u)}\frac{d}{dx}\right)^{\frac{m}{k}} D_{H,h}^{k,\alpha,a^+}\phi(x) = D_{H,h}^{k,\alpha+m,a^+}\phi(x) \tag{3.9}$$

Proof:

$$\left(\frac{h(u)}{h'(u)}\frac{d}{dx}\right)^{\frac{m}{k}} D_{H,h}^{k,\alpha,a^+}\phi(x) = \left(\frac{h(u)}{h'(u)}\frac{d}{dx}\right)^{\frac{m}{k}}\left(\frac{h(u)}{h'(u)}\frac{d}{dx}\right)^{\frac{n}{k}} I_{H,h}^{k,n-\alpha,a^+}\phi(x)$$

$$= \frac{1}{\mathrm{k}\Gamma_k(n-\alpha)}\left(\frac{h(u)}{h'^{(u)}}\frac{d}{dx}\right)^{\frac{n+m}{k}} \int_a^x \left(\frac{h'(u)}{h(u)}\right)(\ln h(x) - \ln h(u))^{\frac{n-\alpha}{k}-1}\phi(u)du$$

Since $n-1<\alpha<n$ thus $(n+m)-1<\alpha+m<n+m$, now we rewrite the above integral by the terms n+m and $\alpha+m$

$$= \frac{1}{\mathrm{k}\Gamma_k(n+m-\alpha)}\left(\frac{h(u)}{h'^{(u)}}\frac{d}{dx}\right)^{\frac{n+m}{k}} \int_a^x \left(\frac{h'(u)}{h(u)}\right)(\ln h(x) - \ln h(u))^{\frac{(n+m)-(m+\alpha)}{k}-1}\phi(u)du$$

$= D_{H,h}^{k,\alpha+m,a^+}\phi(x)$, thus the proof is complete.

3.8 Proposition:

Let $n, m \in N$, $n-1<\alpha,\beta<n$ and $k<n$, $h \in C^{n+m}[a,b]$, $\lim_{x\to a} h(x)$ *exists* , then:

$$I_{H,h}^{k,\alpha,a^+}(\ln h(x) - \ln h(a))^{\frac{\beta}{k}-1} = \frac{\Gamma_k(\beta)}{k\Gamma_k(\alpha+\beta)}(\ln h(x) - \ln h(a))^{\frac{\alpha+\beta}{k}-1} \tag{3.10}$$

Proof:

$$I_{H,h}^{k,\alpha,a^+}(\ln h(x) - \ln h(a))^{\frac{\beta}{k}-1} = \frac{1}{\mathrm{k}\Gamma_k(\alpha)}\int_a^x \left(\frac{h'(u)}{h(u)}\right)(\ln h(x) - \ln h(u))^{\frac{\alpha}{k}-1}(\ln h(u) - \ln h(a))^{\frac{\beta}{k}-1}du$$

$$= \frac{1}{\mathrm{k}\Gamma_k(\alpha)}\int_a^x \left(\frac{h'(u)}{h(u)}\right)(\ln h(x) - \ln h(a) + \ln h(a) - \ln h(u))^{\frac{\alpha}{k}-1}(\ln h(u) - \ln h(a))^{\frac{\beta}{k}-1}du$$

$$= \frac{1}{\mathrm{k}\Gamma_k(\alpha)}\int_a^x \left(\frac{h'(u)}{h(u)}\right)\big((\ln h(x) - \ln h(a)) - (\ln h(u) - \ln h(a))\big)^{\frac{\alpha}{k}-1}(\ln h(u) - \ln h(a))^{\frac{\beta}{k}-1}du$$

$$= \frac{(\ln h(x)-\ln h(a))^{\frac{\alpha}{k}-1}}{\mathrm{k}\Gamma_k(\alpha)}\int_a^x \left(\frac{h'(u)}{h(u)}\right)\left(1 - \frac{\ln h(u)-\ln h(a)}{\ln h(x)-\ln h(a)}\right)^{\frac{\alpha}{k}-1}(\ln h(u) - \ln h(a))^{\frac{\beta}{k}-1}du \tag{3.11}$$

Let $w = \frac{\ln h(u)-\ln h(a)}{\ln h(x)-\ln h(a)}$, *then* $dw = \left(\frac{h'(u)}{h(u)}\right)\frac{du}{\ln h(x)-\ln h(a)}$, by substitution w, 1-w in Eq.(3.11) we obtain

$$I_{H,h}^{k,\alpha,a^+}(\ln h(x) - \ln h(a))^{\beta-1} = \frac{(\ln h(x) - \ln h(a))^{\frac{\alpha+\beta}{k}-1}}{\mathrm{k}\Gamma_k(\alpha)}\int_0^1 (1-w)^{\frac{\alpha}{k}-1}w^{\frac{\beta}{k}-1}dw$$

We note that the last integral is Beta function, hence

$$I_{H,h}^{k,\alpha,a^+}(\ln h(x) - \ln h(a))^{\beta-1} = \frac{(\ln h(x) - \ln h(a))^{\frac{\alpha+\beta}{k}-1}}{\mathrm{k}\Gamma_k(\alpha)}B(\alpha,\beta)$$

$$I_{H,h}^{k,\alpha,a^+}(\ln h(x) - \ln h(a))^{\beta-1} = \frac{(\ln h(x) - \ln h(a))^{\frac{\alpha+\beta}{k}-1}}{\mathrm{k}\Gamma_k(\alpha)}\frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} = \frac{\Gamma_k(\beta)}{k\Gamma_k(\alpha+\beta)}(\ln h(x) - \ln h(a))^{\frac{\alpha+\beta}{k}-1}$$

4.Conclusion

In this paper, the results illustrate the examination of the properties of fractional integrals for the modified new k-fractional Hadamard integral and derivative operators. This modification of the new k-h-fractional Hadamard integral relies on an exponential kernel with an arbitrary exponent within the integral. Additionally, in relation to the new fractional integral operators. This work is framework for future work in application and fractional differential equations.

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